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On the history of variational methods of non-linear equations investigations and the contribution of Soviet scientists (1920s-1950s)[1]

Abstract. Variational methods for solving nonlinear equations (differential, integral, etc.) are currently perhaps the most common ones. However, the history of their origin and development, both in the USSR and in the whole world, has not been studied enough.

The author attempts to fill this gap, limiting himself mainly to 1920's-1950's, starting with Hilbert's works on justification of Dirichlet principle and with Poincaré's work on the study of closed geodesics on convex surfaces.

The article includes an analysis of the results of Soviet mathematicians – L.A. Lyusternik, L.G. Shnirelman, S.L. Sobolev, M.A. Krasnoselskii, M.M. Vainberg and others– obtained both in the field of solvability and direct methods, and in the field of qualitative analysis. Their achievements are gauged in the context of the development of this topic in the works of their foreign colleagues: L. Tonelli, W. Ritz, L. Lichtenstein, G.D. Birkhoff, M. Morse, A. Hammerstein, M. Golomb, and others.

Along the way, the question of the origin of Sobolev spaces and the mutual influence of functional analysis and variational methods for solving operator equations is investigated.

In conclusion, the author gives an example of applying the qualitative theory of variational methods to the problems of nonlinear mechanics that was realized by the Soviet scientist I.I. Vorovich.

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[1]The main results of this work were reported at the *XXIV Annual Conference of S.I. Vavilov Institute for the History of Science and Technology RAS* (v. Bogatov (2018)) and at the *XIX All-Russian Symposium on Industrial and Applied Mathematics* (v. Bogatov (2018)).

1. Introduction

By *variational methods of solving nonlinear equations* [2] we mean the methods for studying equations equivalent to the nonlinear equation

$$F(u) = 0 \tag{1}$$

and having the form

$$\Phi(u) \to min, \tag{2}$$

where $\Phi(u)$ is a functional and u has an appropriate smoothness. Since more than a century and a half has passed since the beginning of such methods, an analysis of their development is of independent interest (see, for example, [25], [70] -[68], [21]). However, with a few exceptions [59, §4], this analysis does not extend beyond the beginning of the 20th century. As for the role of Soviet scientists, it is indicated (rather briefly) in encyclopedias on the history of domestic mathematics [57, §3-4, §6], [22, §2-3, §5].

The aim of the author is to do more detailed historical-mathematical analysis of the development of variational methods in the USSR in the 1920s-1950s, including answering the questions:

1. What was the starting point in the development of variational methods in the USSR and what is its background?
2. What was the motivation of the evolution of variational methods until the middle of the 20th century?
3. What were the main tools of these methods and the object of research?
4. What is the contribution of Soviet[3] and foreign scientists to the elaboration of variational methods?
5. How did the results in the field of variational methods influence the development of functional analysis?

2. Prehistory. Dirichlet principle

In the middle of the XIX century Carl Friedrich Gauss [27, §29-34], William Thomson [81], and Peter Gustav Lejeune Dirichlet [52] proposed a new scheme for proving the solvability of the boundary value

[2] For brevity, we will write further *variational methods*.

[3] Here, and throughout the paper, by *Soviet mathematicians* we understand mathematicians from the Soviet Union; those whose activity took place in the period of 1922-1991, when the country USSR was in existence.

problem (BVP) for the Poisson equation with Dirichlet boundary condition

$$\Delta u(x) = f, \quad x \in \Omega \tag{3}$$
$$u(x) = g, \quad x \in \partial\Omega \tag{4}$$

where Δu is the Laplace operator, $\Omega \subset \mathbf{R}^3$ and functions u, f, g have an appropriate smoothness.

Considering that the solution of the first boundary value problem for (3) gives a minimum to some functional Φ, they deduced the solvability of this problem from of existence of the minimum of Φ (such a method[4] of proving solvability is a consequence of the equivalence of variational and boundary problems, established by Euler). The novelty was that in this way they tried to prove the *existence of solutions to boundary problems* [70, p. 208].

As is well known (see, for example, [69, p. 295]) since the middle of the 19th century Bernhard Riemann widely used Dirichlet principle in his works. At the same time, in contrast to his predecessors listed above, he realized the necessity of proving the existence of solutions to variational problems and outlined the ways of such a proof. However, his treatment was not fully rigorous from a formal point of view. Therefore Karl Weierstrass considered Riemann's reasoning on the Dirichlet principle as insufficiently substantiated. After that, the Dirichlet principle was abandoned for almost a half of a century.

We emphasize that the proof of the solvability of variational problems, which would constitute a main part of the rigorous justification of the Dirichlet principle, was hardly possible in the 19th century. This was mostly due to the lack of a theoretical basis for extracting a minimizing extremal from the infinite set E of admissible extremals. The obstacles had an abstract (topological) character, as the authors of [50, p. 120] indicated, forcing researchers to study the structure and properties of the functional class E first. The research of representatives of the Italian Mathematical School, Giulio Ascoli [5] and Cesare Arzelà [3] crucial. They characterized compact sets of functions in the space of continuous ones, allowing Arzelà to extend the Weierstrass theorem about the extremal values of continuous functions on the interval to the

[4] At the end of the 19th century the Dirichlet principle was presented as a way to prove the existence of solution to the Dirichlet problem for the equation (3) based on the fact that among all differentiable functions u taking specified values on the boundary Ω, the needed solution delivers a minimum to the Dirichlet integral $I(u) = \int\limits_{\Omega} |\nabla u(x)|^2 dx$ [37, p.98].

case of functionals [28, p.5]. After proving the theorem about a continuous functional attaining its minimum on a set of uniformly bounded and equicontinuous functions, Arzelà tried to revive the Dirichlet principle in [4] (1897), but it was not entirely successful [70, p. 213].The rigorous justification of this principle was given by David Hilbert in two papers in 1900 and 1904 [32]-[33] (see also [34, p.9-34]). Here Hilbert gave a proof of existence of a global minimum of the functional Φ, selecting from the entire class of admissible lines a compact class[5] K on which the minimum is attained. Moreover, he proposed to construct a minimizing sequence of functions $u_n(x)$ converging to the solution of the problem (2) and proved that the class K is not empty. Hilbert saw the advantage of the new method in using only the minimum property and not taking into account the special nature of the problem.

Exploiting his method, Hilbert proved the existence of a geodesic on a given surface and the solvability of the Dirichlet problem for the Laplace equation in a simply connected two-dimensional domain:

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

associated with the attainment of a minimum of a functional $I(u)$.

One of the first who switched to nonlinear variational problems was the Russian mathematician Sergei Natanovich Bernstein [8], [9, p. 115-118]. In 1910 he proved the existence of a minimum of the functional

$$J(u) = \iint_{\Omega} F(x, y, u, p, q)dxdy$$

by the method of analytic continuation (here $p = u_x$; $q = u_y$; $F \geqslant C(p^2 + q^2)^{\alpha}$; $\alpha > 0$).

3. The introduction of semicontinuity in variational type problems

As is known, the problem of substantiating the Dirichlet principle was the subject of research of many scientists (for details, see, e.g., [68]). One of the most prominent among them was Henri Lebesgue[6].

In his dissertation, published in 1902, he extended the concept of semicontinuity[7], given by René Baire for the real functions [6, p. 141], to functionals [51, §93, p.344].

[5] This class consisted of functions with bounded Dirichlet integral.

[6] A member of the USSR Academy of Sciences since 1929.

[7] The function $y = f(x)$ is called *lower semicontinuous* if $\forall(\varepsilon > 0)\forall\,(x \in D(y))\,[f(x + h) \geqslant f(x) - \varepsilon]$, where h is sufficiently small.

The advantage was seen in the fact that for the existence of the minimum of a real function $f : [a, b] \to \mathbf{R}$, the continuity of the function is not required. Lower semicontinuity is enough. Let us transform it to a more understandable form[8]:

$$f(x) \leqslant \varliminf_{n\to\infty} f(x_n) \quad \text{for} \quad x = \lim_{n\to\infty} x_n. \tag{5}$$

Now let x_n be a number sequence in $[a, b]$ such that

$$\lim_{n\to\infty} f(x_n) = \alpha,$$

where $\alpha = \inf_{x\in[a,b]} f(x)$. Since $[a, b]$ is a bounded segment, there is a convergent subsequence $x_n^{'}$ such that

$$\lim_{n'\to\infty} x_n^{'} = x^*.$$

The closedness of $[a, b]$ implies $x^* \in [a, b]$.

Further, from the inequality (5),

$$f(x^*) \leqslant \varliminf_{n'\to\infty} f(x_{n'}) = \alpha,$$

we deduce that $f(x)$ attains its minimum α at x^*.

Let us now turn to the functionals. The functional $J{=}J_C$ was called (by Lebesgue) *lower semicontinuous* on the set of curves M from the domain of J, if whatever the number $\varepsilon > 0$, for any curves C_1,$C_2 \in M$ close to each other the following inequality holds

$$J_{C_2} \geqslant J_{C_1} - \varepsilon. \tag{6}$$

This idea of Lebesgue found its development in the studies of the Italian mathematician Leonida Tonelli, a student of C. Arzelá. Starting from the paper [82](1911) and for 10 years[9], Tonelli developed the theory of calculus of variation based on the semicontinuity property. The result was the two-volume edition of "*Fondamenti di Calcolo delle Variazioni*" [83]-[84], parts of which were published in 1921 and 1923, respectively.

The main reasons that prompted Tonelli to devote so close attention to this topic were revealed by him at the International Congress of Mathematicians in Toronto in 1924 [85]:

[8]Here we follow a modern interpretation of semicontinuity, close to [96, Chap.38]

[9]Details and bibliography can be found, for example, in [18, p.24-35].

1. The possibility of extending the Weierstrass theorem to the most general class of functionals of the form

$$I(u) = \int_C F(x, u(x), u'(x))ds, \tag{7}$$

where C is an admissible curve or surface.

2. Transfer of consideration of variational type problems to the field of functional analysis.

3. Keeping the results as simple as possible.

4. Getting the independence of the calculus of variations from the theory of differential equations regarding questions of solvability.

Without going into other details[10], I would like to elaborate on the property of convexity of the function $F(x, u, u')$ with respect to u'. This is because that it was a prerequisite for semicontinuity, as was shown by Tonelli in [84, Ch. VI, X]. In modern language, Tonelli proved[11] that if the functional I defined in (7) is sequentially semicontinuous with respect to the topology of uniform convergence, then the function $F(x, u, z)$ is convex in the variable z.

Since nonconvex functionals of the form (7) may also appear in applications[12], their study is also of scientific interest. One of the first to undertake this work was the young Soviet scientist Nikolai Nikolaievich Bogolyubov (obviously, under the influence of Tonelli [24]). Using the original procedure, in 1929, Bogolyubov constructed a convex functional related to the original one (currently called the *relaxation functional*) and proved the equality between the lower extrema of these functionals (*relaxation theorem*) [13] - [14]; (see also [16, p.102-142]). This theorem gives a necessary and sufficient condition for the existence of a minimal nonconvex solution of a variational quasi-regular problem. Bogolyubov, in particular, showed that under natural conditions the non-convex functional of the simplest variational problem

$$\inf\Big\{\int_a^b F(x, u, u')dx; \quad u(a) = A, \ u(b) = B\Big\}$$

[10] They were well described by Tonelli himself in the review "Functional Analysis and Calculus of Variations" [86].

[11] A clear demonstration of this fact is given in [24, p.7-8].

[12] This became apparent in the mid-1970s in the framework of the nonlinear theory of elasticity [23].

can be replaced by a convex one (with the same minimum).

Although the proof and the methods used were completely new and in 1932 he published monograph [15] in which he revealed the essence of his research in more detail, these results by Bogolyubov did not have a worthwhile sequel. Most likely, this was due to the fact that since the 1930s Bogolyubov devoted himself entirely to solving actual problems[13] of theoretical and mathematical physics [1].

4. Direct methods of calculus of variations in nonlinear problems

The method proposed by Hilbert for the substantiation of the Dirichlet principle was called the *direct method of calculus of variations*. It suggested the possibility of constructing a sequence of functions $u_n(x)$, in a certain sense, *approximating* the unknown function and minimizing the functional Φ. It does not matter at all whether or not the initial problem is linear or the functional Φ is quadratic,. In addition, if the choice of $u_n(x)$ is efficient enough, then we can get a solution to problem (1) with a given accuracy.

By the beginning of XX century, many problems of mathematical physics were solved approximately by expanding the desired function in a Fourier series. A Göttingen mathematician Walther Ritz found it possible to use this idea to obtain an " engineering method[14]" for solving variational problems. In the paper [72] (1909) he presented the terms of the minimizing sequence $u_n(x)$ as some analog of partial sums of the Fourier series:

$$u_n = \sum_{i=1}^{n} a_i \varphi_i(x), \tag{8}$$

where $\varphi_i(x)$ are " coordinate" functions and a_i are numerical coefficients chosen so that the fuctional Φ attains minimum, i.e. they can be obtained by solving the system of equations

$$\frac{\partial \Phi_n}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n$$

where $\Phi_n = \Phi(a_1, ..., a_n)$.

Then, if we denote $\lim\limits_{n\to\infty} \Phi_n = \Phi^*$, it remains to show that $\min\limits_u \Phi(u) = \Phi^*$. In other words, to substantiate the Ritz method for continuous

[13]This activity was so successful that, at the age of 39, Bogolyubov was elected an academician of the Ukrainian SSR Academy of Sciences, and in the 1970s he was nominated for the Nobel Prize.

[14]This terminology was introduced by Henri Poincaré [50, p.128].

functionals, it was necessary to prove[15] that the limit function $u^* = \lim\limits_{n\to\infty} u_n$ does indeed minimize the functional Φ in some function space.

The justification of the Ritz method for nonlinear (one-dimensional) problems was performed by a German mathematician of Polish origin Leon Lichtenstein [53]. In 1915 he considered the boundary value problem

$$\begin{cases} y'' + f(x, y(x)) = 0, \\ y(0) = y(\pi) = 0, \end{cases} \tag{9}$$

which was reduced to the problem of minimizing the functional

$$\Phi(y) = \int_0^\pi [\frac{(y'(x))^2}{2} - F(x, y(x))]dx,$$

where $F(x, y) = \int_0^y f(x, s)ds \leqslant A$,
where A is some real number and $x \in [0, \pi]; y \in \mathbf{R}$.

The admissible set of functions was a set corresponding to the modern class $C_0^1[0, \pi]$.

Lichtenstein elaborated an approach containing a number of components of the modern direct method of calculus of variations for the study of (9), namely[16]:

1. Minimizing sequence$\{y_k\}$ is considered.
2. The boundedness of $\{y_k\}$ and $\{\int_0^\pi (y_k')^2 dx\}$ is proved. Based on this, a subsequence is obtained that converges uniformly on $[0, \pi]$ to some $\boldsymbol{y}$, which is included in the " energy" equality[17].
3. By using Green function of the linear problem corresponding to (9), it is deduced that previously found weak solution $\boldsymbol{y}$ has necessary smoothness.

Ritz method was adopted by the German mathematician Adolf Hammerstein in 1930 to prove the unique solvability of the equation

$$y(x) = \int_a^b K(x, s) f(s, y(x))ds, \tag{10}$$

where K is a symmetric, positive for all $x \in [a, b]$ and $y \in \mathbf{R}$, piecewise continuous bounded kernel function, and f is a continuous function [31].

[15]Ritz carried out this proof for a number of linear problems corresponding to positive definite quadratic forms Φ_n. The case of an indefinite quadratic form Φ_n in the one-dimensional situation was studied in detail by Nikolai Mitrofanovich Krylov and Yakov Davidovich Tamarkin in [45] ([46, p. 132-145]).

[16]The details see in [59, p. 453-454].

[17]This $\boldsymbol{y}$ can therefore be perceived as a weak solution of the problem (9).

If $0 < \lambda_1 < \lambda_2 < ...$ determine the eigenvalues of the linear problem

$$y(x) = \lambda \int_a^b K(x,s)y(s)ds, \tag{11}$$

then for some constants

$$C > 0 \quad \text{and} \quad \alpha < \lambda_1 \tag{12}$$

one can assume that the indefinite integral F over f satisfies the condition

$$F(x,y) \leqslant \frac{\alpha y^2}{2} + C \tag{13}$$

for any $x \in [a,b]$ and $y \in \mathbf{R}$.

To prove the solvability of equation (10) under the above assumptions, Hammerstein considerd a sequence of finite-dimensional problems of minimizing functionals H_n defined in R^n as follows:

$$H_n(c_1, c_2, \ldots, c_n) = \frac{1}{2}\sum_{j=1}^{n} \lambda_j c_j^2 - \int_a^b F(x, \sum_{j=1}^{n} c_j \varphi_j(x))dx,$$

where $\varphi_1(x), \varphi_2(x), \ldots,$ is an orthogonal system of eigenfunctions corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots$ of linear problem (11).

Relying on (12) and (13), one can show [74, § 15] that the functional H_n is bounded from below, therefore, it has the minimum value μ_n, attained at some vector

$$C^{(n)} = (c_1^{(n)}, \ldots, c_n^{(n)}) \in \mathbf{R}^n,$$

which generates a sequence of functions $\{y_n(x)\}$, with terms determined by the formula (8). The system of equations for determining the vector $C^{(n)}$ is equivalent to the original integral equation.

Hammerstein proved the existence of a subsequence $\{y_{n_k}(x)\}$ converging uniformly on $[a,b]$ to the solution y of equation (10). He obtained uniqueness for the case when the condition (13) was replaced by the condition that the function $f(x,y)$ is nondecreasing in the second variable.

5. Transition to a new level: the study of variational nonlinear problems in an abstract form

In the mid-1920s it became clear that the spectral theory of Hilbert spaces is an adequate mathematical tool of quantum mechanics. In particular, a quantum mechanical system can be described in terms of linear

symmetric operators in a Hilbert space, the eigenvalues of which are the energies of the eigenstates of the system, and the eigenfunctions are the corresponding stationary states of the system. This circumstance stimulated a growing interest in research of abstract theory of operators in a Hilbert space, and John von Neumann's works[18] in 1927-1930 were among the first ones. The publication of monographs of Stefan Banach [7] "Théorie des operations linéaires" in Poland and of Marshall Stone [79] "Linear Transformations in the Hilbert Space" in the USA (both in 1932) also helped the penetration of functional-analytic ideas into the minds of mathematicians of that time.

In those years, the Polish school of functional analysis was gaining strength and one of its most prominent representatives[19] was Otto Nikodym [66]. He had a wide range of scientific interests, among which there was the Dirichlet principle. Nikodym's research in this area was outlined in 1931 at a meeting of the Warsaw branch of the Polish Mathematical Society [67]. He showed that

In each convex and closed subset M of the Hilbert space H there is exactly one element with the minimum norm.

Nikodym used this fact to prove the existence of exactly one (up to measure zero) function, almost everywhere harmonic, that delivers minimum to the Dirichlet integral:

$$I(u) = \int_D |\nabla u|^2 dx \to \min. \tag{14}$$

As a continuation of this topic, Nikodym developed a method of orthogonal projection onto a convex set M to find the minimum of a functional in H and applied it to the proof of the unique solvability of the Dirichlet problem for elliptic equations [67]. Despite their fruitfulness, these ideas of Nikodym were not disseminated further [80, p.30]. The same fate befell the joint work of other prominent representatives of the Polish school of mathematics – Stanisław Mazur and Juliusz Schauder. At the International Congress of Mathematicians in 1936, they gave a report in which the following analogue of the classical Weierstrass theorem was presented, summarizing the above-described result of Nikodym [38, p.502]; [61]:

Let K be a closed convex subset of a Banach space E and $\varphi(\cdot)$ be a continuous convex functional on K. If $\{x_n\} \in K$ is a sequence weakly

[18] One of his most famous papers is "Allgemeine Eigenwerttheorie Hermitescher Funktional-operatoren" [91].

[19] He was among the sixteen scientists who created the Polish Mathematical Society [47, p. 51].

convergent to x_0 *and* $\lim\limits_{n\to\infty}\varphi(x_n)=\mu$, *where* $\mu=\inf\limits_{K}\varphi(x)$ *then* $x_0\in K$ *and* $\varphi(x_0)=\mu$. *The minimum of will be attained in* K.

The speakers stated that this theorem can be used to prove the solvability of problems like (11), but this was not confirmed in their subsequent publications [30, p.59], [38, p.503]. Development of these areas in other mathematicians' works happened much later (see, e.g., [19, p.31-32]).

Hilbert spaces in the 1930s remained the most attractive ones among function spaces. That was because of easy construction of an orthonormal basis in them (e.g., a trigonometric system of functions) and entering the coordinates for the "points" $f(x)$ of this space as coefficients of the expansion of $f(x)$ in the Fourier series. After that, we can already speak about analytic geometry in a Hilbert space E, based on the analogy with Euclidean geometry. In particular, variational problems, corresponding to linear equations, reduce to the study of extremes of quadratic forms in E and second-order hypersurfaces. An important role here was played by *completely continuous*[20] operators, which made it possible to draw a line between the finite-dimensional and infinite-dimensional case[21].

Hammerstein's student Michael Golomb made great strides in the study of nonlinear variational problems in a Hilbert space E. He was the first (1935) who introduced [29] an abstract analog of equation (10):

$$y-H^2G(y)=0. \tag{15}$$

Here $H:E\to E$ is bounded, completely continuous operator, such that for some $h>0$ and $\forall y\in E$ the following inequality holds

$$(Hy,Hy)\leqslant\frac{1}{h}(y,y).$$

Moreover, the operator G is defined on the closure E' of the range of H, is continuous on E' and is the gradient of a real function F on E' such that

$$F(y)\geqslant-\frac{k}{2}(y,y)-C\qquad\forall y\in E'$$

with some number $k:0<k<h$ and a constant $C>0$.

[20] A nonlinear operator $A:E\to E$ acting in a Banach space E is called *completely continuous* if it is continuous and transforms each bounded set into a compact one (Frigyes Riesz, J. Schauder).

[21] For more details see, e.g., [20, Ch. VII, p. 261-262].

The solution of equation (15) is obtained by minimizing the functional

$$\Psi(y) = (y, y) - 2F(Hy)$$

over the space E'.

Let us emphasize that the existence of a minimizing sequence was proved by Golomb *without passing to finite-dimensional subspaces* and was based only on the weak compactness of the sphere in the Hilbert space and the properties of the operator H.

6. Variational methods as a "catalyst" for the emergence of new functional spaces[22] (as shown by the example of S. L. Sobolev works)

In 1937, the direct method of the calculus of variations was used by Sergei Lvovich Sobolev for proving solvability of the following BVP for the polyharmonic equation:

$$\Delta^m u = 0 \quad x \in \Omega, \tag{16}$$

where $\Delta^m u = \left(\sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)^m (u)$;
$\Omega \in \mathbf{R}^n$ is a domain which has a boundary $S = S_{n-1} \cup S_{n-2} \cup ... \cup S_0$ such that S_{n-i} is piecewise a manifold of dimension $(n-i)$ [75]. The function $u(x)$ and all of its derivatives of order from 1 up to $(m - \left[\frac{s}{2}\right] - 1)$ take (on average) fixed values on every surface S_{n-i}.

In fact he applied the modified Hilbert scheme:

1. A functional sequence $\{u_j\}$was chosen whose terms satisfy boundary conditions. It converges under the Dirichlet integral

$$\lim_j D_m(u_j) = d,$$

in a suitable Hilbert space. Here

$$D_m = \int\limits_{\Omega} \sum_{\sum \alpha_i = m} \frac{m!}{\alpha_1! \dots \alpha_n!} \left(\frac{\partial^m u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}\right)^2 dV$$

and $d = inf D_m(u)$.

[22]This topic was suggested to the author by S.S. Petrova in the section of the history of mathematics on the XXIV Annual Conference of IHST RAS.

2. The convergence in the mean square of the sequence $\{u_j\}$ as well as of its derivatives of order up to m was proved[23]:

$$D_r(u_i - u_j) \to 0, \quad \forall\ r = 0, 1, ..., m$$

whenever i, j tend to infinity.

3. It was deduced that the limit function $u^* = \lim\limits_{j\to\infty} u_j$ satisfies equation (16) and the boundary conditions both in the *weak* sense,

$$\int_D u^* \Delta^m \varphi dx = 0 \tag{17}$$

for any smooth finite function φ, and in the classical sense.

A hallmark of Sobolev approach was, therefore, the introduction of integral identities (17) to the scheme of the direct method, which expand the interpretation of solutions[24]. This naturally led to the need for introducing a new concept of a derivative.

The study of general differential expressions of the form

$$Lu = f, \tag{18}$$

where L is a differential operator acting in a Hilbert space H, led Sobolev to considering the relations

$$(Mv, u) = (v, f). \tag{19}$$

Here M is the Lagrange conjugate[25] of L, and the functions $u(x)$ and $v(x)$ belong to the class $\overset{\circ}{E}$ of finite smooth functions. (v, f) denotes the scalar product, which can be interpreted as a linear functional f in the class $\overset{\circ}{E}$. The concept of a (*generalized*) derivative was introduced by Sobolev through the identification of the functional $u(w) = (w, u)$, where $w = Mv$, with the function u (see [76, 77, §6]).

Sobolev showed that success in the formulation and investigation of boundary value problems essentially depends on the choice of the corresponding space from the scale of functional spaces which now bear his name. The methods applied by Sobolev in [75], [76, 77] were developed by him in a number of subsequent works (see the bibliography,

[23] This step has been carried out by the Friedrichs algorithm [26].

[24] Not only Sobolev arrived at this expansion; for historically-mathematical analysis of the situation, see, for example, in [64].

[25] M appears when we integrate by parts expression for $u(x)$, $v(x) \in \overset{\circ}{E}$: $(Lu, v) = (u, Mv)$.

for example, in [12]). They yielded important "embedding theorems" which are now among the main methods of functional analysis used in the theory of partial differential equations [2].

As it turned out, the introduction of the concept of a generalized derivative made it possible to go far beyond the bounds of direct methods for solving problems of variational type. In the transformed scheme of the proof of solvability[26] Sobolev shifted emphasis from the equation(19) to the result of its integration by parts (18). The next step envisaged the construction of an approximate solution (as a rule, the Ritz method or its modifications were used) and the proof of its convergence to the generalized solution u^*. At the last stage, the question of the smoothness of u^* was discussed with the help of embedding theorems.

As a model example, we consider the initial-boundary problem for the nonstationary Navier-Stokes equation[27] in a bounded domain $\Omega \in \mathbf{R}^3$ with fixed boundary S, following [48, Ch. VI]:

$$\frac{\partial \bar{v}}{\partial t} - \nu \Delta \bar{v} + \sum_{k=1}^{3} v_k \frac{\partial \bar{v}}{\partial x_k} = -grad\, p + \bar{f}(x,t) \tag{20}$$

$$div\, \bar{v} = 0, \quad \bar{v}|_S = 0, \quad \bar{v}|_{t=0} = \bar{a}(x). \tag{21}$$

It is also assumed that

$$div\, \bar{a}(x) = 0, \quad \bar{a}|_S = 0. \tag{22}$$

Olga Alexandrovna Ladyzhenskaya and Andrei Alexeevich Kiselev showed [36] that it is most natural to consider problem (21) - (22) in a generalized formulation, for which it is possible to prove unique global solvability in Sobolev-type spaces.

According to Ladyzhenskaya, [48, p. 136]

"*... Sobolev spaces organically connected with the variational form of hydrodynamical problems, and the basic law of energy dissipation and other regularities of the moving fluid are expressed in their norms. In other words, the main a priori estimates, on which all studies are based, are formulated in the norms of these spaces*[28]."

This makes it possible, in particular, to prove the existence and uniqueness of the generalized solution of the problem (21) - (22) in a

[26] There was an opportunity to consider both linear and quasi-linear equations. For details see the preface in [49].

[27] It describes the motion of an incompressible fluid.

[28] Translation from Russian original was made by the author of this article.

simple way for sufficiently smooth initial data and for any $t \geqslant 0$ in the case of potential external forces or axial symmetry of the domain Ω [48, Ch.VI,§2-3].

7. Qualitative methods of analysis of nonlinear problems admitting variational formulation. Let us return to the variational type problems of the beginning of the 20th century, which can be reduced to the minimum of some functional Φ. One of these problems, originating from the three-body problem, is the problem of geodesic lines on convex surfaces. It was posed by Poincaré in 1905 and partially solved by him in [71]. In particular, he showed that the minimum principle is sufficient for detecting closed geodesics on surfaces of genus greater than zero (i.e., on the surfaces on which not all lines are contractible to a point).

For a surface S of genus zero, Poincaré used the so-called *method of analytic continuation*. He entered the parameter t into the surface equation, which was chosen so that for one of the values of t the surface would turn into an ellipsoid, and into the surface S for the other. Poincaré studied the variations of geodesic lines, together with the change of the parameter t. He conjectured that the number of closed geodesics on the surface of the genus zero is greater than or equal to three (which is true for an ellipsoid).

If we take the more general variational type problem for the surface S of zero genus,

$$\Phi(u) = \int_S F(u) d\sigma \to min$$

then in the case of a positive definite function $F(u)$, the closed extremal does not deliver the minimum to $\Phi(u)$ (the minimum will be attained on the curves that degenerate into points). Topological studies of stationary points and minimax-type extremals on convex surfaces were started by the American mathematician George David Birkhoff in 1917 [10]. Considering the second variation of $\Phi(u)$ in the case when $\delta\Phi = 0$, we find different behavior of the function in the neighborhood of the stationary point depending on the sign of the eigenvalues of $\delta^2\Phi$. Birkhoff studied the case when $\delta^2\Phi$ has only one negative eigenvalue; a more general situation was described by his colleague and compatriot Marston Morse in 1925 [62].

The topological approach to solving variational problems started by Poincaré and continued by Birkhoff and Morse has been actively used also in the USSR since the mid-1920s. Thus, in 1924, Pavel Samuilovich Urysohn made a report at a meeting of the Moscow Mathematical Soci-

ety in which he gave a proof of the existence of two closed geodesics on surfaces of genus zero. The proof was based on the minimax principle; it was not published [56, p.13].

By the beginning of the 20th century it was known that many properties of functions and functionals depend on the topological structure of the space where they are defined. In particular, these properties include the *number of critical points*[29], as well as the number of solutions of variational problems. For example, on a sphere it is easy to define a triply differentiable function that has only two critical points, a minimum and a maximum, but functions defined on the torus have at least three critical points.

The important question of estimating the number of solutions of the problem of functional extremum should first be solved for functions. It is necessary to distinguish the *geometrically* and *analytically* different stationary points.[30] Among the first scientists who mastered the technique of estimating the multiplicities of analytically different stationary points was Morse [63], [17, 73, §4]. Another method has been applied to the number of geometrically different singularities of the functions. A lower bound for them was obtained with the help of a homotopic invariant, *a category*, introduced by the Soviet mathematician Lazar Aronovich Lyusternik in 1930 [54]:

For a closed set A contained in the space M the minimum number of closed sets covering A in total, each of which is contractible to a point by a continuous deformation in M, is called the category $(cat_M A)$ of A with respect to the space M.

It turned out that the category of the manifold M with respect to M itself is not less than the number of geometrically different critical points of a twice continuously differentiable functions defined on M.

L.A. Lyusternik and Lev Genrikhovich Shnirelman extended the concept of category and main results about it to the case of functionals, which allowed them to estimate the number of solutions of variational problems [56]. As applications, they considered problems of extrema of positive definite functionals defined on certain families of curves, in particular the problem of geodesic lines. Defining the concept of a category for a family of lines on a surface, they showed that on a surface with zero genus one can construct a family of closed lines with $cat_M A \leqslant 4$, which ensured the existence of no more than four solutions of the corresponding variational problems (e.g. a closed geodesics problem). One

[29] The points at which the differential of the function is equal to zero.

[30] Two analytically different singular points may coincide geometrically.

of these solutions is trivial (an isolated closed point), the other three give three closed extremals (for example, closed geodesics). Thus, the Poincaré problem of three geodesics was completely solved [56], [58].

Another application of Lyusternik- Shnirelman theory of categories to problems of variational type was obtained by Lyusternik ten years later[31]. In 1939, he estimated the number of critical points of functionals in a Hilbert space H, generalizing the property of a quadratic weakly continuous functional to have a countable system of critical points [55].

More precisely, he discovered the following interesting fact: *on each sphere of the Hilbert space H the gradient of an even homogeneous functional has at least a countable number of eigenvectors*[32].

The property of the functional being even allowed Lyusternik to proceed with the study of the functional on projective spaces obtained by identifying the antipodal points of the mentioned sphere. Category theory was used to estimate the number of critical points.

Lyusternik's theorem was continued in the works of Mark Alexandrovich Krasnoselskii (see below) and Mordukhai Moiseevich Vainberg [88], as well as Iosif Izrailevich Vorovich [92].

8. The development of variational-qualitative methods in the works of M.A. Krasnoselskii. After the publication of Golomb's work it became clear that the variational methods are applicable to the case where the studied equation can be reduced to the the form

$$\psi = G\psi, \tag{23}$$

in which G is the gradient operator of the functional F defined in Hilbert space $\mathcal{H}$. In this case, the solutions of equation (23) are the critical points of the functional

$$\Phi(\psi) = \frac{1}{2}(\psi, \psi) - F(\psi).$$

In the late 1940s, this topic was addressed by the Kiev mathematician M.A. Krasnoselskii, one of the students of Bogolyubov-Krylov School of Nonlinear Mechanics [73]. Krasnoselskii derived an important criterion of the existence of critical points of functionals, systematizing and generalizing the results of his predecessors, Hammerstein and Golomb.

To formulate the indicated criterion, we need some definitions.

A functional Φ, defined on Hilbert space $\mathcal{H}$, is called *increasing,* if

$$\lim_{\|y\|\to\infty} \Phi(y) = \infty.$$

[31] A new stage of its development began in the 1960s. See details in [60, p. 330-332].

[32] An explanation of this fact can be found in [96, p. 103-105].

A functional F, defined on Banach space E is said to be *differentiable with bounded remainder on a set* $M \subset E$ if its increment is represented as

$$F(y+h) - F(y) = l(h) + \omega(y,h),$$

where $l(h)$ is linear continuous functional on E, $l \in E^*$ and the fraction $\dfrac{\omega(y,h)}{\|h\|}$ tends to zero uniformly (with respect to $y \in M$) when $\|h\| \to 0$.

And now Krasnoselskii principle [40, p.950]:

THEOREM 8.1 *Let F be an increasing functional which is differentiable with bounded remainder in all balls of Hilbert space $\mathcal{H}$.*

Let the gradient G of the functional Φ be represented as $G = G_1 + G_2$, where G_1 is a linear continuous operator having a continuous inverse, and G_2 is a completely continuous operator.

Then there is a point $y_0 \in \mathcal{H}$ at which the functional Φ attains its minimum. Moreover, $G(y_0) = 0$.

A great merit of Krasnoselskii is that he was able to get new results of a *qualitative* nature based on variational methods. One of these results was the *principle of linearization in the problem of bifurcation points.*

The equation $A\varphi = \lambda\varphi$ (with $A\theta = \theta$, where θ is the zero element of a Banach space E) usually has no small non-trivial solutions for small λ, i.e. there are no small eigenvalues corresponding to eigenfunctions with a small norm. But with increasing λ, small eigenfunctions can appear.

The number λ_0 is called *a bifurcation point* of a nonlinear operator A, $(A\theta = \theta)$ acting in E if $\forall \varepsilon, \delta > 0$ there exists an eigenvalue λ and an eigenfunction $\varphi : A\varphi = \lambda\varphi$, such that

$$|\lambda - \lambda_0| < \varepsilon, \|\varphi\| < \delta.$$

Simply put, bifurcation points define the boundaries of those values of λ for which small eigenfunctions cannot appear.

As an example, considered by Krasnoselskii in [43], we give the problem of the longitudinal bending of a hinged rod.

The shape of a compressed rod of unit length and variable stiffness s under the influence of a force λ (see Fig. 1) is determined by the boundary value problem

$$\begin{cases} y''(s) + \lambda\rho(s)y(s)\sqrt{1 - y'^2(s)} = 0, \\ y(0) = y(1) = 0. \end{cases} \tag{24}$$

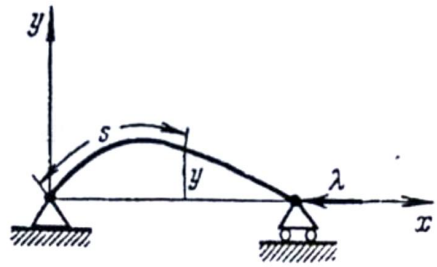

Figure 1: Rod bending by a concentrated force [11, p. 338].

If we make the substitution $y''(s) = -\varphi(s)$, then we can reduce the boundary value problem (24) to a nonlinear integral equation (see, e.g., [11, Ch. VI,§3.4]) :

$$\varphi(s) = \lambda\rho(s)\int_0^1 K(s,t)\varphi(t)dt\sqrt{1-\left[\int_0^1 K'_s(s,t)\varphi(t)dt\right]^2}, \qquad (25)$$

where $K(s,t)$ is the Green function of the operator $Ay = y''$ under zero boundary conditions.

The equation (25) has a trivial solution for all values of the parameter λ. For some values of the load λ, this equation may have nontrivial solutions that determine the forms of buckling. The load λ_0 is called the *Euler critical load* if, for some loads close to λ_0, the equation (24) has small nontrivial solutions, i.e. λ_0 is the bifurcation point for the rod bending equation.

For further analysis, it is more convenient to linearize the problem, which leads to the equation

$$\varphi(s) = \lambda\rho(s)\int_0^1 K(s,t)\varphi(t)dt. \qquad (26)$$

If $\psi(s)$ is a nontrivial solution of the equation (25) at $\lambda = \lambda_0$, then the function

$$y(s) = \int_0^1 K(s,t)\psi(t)dt$$

will be a solution of the equation

$$y''(s) + \lambda_0\rho(s)y(s) = 0$$

with zero boundary conditions.

Representing equation (26) in the form $\varphi = \lambda B\varphi$ and equation (25) in the form $\varphi = \lambda A\varphi$, one can note that all bifurcation points of the

nonlinear operator A are characteristic values of the linear operator B (this, in particular, was shown for specific equations by Hammerstein [31] and Rudolf Iglisch [35]). The converse statement is not always true [42, p. 196]. But if we impose on the operator A the requirement of differentiability[33] and potentiality (in addition to complete continuity), then linearization becomes valid. This was proven by Krasnoselskii[34]:

THEOREM 8.2 *Let G $(G(\theta) = \theta)$ be a nonlinear completely continuous operator that is the gradient of a weakly continuous functional Φ $(\Phi(\theta) = \theta)$ which is smooth in some neighborhood of the point θ. Let the operator G have the Fréchet derivative B at the point θ such that B is a completely continuous self-adjoint operator.*

Then each eigenvalue of the linear operator B is the bifurcation point of the operator G.

Having rejected the requirement of smoothness of the functional Φ, Krasnoselskii proved the existence of a single bifurcation point, based on the properties of functionals that are close to quadratic [41, §1–2]. Precisely speaking, he showed that *the largest positive eigenvalue of the operator B is the bifurcation point of the operator G.*

Using variational methods, Golomb proved new theorems of the existence of nonlinear operators' eigenfunctions. In particular, he showed that in the space L_2 *potentiality of the operator implies it has a continuum of different eigenvectors* [29, p.65].

Extending the result of Golomb, Krasnoselskii proved the following theorem [42, p.343]:

Let a functional Φ defined on a ball $B_{\rho_0}(\|\varphi\| \leqslant \rho_0)$ of a Hilbert space H, be weakly continuous and differentiable on B_{ρ_0}.

Then, in every ball $B_\rho(0 < \rho \leqslant \rho_0)$ the operator G which is the gradient of the functional $\Phi(\varphi)$, has a continuum of eigenvectors.

The application of this theorem to the study of specific equations $A\varphi = \varphi$ consists in constructing a weakly continuous functional in the space L_2 whose eigenfunctions can be used to obtain the eigenfunctions of the operator A.

[33] An operator A has the Fréchet derivative B at the point θ (more briefly, an operator A is *differentiable*) if there exists a linear operator B such that

$$\lim_{\|\varphi\|\to 0} \frac{\|A(\theta+\varphi) - A(\theta) - B(\varphi)\|}{\|\varphi\|} = 0.$$

[34] Here he used rather complex constructions based on the ideas of Lyusternik and Shnirelman [41, §5].

Further development of variational methods was carried out by Krasnoselskii in the direction of the proof *of theorems on estimating the number of eigenvectors of the functional's gradient.* Krasnoselskii proposed a proof of the corresponding Lyusternik's theorem (see above, the end of Section 6), based on the use of a new topological invariant: *the genus of a set.* It coincides on a sphere with the category of the image of a set in the projective space [39, p.157].

Generalizing the result of Lyusternik, Krasnoselskii got the following theorem [42, p. 367]

Let a weakly continuous smooth[35] *functional* F *defined on the ball* B *of a Hilbert space* $\mathcal{H}$ *satisfy the following requirements:*

a) evenness on S: $F(-\varphi) = F(\varphi),\ \varphi \in S$;

b) non-negativity: $F(\varphi) \geqslant 0, \quad \varphi \in B$;

and takes the zero value only at the zero of the space $\mathcal{H}$ *;*

c) if G *is the gradient of the functional* $F(\varphi)$*, then* $G\varphi \neq \theta$ *for* $\varphi \neq 0$ *and* $G\theta = \theta$.

Then G *has on the sphere* S *at least a countable number of different critical points, which correspond to different critical values*[36] *of the functional* F.

An example of an application of this theorem was the study of the odd Hammerstein operator H_1 with a positive definite kernel $K(x, y)$:

$$H_1 u(x) = \int_D K(x, y) f(y, u(y)) dy. \tag{27}$$

Krasnoselskii transformed it into the operator $H^* \mathbf{f} H$ so that each function $H\varphi$, where φ is an eigenfunction of the operator $H^* \mathbf{f} H$, is an eigenfunction of the linear operator $B = H^* H$, where

$$B\psi = \int_\Omega K(s, t)\psi(t) dt; \tag{28}$$

H is a completely continuous linear operator, $H : L_2 \to L_p$;

H^* is adjoint to H, $H^* : L_q \to L_2$;

$\boldsymbol{f}$ is the nonlinear *Nemytskii* operator[37] [65]: $\mathbf{f}\varphi(t) = f[t, \varphi(t)]$, $\mathbf{f} : L_p \to L_q$.

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[35] Instead of smoothness, one can assume that the functional F is increasing [44, p.21].

[36] A point $\varphi \in S$ is called a *critical point of the functional* F on the sphere S if φ is an eigenvector of the gradient G of this functional [39, p.159].

[37] From the point of view of mathematical analysis, the expression $f(y, u(y))$ in the formula (27) can be considered as a *function* of two variables. From a functional-analytical point of view, the same expression can be interpreted as an *operator* $\mathbf{f}$.

The operator $H_1 = H^*\mathbf{f}H$ is the gradient G of the functional F which has the form

$$F(H\varphi) = \int_\Omega \left\{ \int_0^{H\varphi(s)} f(s,u)du \right\} ds.$$

The requirements a) - c) of the theorem are satisfied if the following conditions hold:

$$f(s,-u) = -f(s,u); \qquad (s \in \Omega, u \in (-\infty, +\infty)) \tag{29}$$

$$f(s,u) \cdot u > 0 \qquad (s \in \Omega, u > 0). \tag{30}$$

As a result, Krasnoselskii showed [42, p.380-381] that under the conditions of the previous theorem and conditions (29) and (30), *the eigenfunctions φ_n of the Hammerstein operators with positive definite kernels constitute no less than a countable set. They have a form*

$$\varphi_n(s) = \sum_{i=1}^{\infty} \xi_i^{(n)} \psi_i(s),$$

where $\psi_i = H\varphi_i$ are eigenfunctions of the operator B (see (28) with corresponding eigenvalues λ_i; $\{\xi_i^{(n)}\}$ is real number sequence such that

$$\sum_{i=1}^{\infty} \lambda_i \left(\xi_i^{(n)}\right)^2 < \infty.$$

Moreover, for any $C > 0$, the equality[38]

$$\sum_{i=1}^{\infty} \lambda_i \left(\xi_i^{(n)}\right)^2 = C^2, \quad n = 1, 2, ... \tag{31}$$

holds.

The possibility of applying Lyusternik's theorem to the problem of eigenfunctions of the Hammerstein operator and conditions (29) - (30) were first indicated by Vladimir Ivanovich Sobolev in 1950 [78].

Note that Krasnoselskii additionally investigated the stability of the critical values of an even functional, proving that these values do not disappear under small perturbations by odd functionals [42, Ch. VI, §4].

9. The development of variational methods in the works of M.M. Vainberg. The Moscow mathematician M.M. Vainberg (a student of Victor Vladimirovich Nemytskii) was engaged in the construction of a variational theory of nonlinear operators in the 1950s in parallel

[38] Equality (31) can be interpreted as the equation of an ellipsoid in the Hilbert space $\mathcal{H}$: $\sum\limits_{i=1}^{\infty} \frac{(x_i, x_i)}{a_i^2} = 1$, where $a_i > 0$ and $\forall\ x \in \mathcal{H}$ the unique composition $x = x_1 + ... + x_n + ...$ is true [88, p.126].

with Krasnoselskii. Without being able (and necessary) to describe in detail all of his results [87]-[89] here, we will list *the main* problems that Vainberg managed to solve during the period of consideration:

1. Finding the necessary and sufficient conditions for the potentiality of differentiable and non-smooth operators (this problem has got a complete solution). The conditions found resembled the known conditions for the potentiality of a vector field in a finite-dimensional space.
2. The possibility and validity of replacing the equation $\Phi(x) = 0$ where the operator Φ is not potential, by an equivalent equation $\Psi(x) = 0$ where the operator Ψ is potential. This problem got a positive solution in a real Hilbert space H and in the spaces L_p, where $p > 2$ for operators of the form $\Phi(x, \lambda) = AF(x) - \lambda x$, where A is a linear operator satisfying some conditions, F is a potential operator, is a real parameter.
3. Finding conditions for the solvability of equations of the form $x = BF(x)$, where $B(x)$ is a self-adjoint positive (*quasidefinite*) operator defined in H and $F(x)$ is a potential operator.
4. The definition of conditionally extremal and conditionally critical points of functionals defined in a *Banach* space and the derivation of necessary and sufficient conditions for their existence. A transfer of the principle of a critical point from finite-dimensional to infinite-dimensional Banach spaces E.
5. Investigation of the problem of eigenfunctions of the product BF of small and large norms for cases of positive and quasidefinite operators B acting in H and questions on the spectrum of the operator BF. Finding out the conditions of the existence of at least one bifurcation point of the operator BF.
6. Investigation of the properties of Hammerstein, Nemytskii and Lyapunov-Lichtenstein[39] T operators: continuity, differentiability and potentiality. Application of general theorems to the proof of the solvability of corresponding equations, the study of their spectrum and behavior of eigenfunctions under various conditions on their kernels.

10. Conclusion. Variational methods for solving nonlinear equations can be divided into direct and qualitative ones. The former origi-

[39] $T(u(s)) = \sum_{n=0}^{\infty} \int_0^1 \cdots \int_0^1 K_{2n+1}(s; s_1, ..., s_{2n+1}) u(s_1) ... u(s_{2n+1}) ds_1 ... ds_{2n+1}$

nate from the Dirichlet principle (Hilbert); the latter are based on the minimax principle (Birkhoff). Direct methods, in turn, developed in two areas - " *functional*" (Tonelli), which built on the semi-continuity property of minimized functionals, and "*operator*" (Golomb), which were based on the property of complete continuity and potentiality of operators. Herewith, by the end of the 1930s to the beginning of the 1940s, the "operator direction" began to dominate both in Europe (with the exception of Italy) and in the USSR. This, apparently, can be explained by the rapid development (in the indicated period) of the theory of operators in Banach and Hilbert spaces, which includes the theory of functionals as a special case.

Some ideas of Tonelli expressed in the context of understanding the mutual influence of functional analysis and calculus of variations in 1940 [86, p. 296] were developed by Soviet mathematicians. Namely, in order to study the functional F to be minimized in the class of functions K, it is advisable to consider

a) another functional G, with better properties, which has the same minima as F;

b) another, narrower class of functions for which the functional F possesses the necessary properties[40].

We can assume that the implementation of item a) was carried out by Vainberg (see above: the replacement of the equation $\Phi(x) = 0$ by the equation $\Psi(x) = 0$), and the implementation of item b) was done by Sobolev (embedding theorems). Sobolev spaces, which emerged as a "by-product" in the development of variational methods, quickly became their classic tool. The direct use of the properties of semicontinuous functionals in the study of nonlinear operator equations experienced its rebirth in the USSR in the early 1970s (see, e.g., [90, Chap. III]).

It can be said that by the end of the 1950s, in the field of qualitative methods for studying problems of variational type Soviet mathematicians took the leading positions. This is evidenced, in particular, by the fact that in most monographs on nonlinear functional analysis, among references to the primary sources the first and often the only ones are the works of Lyusternik, Shnirelman, Krasnoselskii and Vainberg (see, e.g., [96, p.350]). In addition to the studies of these scientists associated with the concept of a category and their further development, I would also like to note the achievements of the Soviet mechanician I.I.

[40]The most important property, according to Tonelli, was the semicontinuity of the functional F.

Vorovich in the second half of the 1950s [93] - [95]. In [93] he considered a mathematical model of the casing deformation, taking into account the influence of transverse deflections, and reduced the equation of its state to a nonlinear integro-differential equation that allows the operator interpretation:

$$w = Aw, \tag{32}$$

where A is a potential operator. This allowed him to take as a foundation the methods developed by Krasnoselskii and Vainberg (see above) and adapt them to prove the solvability of equation (32) in Sobolev-type spaces.

In addition, in [95], Vorovich considered an important (in practice) problem of shell buckling, whose state is described by equations of von Kármán type. For the Karman plates, he justified the principle of linearization in the problem of losing their stability, similar to the principle of linearization of Krasnoselskii obtained for rods (see Section 7). Going further, Vorovich investigated the behavior of the plates after losing their stability [92], [94], based on the Lyusternik-Krasnoselskii theorem on estimating the number of critical points of the functional.

Thus, thanks to the work of Soviet scientists in the period under review, functional analysis was enriched by a new class of functional spaces (Sobolev spaces), a new type of operators (potential operators) and methods for working with them (including Lyusternik-Shnirelman theory). After such enrichment, it was able to provide tools for the development of variational methods in areas with applications (for example, hydrodynamics and nonlinear mechanics).

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O historii badania równań nieliniowych metodami wariacyjnymi i wkładzie w te badania naukowców sowieckich (lata 1920-1950)

Egor Mikhailovich Bogatov

Streszczenie. Metody wariacyjne rozwiązywania równań nieliniowych (różniczkowych, całkowych itp.) są prawdopodobnie najpopularniejszymi obecnie metodami. Jednak historia ich stworzenia i rozwoju, zarówno w ZSRR jak i na całym świecie nie została wystarczająco zbadane i opisana.

Celem tej pracy jest wypełnienie tej luki. Autor ogranicza się głównie do okresu 1920-1950, poczynając od prac Hilberta na temat uzasadnienia tzw. zasady Dirichleta i pracy Poincarégo nad badaniem zamkniętych linii geodezyjnych na powierzchniach wypukłych.

Artykuł zawiera analizę wyników matematyków sowieckich - L.A. Lusternika, L.G. Sznirelmana, S.L. Sobolewa, M.A. Krasnosielskiego, M.M. Wainberga i innych- uzyskanych zarówno w dziedzinie istnienia rozwiązań, jak i metody bezpośredniej oraz w zakresie analizy jakościowej. Ich osiągnięcia są oceniane w kontekście rozwoju tego tematu w pracach ich zagranicznych kolegów - L. Tonellego, W. Ritza, L. Lichtensteina, G.D. Birkhoffa, M. Morse'a, A. Hammersteina, M. Golomba itd.

Przy okazji pojawiają się informacje o pochodzeniu przestrzeni Sobolewa i badany jest wzajemny wpływ analizy funkcjonalnej i metod wariacyjnych na rozwiązywanie równań operatorowych.

Podsumowując, autor podaje przykład zastosowania jakościowej teorii metod wariacyjnych do problemów mechaniki nieliniowej, które zostało dokonane przez sowieckiego naukowca I.I. Worowicza.

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Słowa kluczowe: Historia nieliniowej analizy funkcjonalnej; metody wariacyjne; zasada Dirichleta; równanie Hammersteina; metoda Ritza; operatory potencjalne; teoria Lusternika-Sznirelmana; przestrzenie Sobolewa; funkcjonały półciągłe; uzasadnienie linearyzacji w problemach bifurkacji punktów; D. Hilbert; L. Tonelli; M. Golomb; L. Lichtenstein; M. A. Krasnosielski; M. M. Wainberg; I. I. Worowicz..

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