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HISTORY AND PHILOSOPHY OF MATHEMATICS

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Exploring Felix Klein's contested modernism

Abstract. An alleged opposition between David Hilbert and Felix Klein as modern *vs* countermodern[1] has been pursued by the marxist historian Herbert Mehrtens and others. Scholars such as Epple, Grattan-Guinness, Gray, Quinn, Rowe, and recently Siegmund-Schultze and Mazzotti have voiced a range of opinions concerning Mehrtens' dialectical methodology. We explore contrasting perspectives on Klein's contested modernism as well as Hilbert's and Klein's views on intuition, logic, and physics. We analyze Jeremy Gray's comment on Klein's ethnographic speculations concerning Jewish mathematicians and find it to be untenable. We argue that Mehrtens was looking for countermoderns at the wrong address.

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1. Introduction. Felix Klein's stature as a modern has been subject to dispute; see Mehrtens [33], Bair et al. [3], Franchella [13], and other commentators. Herbert Mehrtens (1946–2021), henceforth HM, sought to cast Klein in an unenviable role of a countermodern.

[1]Editor's remark: "Modern vs countermodern" refers to a comparison or opposition between modern and countermodern perspectives or ideologies. In this context, it is used to discuss contrasting views or approaches. The term "modern" typically relates to contemporary ideas, while the term "countermodern" suggests a position or approach that opposes or contrasts with modernism. The phrase is often used to explore the tensions or debates between different philosophical or intellectual stances within a given context.

HM was subsequently provided with a platform by Goldstein, Gray, and Ritter to present his views in [34]. HM's approach has influenced a number of authors, including Gray [16], Quinn [41], and most recently Mazzotti [32], as we analyze below.

In his recent obituary for HM, Reinhard Siegmund-Schultze appears to endorse HM's dichotomy of moderns and countermoderns (as well as to a certain extent the assignment of Klein to the latter group). We will examine Siegmund-Schultze's appraisal of HM's methodology in Section 2, and the views of Jeremy Gray and Frank Quinn in Sections 3 and 4, respectively. Sections 5 through 7 analyze the historical context of the 1900s in relation to the Berlin/Göttingen rivalry, and its implications for the contested issue of Klein's modernism. Section 8 presents our conclusions.

2. Siegmund-Schultze on Mehrtens and his critics

Commenting on HM's views concerning intuition and modern mathematicians, Siegmund-Schultze mentions

> [Mehrtens'] juxtaposition of modern mathematicians (Cantor, Hausdorff, Hilbert, etc.) and their critics who relied and insisted on 'Anschauung' (Intuition) (Poincaré, Brouwer, Klein, Bieberbach).[2]

Siegmund-Schultze's comment is a reference to HM's dichotomy of moderns versus countermoderns, with Klein listed on the countermodern side. Siegmund-Schultze seems to have sensed that there is a problem with HM's application of the dichotomy to Klein:

> [Felix] Klein stood *in the middle* between proto-typical 'countermoderns' and 'moderns,' which is basically due to the fact that Klein contributed considerably both to mathematical and technical modernization, but, at the same time, to criticism of both as well.[3]

Did Klein stand "in the middle" as Siegmund-Schultze claims, or was he a modern? Siegmund-Schultze cites unfavorable evaluations of HM's methodology by Rowe and Grattan-Guinness in the following terms:

> Herbert's *pioneering and novel* distinction between "moderns" and "countermoderns" has been misunderstood by some *superficial read-*

[2]Siegmund-Schultze [49, p. 27].
[3]Op. cit., p. 28; emphasis added.

> *ers* of the book, and has been criticized as misleading by some more thorough readers (Grattan-Guinness, 2009; Rowe, 1997).[4]

Siegmund-Schultze's reference is to the criticisms by Grattan-Guinness in [15] and Rowe in [45]. Indeed, neither of the pair would have necessarily endorsed a description of HM's distinction as "pioneering and novel," particularly in application to Felix Klein, as we analyze in Section 2.2.

2.1. Shocked reaction

Siegmund-Schultze's attempt to put a good face on HM's distinction becomes all the more surprising when compared with his initial reaction to HM's book, which was that of shock:

> It came as a shock to me since it was certainly not aiming in the direction of our project.[5]

He and HM were working on a joint project in the 1980s that was supposed to result in a book, when Siegmund-Schultze received a draft copy of HM's book in june 1989 and was apparently shocked by its marxist and semiotic content. Siegmund-Schultze then quotes HM and concludes: "the quotation describes quite clearly Herbert's individual scientific development and gives the essential explanation why *our project had to fail*."[6]

On the subject of HM's dichotomy, Grattan-Guinness wrote, rather critically:

> On the place of modernism in mathematics as a whole, the claim [by Mehrtens] that 'Modernism and Antimodernism make between them the success history of mathematics in the twentieth century' [M, dustjacket][7] requires us to rejudge as failures or footnotes the developments of, for example, topology, mathematical statistics, and numerical methods![8]

[4]Ibid.; emphasis added. Siegmund-Schultze does not specify who such "superficial readers" may have been, nor in what way they may have "misunderstood" Herbert's distinction. We are familiar with only one full-length recent critical study of HM's distinction, namely Bair et al. [3]. However, it is questionable whether Siegmund-Schultze would have had the study [3] in mind, as some of the arguments of [3] overlap with those of Grattan-Guinness and Rowe, which Siegmund-Schultze apparently finds to be "more thorough" than those of the unnamed superficial readers.

[5]Siegmund-Schultze [49, p. 27].

[6]Siegmund-Schultze [49, p. 30]; emphasis added.

[7]This bracketed comment is in Grattan-Guinness' text.

[8]Grattan-Guinness [15, pp. 9–10].

2.2. Rowe on Mehrtens' dialectics

Rowe points out that HM's book "was written during the midst of the *Historikerstreit* of the late 1980s when debates regarding the historical significance of the Holocaust were in full flame."[9] He notes that some of HM's protagonists can be viewed as modern in one sense and countermodern in another, and that

> Klein and Hilbert shared strikingly similar views about what constituted good and bad mathematics.[10]

Rowe also questions HM's claim concerning antisemitism. HM claimed that the movement that embraced *Anschauung* in the tradition of Klein germinated into the virulent form of antimodernism and anti-Semitism.[11] Writes Rowe:

> Far more provocative and central for Mehrtens's whole approach to countermodernism are his claims regarding the impact Klein's views exerted on subsequent events. According to Mehrtens, the movement that embraced Anschauung in the tradition of Klein–a movement that gained momentum after 1905 with the adoption of the Meran proposals that called for the incorporation of anschauliche methods in mathematics instruction–germinated into the virulent form of antimodernism and anti-Semitism espoused by Ludwig Bieberbach, the leading proponent of Aryan mathematics.[12]

Rowe's verdict is unequivocal:

> Unfortunately, Mehrtens tells us only how this transition *might have* happened instead of documenting that it *actually did* happen.[13]

Given the weakness of HM's positing such a link, one can well wonder what the source of HM's animus toward Klein might have been. Rowe provides a hint when he speaks of "*dialectical* elements so prominent in Mehrtens's interpretation"[14] and mentions that HM

[9] Rowe [46, p. 517].
[10] Rowe [45, p. 536].
[11] For a recent reaction by Mazzotti to such a claim, see the main text at note 24.
[12] Op. cit., p. 5.
[13] Ibid.; emphasis added.
[14] Rowe [46, p. 519]; emphasis added.

> paints a harsh picture that underscores moral bankruptcy and dehumanizing influences, as when mathematicians willingly lent their technical expertise to the military.[15]

The academic-industrial complex was the subject of a 1908 cartoon reproduced in HM's book.[16] HM's distaste for the Academy's cooperation with the industry and the military arguably stemmed from HM's political commitments and colored his evaluation of Klein's relation to modernism. As noted by Gray,

> Mehrtens's critique was written in a post-Marxist spirit, influenced by such writers as Foucault.[17]

Rowe adds that "Mehrtens evinces little interest in such 'details' that would clutter up the main picture he wishes to convey."[18] When HM's crude marxism and semiotic obfuscation escalate, Rowe does not shy away from calling him to task:

> Mehrtens goes off the deep end, for example, when he tries to find parallels between the formal languages of modern mathematics and the commando rhetoric of contemporary dictatorships.[19]

2.3. Mehrtens as seen by Epple, Tobies and Mazzotti

Epple is similarly critical of HM's insufficient clarity in the definition of the terms *modern* and *countermodern*:

> Mehrtens' book is an example of a very elaborated kind of external historiography. His sources are mainly the programmatic declarations of the mathematicians involved and the documents of their institutional activities. Mehrtens does not attempt to analyze some of the more advanced productions of modernist or counter-modernist mathematicians ... Thus we are left in a somewhat unclear position if we accept his narrative.[20]

Epple goes on to propose his own interpretation of what the dichotomy of modern vs countermodern could mean. We will not deal with Epple's definition because our main concern is HM's distinction rather than Epple's. HM's definition is characteristically slippery, but at least two features of mathematical modernism seem indisputable:

[15] Rowe [46, pp. 516–517].
[16] For a discussion of the cartoon see Bair et al. [3, Section 4.7].
[17] Gray [16, p. 10].
[18] Rowe [45, p. 4].
[19] Rowe [46, p. 520].
[20] Epple [11, p. 191].

1. a tendency to unify, rather than segregate, fields within mathematics; and

2. in seeking to understand what constitutes mathematical modernism, we cannot ignore what leading 20th century mathematicians thought characterized modern mathematics.[21]

By the unifying criterion, Klein wins hands down, because of his efforts to unify pure and applied mathematics, as detailed by Tobies.[22]

In a similar vein, Tobies writes:

> Contrary to Herbert Mehrtens's (1990) classification of mathematicians as either modern (Hilbert et al.) or anti-modern (Klein et al.), it seems to me as though, in Klein's case, it is possible to speak of a particular sort of modernity on whose basis new domains such as modern numerical analysis, actuarial mathematics, and financial mathematics were able to develop.[23]

Thus, Tobies tends to assign Klein to the *moderns*. Meanwhile, Mazzotti writes:

> [I]n one of the *strongest* claims of the book, [Mehrtens] argues for an essential continuity between this [i.e., Felix Klein's] strain of Wilhelmian mathematics and the anti-modern and anti-Semitic campaign of the proponents of Aryan mathematics under the Nazi regime.[24]

As noted by Rowe, HM did not *argue for* but rather merely *postulated* such an "essential continuity." With regard to the qualifier "strongest" it is unclear whether it is being used in the sense of *outrageous* or in the sense of *compelling* by Mazzotti, who does not conceal MH's influence on his work:

> In this book I have clearly followed Mehrtens's invitation to explore the politics of mathematical modernity, . . .[25]

[21] More on this in the main text at note 61.

[22] Tobies [53, Chapters 7.7 and 8.1].

[23] Tobies [53, p. 10].

[24] Mazzotti [32, p. 277, n. 20]; emphasis added.

[25] Mazzotti [32, p. 238]. Mazzotti describes Mehrtens' [33] as "an ambitious and original book . . . that deploys high-powered semiotic and philosophical tools to study the discourse of the autonomy and meaning of mathematics" [32, p. 237].

In contrast to Grattan-Guinness, Rowe, and Tobies, Jeremy Gray adopts a more sympathetic view toward HM's approach to Felix Klein. In a 2008 book, Gray voices criticisms of Klein that echo HM's. We will examine Gray's portrayal of Klein in Section 3.

3. Gray on Klein's 'notorious' paragraph. Like HM, Jeremy Gray attempts to tackle modernism in his 2008 book [16]. Grattan-Guinness expresses scepticism with regard to Gray's definition of modernism in the following terms:

> [W]hile Gray relates his definition to a text by Guillaume Apollinaire on cubist art, it is surely incomplete and so too wide; for example, there is no reference to mathematics at all, and no time dependence.[26]

HM wrote that

> Theodor Vahlen ... cited *Klein's racist distinctions* within mathematics, and sharpened them into open antisemitism.[27]

Even though HM is only mentioned once in a footnote in Gray's Section 4.2.1.1 on pages 197–199, we will see that HM's methodology exerted significant influence on Gray's portrayal of Klein. Most commentators today would agree that Klein's distinctions, as discussed below, were *racial* rather than *racist*. The implied connection between Klein's offhand comments on distinct ethnic styles in mathematics, on the one hand, and anti-Semitism, on the other, bears closer scrutiny. In this section, we will analyze Jeremy Gray's take on Klein's comments.

3.1. 'Teutons' and Jews. As is well known, Klein was instrumental in securing positions for a number of qualified Jewish mathematicians in German universities.[28] Regrettably, Gray tends to downplay Klein's valiant efforts on behalf of Jewish mathematicians. One would suspect little of such efforts by reading Gray:

> Then came a paragraph that became notorious.[48] [here 48 is a footnote in Gray's book; see Section 3.2] Klein speculated that the exactness of spatial intuition might vary between individuals along racial lines. *Teutons* seemed to have strong, naive spatial intuition, while the critical, purely logical sense was more fully developed in the Latin and Hebrew races.[29]

[26] Grattan-Guinness [15, p. 4].
[27] Mehrtens [33, pp. 310–311]; translation ours; emphasis added.
[28] See e.g., Rowe [42].
[29] Gray [16, p. 198]; emphasis added.

It is worth analyzing the accuracy of Gray's passage in detail. In fact, there are some significant inaccuracies in his narrative. At variance with Gray's attempted summary, Klein himself never spoke of *Teutons*, but only of *the Teutonic race*. Klein's passage reads as follows:

> Finally, it must be said that the degree of exactness of the intuition of space may be different in different individuals, perhaps even in different races. It would seem as if a strong naive space-intuition were an attribute pre-eminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races. A full investigation of this subject, somewhat on the lines suggested by Francis Galton in his researches on heredity, might be interesting.[30]

In Klein's case, there was no difference between words and actions: whatever ethnographic comments he may have made (and whatever their depth or lack thereof), the comments were all positive rather than critical; namely, he pointed out the positive aspects of having distinct styles in mathematics, as in the case of his comments on Sylvester.[31] His words were consistent with his actions on behalf of many Jewish mathematicians, as noted by Tobies (see Section 3.2).

In the passage quoted above, Klein is not using the term *Teutons*, which would be the name of a *people* but rather, Klein refers to *the Teutonic race* which indicates that Klein is simply dabbling in the ethnographic theorizing that was common for this historical period. Gray continues:

> A fuller investigation along the lines of Galton's study of heredity might be interesting, he [i.e., Klein] concluded. Given Klein's evident preference for intuition, it is *hard not to read* this passage as being somewhat anti-Semitic.[32]

Gray finds that it is "hard not to read" Klein's paragraph as "being somewhat anti-Semitic." Apparently some find it *easy to read* the passage as anti-Semitic. We find it hard not to ask, *who* exactly would find it easy to read Klein's paragraph in this light? Certainly not the historian Segal [48].[33] While Ludwig Bieberbach (who appears to have truly believed in his crude aryanism) would have surely found it easy to

[30] Klein [26, pp. 45–46].
[31] See Bair et al. [3, Section 2.2].
[32] Gray [16, p. 198]; emphasis added.
[33] Regarding Segal's position, see further in Bair et al. [3].

read Klein's passage in that light, we reject the attribution of indirect culpability to Klein by Gray following HM. The idea that a modern historian would find such an association of ideas hard to resist strikes us as peculiar. Gray continues:

> [1] Moreover, Kronecker, Klein's examplar of the purely logical mathematician, was Jewish; Peano, a Latin, had already been mentioned by Klein as an axiomatic mathematician. [2] Forty years later, these remarks were brandished by the Nazi Bieberbach as 'proof' that 'good' German mathematicians such as Klein and Bieberbach were racially superior.[34]

As per Gray's sentence [1], Klein paints Kronecker and Peano as logicians and axiomatizers; as per sentence [2], Klein's remarks allegedly provided grist for the mill of Bieberbach's racist propaganda. Does Gray take pains to distance himself from the inference that Klein could bear partial responsibility for enabling such propaganda? On the contrary: Gray seems to come dangerously close to falling into the same trap as HM in confusing possibility and actuality (as noted by Rowe):[35]

> Though Bieberbach made poisonous nonsense of it, there is no denying that Klein's remarks partake of the *nineteenth-century glibness* with which whole races are pigeon-holed. Psychology has ever lent itself to the representation of contemporary prejudice as the latest fruit of science.[36]

There may be no denying the glibness, but one can certainly deny that Klein's passage was "somewhat anti-Semitic," contrary to Gray's glib claim. The glibness with which a 21-century commentator dismisses an entire academic discipline (namely, *Psychology*) as lending itself "to the representation of contemporary prejudices" is surprising.

3.2. A belated footnote on academic politics

In footnote 48 on the same page, Gray tries to hedge his bets with regard to Klein:

> [48]It is discussed in Rowe 1986[37] and Mehrtens 1990, 215–218, who also note that elsewhere in his *Entwicklung* Klein is clear that the emancipation of the Jews in Germany in 1812 was quickly

[34]Gray [16, p. 198]; numerals [1] and [2] added.
[35]See the main text at note 13.
[36]Ibid.; emphasis added.
[37]This is a reference to Rowe [42].

> fruitful for mathematics, and that Klein was not anti-Semitic in his *academic politics*, rather the opposite.[38]

Gray's apparent conclusion is that it is only in his "academic politics" that Klein wasn't antisemitic, retaining the unstated possibility of applying the slur in other areas. Regrettably, Gray fails to distance himself sufficiently from HM's comments on alleged affinities between Bieberbach's views on *Jewish mathematics* – and those of Klein. Writes Tobies:

> Klein's classification here was not intended to denigrate any mathematical approach or "race." He assigned equal value to the different ways of achieving knowledge. Nevertheless, after his death, statements of this sort were politically appropriated and misconstrued for anti-French or anti-Semitic purposes.[39]

Tobies' appraisal appears the more balanced.

Gray claims further that "Klein's own mathematics always inclined to the sort of imprecision [Oskar] Perron was adamantly against"[40] but gives no evidence to back up his claim, and instead hedges his bets in the following terms:

> Even if Perron had not had Klein in mind [in Perron's 1911 lecture], Klein's example was the kind of thing he was against[41]

but again without presenting any evidence. On the contrary, we find Perron's view concerning the limited usefulness of geometric intuition[42] in agreement with Klein's view on this topic, as laid out in his Evanston lecture on space-intuition in 1893, where he warned of the pitfalls of naive intuition which can be unreliable.[43] See also Section 7.

4. Quinn's portrayal of Klein

Another author who appears to have been influenced by HM's interpretation, via Gray, in his thinking about Klein is Frank Quinn. In this section we examine the image of Klein as allegedly a traditionalist who opposed the new methods in mathematics and clung to 19th century values, that emerges from Quinn's writings. Quinn's more specific target is Klein's book *Elementarmathematik vom höheren Standpunkte aus*

[38] Gray [16, p. 198]; emphasis added.
[39] Tobies [53, p. 491].
[40] Gray [16, p. 277].
[41] Ibid.
[42] See Gray [16, p. 275].
[43] See Klein [28, p. 225]. For a detailed analysis, see Franchella [13, pp. 1061–1062].

[30]. Quinn's article [41] did not go unnoticed: it was selected for the collection *The Best Writing on Mathematics 2013* published by Princeton University Press.[44] We will argue that it was, alas, *not the best*. Given below is a sample of Frank Quinn's comments on Klein.

1. "[D]uring the transition [period] some traditional mathematicians, most notably Felix Klein, were very influential in education and strongly imprinted nineteenth-century values on early twentieth-century education reforms. . . "[45]

2. "[Felix Klein's] 1908 book *Elementary Mathematics from an Advanced Viewpoint* was a virtuoso example of *nineteenth-century methods* and did a lot to cement their place in education. The 'Klein project' . . . is a contemporary international effort to update the topics in Klein's book but has no plan to update the methodology. In brief, traditionalists lost the battle in the professional community but won in education."[46]

3. "Felix Klein was still denouncing the new methods in the 1920s, and because his views were not only unrefuted but almost unchallenged, outsiders accepted them as fact."[47]

4. "Traditionalists like Poincaré and Felix Klein . . . objected because they wanted to use physically or experientially based methodology."[48]

5. "Klein was not at all bothered by non-Euclidean geometry. However this may have been because he saw a place for them in a deeper understanding of the physical world; he remained committed to the physical-realist philosophy. The new [i.e., Hilbert's] view is that mathematics is a world of its own, with wonderful things not dreamt of in their philosophy."[49]

We will summarize Quinn's position in Section 4.1.

[44] See Pitici [38, pp. 175–190].
[45] Quinn [39, section 1.5, p. 4].
[46] Quinn [41, p. 33]; emphasis added.
[47] Ibid., p. 36.
[48] Quinn [39, section 5.2.3, p. 39].
[49] Quinn [39, Section 8.1.1, p. 47]. Here Quinn both misrepresents Klein's views and distorts Hilbert's by implying that the latter would subscribe to a view of mathematics as "a world of its own;" see Section 7.

4.1. Summary of Quinn's position

As sourced above, Frank Quinn claims that Felix Klein (i) practiced 19th century methods as opposed to Quinn's rigorous Core Mathematics,[50] (ii) opposed Weierstrassian methods, and (iii) wanted to use a physically based methodology. As detailed in [3], the truth about Klein is precisely the opposite: he *rejected* 19th century methods, welcomed the Weierstrassian revolution,[51] and clearly understood that mathematics deals only with *representations* of reality. Writes Corry:

> [I]n 1872, Felix Klein–using ideas developed by Cayley and aware of the independence of the parallel axiom in Euclidean geometry–reformulated the whole issue in terms of axiomatic interdependence: projective geometry should be axiomatically defined, so that the various metrical geometries can be derived by addition of new axioms.[52]

4.2. Quinn's *footwork* claim

Some of Quinn's most puzzling claims concern Klein's students:

(a) Early in the twentieth century Klein was no longer a leader in mathematics because no one would follow him: talented young people saw giving up physical intuition in exchange for technical power as a good deal and *voted with their feet*. It was then that he turned his focus to education...[53]

(b) Traditionalists, including Felix Klein ... had no audience in the professional community because young people *voted with their feet* for the new ways.[54]

How would one evaluate Quinn's *footwork* 2011 claim in items (a) and (b)? The useful *Mathematics Genealogy Project* is in the public record, and records no fewer than nine students who defended their PhD under Klein

[50] Quinn's Core Mathematics involves a combination of Weierstrassian *Epsilontik* and a focus on precise *definitions* in an explicit axiomatic framework. Quinn champions the late 1800s as the formative period of Core Mathematics, mentioning the work of Weierstrass, Peano, Frege, and Hilbert.

[51] Weierstrass (1815–1897) of course lived in the 19th century, but the Weierstrassian revolution in rigor in the foundations of analysis is universally considered part of the modernist transformation of mathematics in the 20th century.

[52] Corry [7, p. 155].

[53] Quinn [39, Section 8.1.3, p. 47]; emphasis added.

[54] Quinn [40, Section 1.5, p. 9]; emphasis added.

in the first decade of the 20th century. They are Carl Wieghardt, Conrad Müller, Max Winkelmann, Anton Timpe, W. Ihlenburg, Julio Rey Pastor, Hermann Rothe, Ludwig Bieberbach, and Erwin Freundlich.[55] Klein's more than 77500 mathematical descendants (as of january 2024)[56] apparently have not noticed the general stampede away from Klein implied by Quinn.

Moreover, Hermann Weyl, who obtained his doctorate under David Hilbert, received detailed assistance and mathematical advice from Klein in writing his distinctly modern book on Riemann surfaces [56].[57]

Quinn's *footwork* claim dovetails with Quinn's *tale*, but such footwork does not seem to be part of the historical record. Quinn's claim concerning young people allegedly walking away from Felix Klein appears to be contrary to historical evidence.

Klein indeed emphasized the importance of intuition, particularly in education, as he will in more detail in his book [30]. However, Klein's stance on education is similar to that of many a modern mathematician who rejects Quinn's ideology on mathematics education.

Thus, Saunders Mac Lane wrote:

> The sequence for the understanding of mathematics may be: *intuition, trial, error, speculation, conjecture, proof.*[58]

Dismissing Klein's take on education as a throwback to the 19th century, as Quinn does, is as unfounded as dismissing Michael Atiyah's[59] on the same grounds. In a related vein, Terry Tao distinguishes three stages in a student's mathematical education: (1) pre-rigorous stage; (2) rigorous stage; (3) post-rigorous stage. Tao describes the pre-rigorous stage as follows:

> In [the pre-rigorous stage] mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and handwaving. (For instance, calculus is usually first introduced in terms of slopes, areas, rates of change, and so forth.) The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years.[60]

[55] See https://www.genealogy.math.ndsu.nodak.edu/id.php?id=7401

[56] Of this number, over 41000 are through Hilbert, who was Klein's mathematical grandson through Lindemann.

[57] See especially pages V and IX there and [3, Section 1.10] for additional details.

[58] Mac Lane in [1, p. 191]; emphasis in the original.

[59] Atiyah et al. [1].

[60] Tao [50].

By Quinn's lights, Tao's first stage would apparently amount to a counterproductive throwback to the 19th century.[61]

5. Hilbert and Klein: allies or adversaries?

While it is true that Klein defended an older mathematical tradition than the one envisioned by Hilbert, the view of Hilbert and Klein as adversaries in an alleged struggle of *modern* versus *countermodern*, found in the writings of Mehrtens [33], Gray [16], and Quinn [41] as analyzed in Sections 2, 3, and 4, can be challenged. Klein's determined campaign to bring Hilbert to Göttingen finally paid off in 1895:

> Klein clearly knew what he wanted as well as what he was getting, and, for his part, Hilbert gladly joined forces with him in his long-standing battle with the Berliners, a fight that continued to rage during the post-Kroneckerean era.[62]

5.1. United against the Berlin school

Rowe notes that Klein and Hilbert were *allies* in resisting a "purist trend toward specialization" emanating from the Berlin school, dominated by Weierstrass, Kummer, and Kronecker:

> [Klein and Hilbert] forged an intellectual partnership that not only overturned the balance of power within German mathematics but also significantly altered its research orientation and the relationship between mathematics and other scientific and technological disciplines.[63]

The Berlin school was characterized by a "belief that arithmetic was the true font of all mathematical knowledge and that other branches of mathematics were to be regarded as subordinate to it."[64] Indeed,

> It was this *purist trend toward specialization* in mathematical research that Klein and Hilbert resisted, turning instead to a broader, all-embracing approach that stressed the vitality of unifying ideas ... Both were universalists with grandiose visions for the mathematics of the future, visions they felt were threatened by institutional and intellectual forces alike. And although their aspirations for mathematics diverged in certain important respects,

[61] The pertinence of taking into account the views of leading 20th century mathematicians was already mentioned in the main text at note 21.

[62] Rowe [46, pp. 10–11].

[63] Rowe [43, p. 186].

[64] Rowe [43, p. 187].

> underlying them was a fundamental agreement on most key issues.[65]

Echoes of such Berlin–Göttingen tensions can be detected in Hilbert's 1900 Paris lecture (its idea is traceable to an invitation by Hermann Minkowski):

> While insisting on rigour in the proof as a requirement for a perfect solution of a problem, I should like, on the other hand, to oppose the opinion that only the concepts of analysis, or even those of arithmetic alone, are susceptible of a fully rigorous treatment. This opinion, occasionally advocated by *eminent men*, I consider entirely erroneous.[66]

The "eminent men" here is a thinly veiled allusion to the Berliners. Having spelled out the error of the analysis *über alles* view as he saw it, Hilbert continued to emphasize the vital link to geometry and mathematical physics:

> Such a one-sided interpretation of the requirement of rigour would soon lead to the ignoring of all concepts arising from geometry, mechanics and physics, to a stoppage of the flow of new material from the outside world, and finally, indeed, as a last consequence, to the rejection of the ideas of the continuum and of the irrational number. But what an important nerve, vital to mathematical science, would be cut by the extirpation of geometry and mathematical physics! On the contrary I think that wherever, from the side of the theory of knowledge or in geometry, or from the theories of natural or physical science, mathematical ideas come up, the problem arises for mathematical science to investigate the principles underlying these ideas and so to establish them upon a simple and complete system of axioms, that the exactness of the new ideas and their applicability to deduction shall be in no respect inferior to those of the old arithmetical concepts.[67]

The "eminent men" criticized by Hilbert were Weierstrass, Kummer, Kronecker (all three were dead by the time of Hilbert's Paris lecture), Frobenius, Schwarz, and others of the Berlin school. Writes Rowe:

[65] Ibid.; emphasis added.
[66] Hilbert [20, p. 442]; emphasis added.
[67] Ibid.

> The Berlin mathematicians had long treated geometrical arguments as foreign to the rigorous standards demanded by algebra and analysis ... Diagrams and other aids to the imagination were considered taboo. Hilbert's aim was to dispel this orthodoxy by showing that the foundations of geometry were every bit as rigorous as those based on the properties of number systems.[68]

As we will see in Section 6.2, Klein was passed over in favor of Schwarz for promotion at Berlin. We will compare their attitudes toward set theory in Section 5.2.

5.2. Fraenkel on Schwarz on set theory

In the area of set theory, we have a memorable recollection of Abraham Fraenkel (of ZFC fame and impeccable *modernist* credentials) concerning a 1912 seminar at Berlin where Fraenkel (as a student) gave a lecture on set theory. Fraenkel documents Hermann Schwarz's reaction in the following terms:

> Schwarz took the most convenient route. He gave interested students complete freedom to report on whatever subjects they chose during the seminar. In unconscious anticipation of my future developments, I chose as my subject the elements of set theory, which was virtually unknown in Berlin. My two-hour lecture stimulated lively participation among the students. The professor then announced in his closing remarks that he too had heard about Georg Cantor's questionable theories, but felt he had to warn young students about them.[69]

Note that such memorable references to "Georg Cantor's questionable theories," assorted with the need to "warn young students about them," were uttered by Schwarz at Berlin, not Klein at Göttingen. As we know, Klein fully endorsed Cantor's theory; see Section 5.4.

5.3. Fraenkel, Lie, Noether, Zermelo

Fraenkel offered the following remarkable recollection on Klein:

> My professional career began in march 1919 with ... an invitation to Göttingen to Privy Counselor Felix Klein, almost 70 years of age, but still active as the "foreign minister" of German mathematics.[70]

[68] Rowe [46, p. 14].
[69] Fraenkel [12, pp. 94–95].
[70] Fraenkel [12, p. 127].

We may therefore add to the list of Klein's credits, the fact of having helped launch Fraenkel's professional career. This is a remarkable accomplishment to Klein's credit. Klein invited Fraenkel to contribute an article on *Number* for a collection on Gauss that Klein was publishing. Since Fraenkel was a rising star in the new field of axiomatic set theory, the fact that Klein asked Fraenkel to write an article on *Number* suggests that Klein still adhered to his view concerning the importance of the *arithmetisation of analysis*, a term he coined in his 1895 address; see [27]. Possibly Klein invited Fraenkel to write an article on *Number* because Klein considered set theory to be the foundation of such an arithmetisation–a markedly modernist stance.

Klein was instrumental in securing Sophus Lie's appointment at Leipzig (against Weierstrass' opposition; see [3]). We have a detailed testimony on how the moderns Klein and Lie reacted to Weierstrass' lectures in 1869. Wrote Klein:

> As far as I remember — I arrived at Berlin in 1869 and stayed there from 1869 to 1870 — Weierstrass had the status of an absolute authority in whose teachings the listeners acquiesced, [taking them] as an indisputable norm, oftentimes without having properly comprehended them in a deeper sense. Doubts weren't allowed to arise, and checking was difficult, already because of Weierstrass' giving extraordinarily few literature references. In his lectures he had set himself the goal of presenting a system of neatly ordered and coherent thoughts. Thus he started with methodical construction, from the ground up, and, in pursuit of his ideal of not leaving out anything, he set out to proceed in such a way as to later only ever have to refer to himself.[71]

Klein continued:

> Back then, I myself—now I regret it—did the same as did [Sophus] Lie: out of contrariness, I attended none of Weierstrass' courses,

[71] In the original: "Nach meinen Erinnerungen — ich kam 1869 nach Berlin und war 1869/70 dort — war Weierstraß' Stellung die einer absoluten Autorität, deren Lehren die Zuhörer hinnahmen als unanfechtbare Norm, oft ohne sie im tieferen Sinn recht aufgefaßt zu haben. Ein Zweifel durfte nicht aufkommen, eine Kontrolle war schon deshalb schwer möglich, da Weierstraß außerordentlich wenig zitierte. Er hatte es sich in seinen Vorlesungen zum Ziel gesetzt, ein System wohlgeordneter Gedanken im Zusammenhang vorzutragen. So begann er mit einem methodischen Aufbau von unten herauf und, seinem Ideal der Lückenlosigkeit nachstrebend, richtete er den Gang so ein, daß er in der Folge nur auf sich selbst zurückgreifen brauchte." Klein [29, p. 284].

> but rather always fought for my own ideas in [Weierstrass'] seminar. But back then I did hand-copy a transcript of a course of Weierstrass' on elliptic functions, and I used it years later when working on this subject.
>
> Gradually Weierstrass gained a reputation all over the scientific world as an incomparable authority (cf. Mittag-Leffler, [International] Congress [of Mathematicians in] Paris [1900], page 131, where Hermite writes: 'Weierstrass is the master of us all').
>
> And yet, Weierstrass, too, was finally not spared the disappointment of having to watch his doctrines being challenged (cf. letter to [Sofia] Kovalevskaya dated 24 March 1885, communicated by Mittag-Leffler, Acta Math., Volume 39, p. 194ff.).[72]

In this connection, Renate Tobies writes:

> Klein and Lie discovered that they had many interests in common. Like Klein, Lie, too, shunned Weierstrass' lectures.[73]

In 1919 Klein and Hilbert habilitated Emmy Noether, another central figure in the development of modern mathematics, who strangely is only briefly mentioned by Gray and not at all by HM.[74]

Ernst Zermelo, one of HM's *moderns*, was habilitated by Hilbert and Klein in 1899.[75] It was at Göttingen, not at Berlin where he had studied, that Zermelo first heard about set theory in a lecture given by Schoenflies in 1898.

---

[72]"Ich selbst habe damals—jetzt bedaure ich es—ebenso wie Lie, aus Widerspruchsgeist kein Kolleg bei Weierstraß gehört, sondern im Seminar immer nur eigene Gedanken verfochten. Aber eine Vorlesung von Weierstraß über elliptische Funktionen habe ich mir damals abgeschrieben und Jahre später bei meinen Arbeiten über diesen Gegenstand oft benutzt.

Allmählich gewann Weierstraß Geltung in der gesamten wissenschaftlichen Welt als unvergleichliche Autorität (vgl. dazu Mittag-Leffler, Pariser Kongreß 1900, S. 131, wo Hermite sagt: 'Weierstraß est notre maître à tous').

Und dennoch ist zuletzt auch Weierstraß die Enttäuschung nicht erspart geblieben, daß er seine Lehren angefochten sehen mußte (vgl. Brief an die Kowalewska [sic] vom 24. März 1885, mitgeteilt von Mittag-Leffler, Acta Math. Bd. 39, S. 194ff)." (Ibid.)

[73]In the original: "Klein und Lie entdeckten viele gemeinsame Interessen. Wie Klein mied auch Lie die Vorlesungen von Weierstraß" Tobies [51, p. 22].

[74]In the foreword to her *Habilitation* thesis that contains the famous Noether's theorems, she writes that "Klein's second note and the present developments have been mutually influenced by each other" [35].

[75]Ebbinghaus [10, p. 32].

Thus Klein appreciated the work of, and provided support for, numerous modern mathematicians including Fraenkel, Lie, Noether, and Zermelo – in some cases, against the opposition of the Berliners.

5.4. Klein and Hilbert at Göttingen

Klein and Hilbert joined forces in turning Göttingen into a broad-based, leading mathematical center offering a versatile alternative to what was becoming a stifling influence of Berlin's focus on arithmetic analysis. Similarly, they jointly transformed the *Mathematische Annalen* into a leading research venue. Contrary to Quinn's claim (see Section 4), the emerging set-theoretic foundations of the "late 1800s" found an avid supporter in the person of Felix Klein:

> One of the keys to th[e] success [of *Mathematische Annalen*] was Klein's knack for promoting the works of mathematicians who either were estranged from or stood outside the mainstream influence of the Berlin school: two prominent examples being Georg Cantor and David Hilbert.[76]

Klein published no fewer than 15 of Cantor's papers in *Mathematische Annalen*.[77] HM acknowledges that Klein also used Cantor as a referee for the journal.[78] In fact, some of the mathematicians that would surely be listed as *modern* by HM, Gray, and Quinn turn out upon closer inspection to be far more retrograde than Felix Klein on certain important mathematical issues. While the essence of mathematics may be its freedom according to Cantor so that all mathematical theories are in principle equally free, it turned out that some mathematical theories were more equally free than others:

> [Cantor] once called infinitesimals the "cholera bacillus of mathematics" and on another occasion referred to them as "paper magnitudes that have no other existence other than to be on the paper of their discoverers and disciples." Hilbert and Klein, on the other hand, both recognized the validity of non-Archimedean systems in geometry...[79]

In this sense Felix Klein was arguably more modern than Georg Cantor.[80]

[76] Rowe [43, p. 192]. See further in [47].
[77] See Tobies [54, p. 38].
[78] Mehrtens [33, p. 207].
[79] Rowe [44, pp. 195–196].
[80] See further in Kanovei et al. [23].

If reading Quinn leaves one with the impression that Hilbert didn't share Klein's sentiment concerning the importance of intuition, one may be pleasantly surprised to read the following comment by Hilbert:

> In mathematics, as in all scientific research, we encounter two tendencies: the tendency toward abstraction - which seeks to extract the logical elements from diverse material and bring this together systematically - and the other tendency toward *intuition* [Anschaulichkeit], that begins instead with the lively comprehension of objects and their substantial [inhaltliche] interrelations ... The many-sidedness of geometry and its connections with the most diverse branches of mathematics enable us in this way [namely, through the anschauliche approach] to gain an overview of mathematics in its entirety and an impression of the abundance of its problems and the rich thought they contain.[81]

Hilbert valued intuition and its role in geometry, and wanted to maintain that tradition in the 1920s when he taught his course on *Anschauliche Geometrie* four times before it was published with Cohn-Vossen in 1932 as [21]. Writes Rowe:

> [Klein and Hilbert] were both elitists who were not afraid to talk to the educated layman in simple terms ... It just was not fashionable for German scholars to "stoop to the masses"–indeed it was regarded as a betrayal of the ideals of *Wissenschaft*. Klein paid a heavy price for doing so, having to suffer the scourn [sic] of the famous mathematicians of the Berlin school–Weierstrass, Kronecker, and Frobenius–who regarded him as a *charlatan*.[82]

The unjustified scorn for Klein felt by Weierstrass, Kronecker, and Frobenius may have ultimately been an influence on the attitudes displayed by a later generation of commentators such as HM, Gray, and Quinn.

6. Wrong address for countermodernism

HM thought that he found a countermodern at Göttingen, in the person of Felix Klein. We argue that HM would have been more successful in his search for countermoderns at another address.

[81] Hilbert as cited by Rowe in [43, p. 199]; emphasis added.

[82] Rowe [44, p. 202]; emphasis added. Helmholtz, citing Kronecker, explicitly referred to Klein as a "charlatan" and none of the luminaries present, including Weierstrass, seem to have objected (see Section 6.1).

6.1. A meeting in Weierstrass' apartment in Berlin

On 14 january 1892, Adolf Hurwitz addressed a letter to Klein. Kronecker having died in 1891, a position became available at Berlin. Hurwitz wrote:

> you can imagine how deep an impression the passing of Kronecker has made on us here [in Königsberg]. Just about anyone in the *unprejudiced* mathematical world agrees that only you can fill the gap that has arisen in Berlin. The only uncertain [step in the process of appointing Klein to Berlin] is whether [the decision-makers] inside the ministy are able to withstand the countercurrents.[83]

Hurwitz appears to have expected the ministry to support Klein, and anticipated resistance from "countercurrents" among mathematicians at Berlin.

Eight days later, Berlin attitudes toward Klein were clearly on display during a discussion that took place among leading Berlin faculty members (the meeting took place in Weierstrass' apartment due to his poor health). Writes Rowe:[84]

> On 22 January [1892] a committee of Berlin faculty members met to propose candidates to fill the vacancies, and on one point there was unanimity of opinion.[85] As excerpts from the committee's protocol indicate, under no circumstances would any of them countenance the candidacy of Felix Klein:[86]
>
> > **Helmholtz**: Kronecker spoke very disparagingly of Klein. He regarded him as a charlatan [*faiseur*].[87]
> >
> > **Weierstrass**: Klein dabbles more. A bluffer [*Blender*].[88]

[83] Hurwitz to Klein, cited in Tobies [52].

[84] Rowe [42, p. 433]. Another translation appears in Grattan-Guinness [15, p. 9, note 22].

[85] Rowe's comment concerning alleged unanimity is not entirely accurate as August Kundt, tenured physicist at the time of the meeting, had spoken favorably of Klein during the meeting.

[86] Rowe's summary that follows is a shortened version of a much longer transcript of the exchange that took place in 1892; see [55] and [4, p. 305–307]. The order of the comments by Helmholz and Weierstrass is reversed by Rowe.

[87] The term *faiseur* rendered by Rowe as "charlatan" is a French loan word referring to someone involved in wheeling and dealing, close in meaning to *forger*, *gambler*, or *cheat*.

[88] A fuller citation of the original is reproduced in [55, p. 4]: "Schwarz bleibt bei der Stange. Guter Vortrag. Klein nascht mehr. Blender." The word *nascht* rendered by

> **Fuchs**: I have nothing against his person, only his pernicious manner when it comes to scientific questions.

Whether or not Klein was a *faiseur* and/or *Blender*, the tone of the 1892 discussion at Berlin is refreshingly frank, and speaks volumes as to who exactly displayed *countermodern* tendencies at the time.

6.2. Apartment meeting following Biermann

The meeting already discussed in Section 6.1 provides evidence of countermodern tendencies at Berlin. The discussion of the candidacy of Klein mentions a book of his (*Lectures on the Icosahedron*, 1884) but fails to mention his even earlier *Erlangen program* dating from 1872. As noted in [3], the modernist nature of the *Erlangen program* is recognized by most commentators – with the exception of HM.

The comments we cited in Section 6.1 (following Rowe) suggest that the participants in the apartment meeting were uncomfortable with Klein's approach to mathematics and perhaps specifically with the modernity of the *Erlangen program*, which was not the traditional kind of mathematics they were accustomed to. Perusing the discussion, it is hard to believe that the candidate Felix Klein mentioned by Weierstrass at the outset:

> at most Klein and Schwarz are eligible. And [we] should only refrain from trying [to get them to Berlin] if they are absolutely unavailable. Other personalities are not to be considered[89]

is the same candidate Felix Klein about whom the ensuing report to the Prussian ministry said:

> [F]irst and foremost it had to be considered [with foresight] that the candidate shall be suited to continue what has been practiced at our university for [several] generations: to lead the students to *serious, self-effacing, and in-depth* work on mathematical problems. . . .[90]

Rowe as "dabbles" more literally means "nibbles." The word *mehr* rendered by Rowe as "more" actually means "rather" in context, as in "Klein dabbles, rather." The word *Blender* rendered by Rowe as "bluffer" more literally means "one who dazzles" with a pejorative implication of having little substance; see Bair et al. [3] for a similar use of the term *Schwindel* by Weierstrass in reference to what he described as "the acolyte society" of Klein, Lie, and Mayer.

[89] Biermann [4, p. 305]; translation ours.

[90] Biermann [4, pp. 307–308]; translation ours; emphasis added.

The principles announced thus far certainly sound reasonable. Their application to the evaluation of Klein's candidacy may surprise the modern reader:

> ... For this reason [we] had to discard such personalities as Professor Felix Klein in Göttingen (born 1849), on whose scientific achievements the opinion of other scholars is very divided, but whose actions as a whole, both through his writings and his teaching, *contradict the tradition of our university* just described.[91]

At the outset, Weierstrass describes Felix Klein, along with Hermann Schwarz, as the two leaders of German mathematics who are the only ones in the running for Kronecker's prestigious chair. By contrast, the final report presents Klein as a character of questionable record both in research and in teaching who cannot even be relied upon to supervise students in "the tradition of the university", being apparently *un*-serious, *not* self-effacing (self-centered?), and *not* in-depth (shallow?).

What emerged at the meeting is that the Berlin traditionalists did not trust Klein with the direction his mathematics (and possibly the *Erlangen Program* specifically) was taking, thereby "contradicting the tradition" of their university. Signing that report, the traditionalists in effect signed away the leadership of the Berlin school in mathematics for decades to come.

7. Geometry, intuition, logic, physics

Klein and Hilbert shared a belief in a decisive role of intuition in mathematics. Thus, Hilbert believed his *Grundlagen der Geometrie* [19] to be a study of our geometric intuition.

7.1. Hilbert's *Grundlagen*

In the Preface to his *Grundlagen*, David Hilbert writes:

> The choice of the axioms [of geometry] and the investigations of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. *This problem is tantamount to the logical analysis of our intuition of space.* (emphasis added)

In Klein's remarks on Hilbert's *Grundlagen*, he stresses the importance of logic in geometrical investigations:

[91] Ibid.; emphasis added.

> I have already emphasised one thing about which most people today are in reasonable agreement. That is that we are concerned here with the leading concepts and statements, which one must of necessity put into the front rank of geometry in order to be able to realise mathematical proofs derived from them by pure logic. This statement does not answer the question as to the real source of these leading concepts and theorems. There is the old point of view that they are the intuitive possession of every person, and that they are of such obvious simplicity that no one could question them. This view, however, was shaken, in large measure, by the discovery of non-Euclidean geometry; for here it is clearly shown that space intuition and logic by no means lead compellingly to the Euclidean parallel axiom.[92]

Having thus analyzed the relation of logic and intuition, Klein continues:

> To the contrary, we saw that, with an assumption, which contradicts the parallel axiom, we come to a logically closed geometric system, which represents actual perceptual relations with any desired degree of approximation. However, it may well be claimed that this parallel axiom is the assumption, which permits the simplest representation of space relations. Thus it is true in general that fundamental concepts and axioms are not immediately facts of intuition, but are appropriately selected idealisations of these facts. The precise notion of a point, for example, does not exist in our immediate sensory intuition, but is only a fictitious limit, which, with our mental pictures of a small bit of shrinking space, we can approach without ever reaching.[93]

Thus Klein, like Hilbert, stressed the role of logic in the study of mathematical intuition.

7.2. Physics as seen by Hilbert and Klein

Mehrtens, Gray, and Quinn sought to portray Klein as looking backward to the 19th century. Their critique imputes a further allegedly *countermodern* shortcoming to Klein, namely an interest in applications to physics. This debate (concerning a proper relation of mathematics to physics) we suspect will never end, and it was the real issue at the root of the debate sparked by the text by Jaffe and Quinn [22], leading

[92] Klein [31, p. 213].
[93] Ibid.

some physicists to coin the term "rigorous mathematics in the sense of Jaffe and Quinn."[94]

Meanwhile, anyone who seeks to recruit Hilbert to a hypothetical *modernist* cause will have to come to grips with Hilbert's profound interest in physics in general and Einstein's relativity theory, in particular.[95] As acknowledged by Gray,

> [Einstein] found he got a better hearing from Hilbert and Klein in Göttingen than he did from his colleagues in Berlin.[96]

Concerning the issue of the proper relation between mathematics and physics, one may well ponder the following story told by one of the world's leading mathematicians in 2014:

> In a mythical country, there was a very skilled blacksmith who could make extraordinary swords, and extraordinary axes. (I visited the weapon museum in Nagoya, and I can tell you that the Japanese blacksmiths were very skilled indeed.) One day, the blacksmith's son used an axe from his father's collection to cut down a dangerous tree. The father was furious, and told him that this axe was not to be used, for it was a museum piece. The tale is easy to understand.[97]

The speaker is none other than Pierre Cartier.

7.3. Differences according to Max Born

Physicist Max Born, one of the fathers of quantum mechanics, was trained at Göttingen. Born commented as follows on the differences in style between Klein and Hilbert:

> [Klein] was the most dazzling lecturer, but he was too brilliant for me. One would have to have been a really pure mathematician to enjoy his lectures, even when he treated physical or technological applications, which he liked to do. ... At this time of my life, I was more inclined to pure mathematical rigor - inclined to epsilontics, as we called it in our own jargon. ... ε almost never appeared in Klein's lectures.
>
> If Klein arrived at such a critical point, he used to say: 'I will sketch for you the idea of the matter; the rigorous proof can be

[94] See e.g., Eager and Hahner [9].
[95] See [3].
[96] Gray [16, p. 326].
[97] Quoted in Paycha [37].

> looked up in one of the following publications', and he then gave a literature reference. He even added a disdainful remark about those people who need a proof of such an obvious fact.
>
> Now, the proofs that Hilbert gave in his lectures were rather sketchy, too, but I didn't ever have any difficulties to reconstruct them according to his clear indications.
>
> In the case of Klein, however, I often got hopelessly lost and was force to read a large number of boring treatises. In their general structure, Hilbert's and Klein's lectures were very different.
>
> Hilbert was a mountain guide, who led one on the shortest and most secure route to the summit; Klein, on the contrary, was rather like a prince, who wanted to show his admirers the size of his kingdom, by leading them on endlessly winding paths, and who stopped on every small hill to give an overview of the ground already covered. I got impatient with this method [of lecturing] and wanted to reach the summit quickly. Therefore I preferred Hilbert.[98]

Born then reveals his punchline:

> Nowadays, my taste has changed. Whenever I have the time to read a mathematical work purely for my own pleasure, I reach for one of Klein's books ... and I take delight in them like [one takes delight] in a work of art, a great novel or a biography. Indeed, some of these books of Klein's are an integral part of the culture in which I grew up, and which is now going to pieces.[99]

Felix Klein's style as described by Born is similar to that of Mikhael Gromov, one of the greatest and most influential geometers living.[100] Clearly, the differences in style between Klein and Hilbert persist in modern mathematics. The attempt by Mehrtens, Gray, and Quinn to characterize one of them as countermodern is therefore unconvincing.

7.4. Hilbert's formalism

Corry warns against simplistic interpretations of Hilbert's views on formalism:

[98] Born [5, p. 131].
[99] Ibid.
[100] For example, Gromov's book [18] has been cited by over 2800 articles; see https://scholar.google.com/scholar?cites=2060554209746836258

> David Hilbert is widely acknowledged as the father of the modern axiomatic approach in mathematics. The methodology and point of view put forward in his epoch-making *Foundations of Geometry* (1899) had lasting influences on research and education throughout the twentieth century. Nevertheless, his own conception of the role of axiomatic thinking in mathematics and in science in general was significantly different from the way in which it came to be understood and practiced by mathematicians of the following generations, including some who believed they were developing Hilbert's original line of thought.[101]

Furthermore,

> Among the important sources of ideas that inspired the New Math[102] was a certain perception of the Moore Method and the attempt to apply to school mathematics what this method had considered to be of high value in the training of research mathematicians.[103]

The difference between Hilbert's conception and that of some mathematicians of the following generations who "believed they were developing Hilbert's original line of thought" is well illustrated by the case of Frank Quinn. While championing what he sees as Hilbert's formalist approach, Quinn reproaches Hilbert for having allegedly

> accepted the slanders by saying 'mathematics is a game played according to certain rules with meaningless marks on paper.' (v. Quinn [41, p. 35].)

Quinn goes on to venture an alternative line of argument to Hilbert's that he feels would have been historically more effective.[104] However, Quinn's claim attaches exaggerated importance to Hilbert's comment. Hilbert may have gotten carried away in the heat of the argument

[101] Corry [8, Abstract].

[102] For Freudenthal's critique of New Math, Piaget, and Bourbaki see Freudenthal [14] and Katz [25].

[103] Ibid., Section 1.

[104] The text of such an alternative runs as follows: "Axiomatic definitions can be artificial and useless, but they can also encapsulate years, if not centuries, of difficult experience, and newcomers can extract reliable and effective intuitions from them. Similarly, fully detailed arguments can be formal and content-free, but fully confronting all details usually deepens understanding and often leads to new ideas. Fully detailed arguments also give fully reliable conclusions, and full reliability is essential for successful use of the powerful but fragile excluded-middle method" [41, p. 35]. We will withhold judgment as to the effectiveness of Quinn's proposal.

against Brouwer to make the comment on *meaningless marks*. Hilbert's comment does not represent the thrust of Hilbert's position, as argued by Avigad and Reck [2]. Hilbert's comment should be understood in the context of his *metamathematical* program rather than as a statement of a *philosophical* position. The alleged philosophical significance of the comment has been exaggerated out of proportion subsequently, by many authors besides HM and Quinn. In defense of Hilbert, Novikov [36] argues that Hilbert was as dedicated to *meaning* as any great mathematician.[105] Meanwhile, Gray declares:

> By holding onto meaning, [Kronecker and Enriques] belong to the group of mathematicians Mehrtens called the *Gegenmoderne*.[106]

Accordingly, "holding on to meaning" constitutes a shortcoming and a litmus test of being a *Gegenmoderne*. But by Gray's criterion, Hilbert would also be among the *Gegenmoderne*, as he arguably adhered to meaning as much as Kronecker, Enriques, and Klein. Pinning modernism to an absence of, or indifference to, *meaning* is a position that is too simplistic to be useful as a historiographic tool.

8. Conclusion. We have argued that Klein and Hilbert were *modernist* allies in a common cause challenging what in retrospect appears to be artificial limitations on mathematical research imposed by some of the mathematical trend-setters at Berlin, which would have been a more appropriate address than Göttingen for a search for countermoderns. Klein's and Hilbert's challenge found a clear expression in Hilbert's famous list of 23 problems, only a minority of which were concerned with arithmetized analysis as emphasized by the Berlin school.

Mehrtens' modern/countermodern dichotomy is artificial and appears to be tailor-made to cast mathematicians like Klein in an unfavorable light. Mehrtens' distaste for Klein stems from Mehrtens' political affiliations (see Section 2.2 for Rowe's comments on Mehrtens' dialectics) rather than his historical scholarship. The cause of Mehrtens' resentment of Klein appears to be Klein's role in the academic-industrial complex of Wilhelmine Germany of the pre-world war 1 period.

Mehrtens' decision to incorporate the interest in applied mathematics as a characteristic of the so-called *countermoderns* is a thinly veiled attempt to design a theoretical framework that would make Klein look retrograde. Mehrtens' dialectical methodology influenced authors ranging from Gray and Quinn to, more recently, Siegmund-Schultze and

[105] See [24] for a related discussion.
[106] Gray [16, p. 274].

Mazzotti. Not all of these authors share Mehrtens' original political motivation. Thus, Siegmund-Schultze reports "shock" as his first reaction upon reading an early draft of Mehrtens' book (see Section 2.1). With Rowe and Tobies, we reject Mehrtens' provocative linkage of Klein's ethnographic ideas to Vahlen-style antisemitism, as well as Gray's and Mazzotti's toleration of such linkage.

We have examined the attempts by Mehrtens, Gray, Quinn, and to a certain extent Siegmund-Schultze to question Klein's modernity. We concur with Grattan-Guinness, Rowe, and Tobies that the portrayal of Klein and Hilbert as opponents in the matter of modernity is contrary to much historical evidence.

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Odkrywanie kwestionowanego modernizmu Felixa Kleina.

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Streszczenie. Herbert Mehrtens, historyk marksistowski, oraz inni badacze podjęli temat rzekomej opozycji między Davidem Hilbertem a Felixem Kleinem, prezentując go jako konflikt między perspektywą nowoczesną a kontrnowoczesną. Uczeni tacy jak Epple, Grattan-Guinness, Gray, Quinn, Rowe, a także ostatnio Siegmund-Schultze i Mazzotti, wyrazili różnorodne opinie na temat dialektycznej metodologii Mehrtensa. Nasze badania skupiają się na kontrastujących perspektywach dotyczących kontrowersyjnego modernizmu Kleina, a także analizie poglądów Hilberta i Kleina na tematy związane z intuicją, logiką i fizyką. Krytycznie analizujemy komentarz Jeremiego Graya dotyczący etnograficznych spekulacji Kleina na temat żydowskich matematyków i uznajemy go za nieprzyjmowalny. Nasz argument głosi, że Mehrtens poszukiwał elementów kontrnowoczesnych w niewłaściwym adresie.

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