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Monotonic solutions of multi term fractional differential equations

Abstract. In this paper, we present an existence of monotonic solutions for a nonlinear multi term non-autonomous fractional differential equation in the Banach space of summable functions. The concept of measure of noncompactness and a fixed point theorem due to G. Emmanuele is the main tool in carring out our proof.

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1. Introduction. A series of papers and monographs were devoted to the investigation of ordinary fractional differential equations (see e.g. [3], [11], [16], [17], [18], [23] and the references therein). Numerous application of such equations have been presented in the recent literature ([2], [8], [20], [22], for instance). As a pursuit of this in the present paper, we deal with the existence of nondecreasing solutions for the non-autonomous multi-term differential equation of the fractional type

$$(1)\quad \left(D^{\alpha_n}-\sum_{i=1}^{n-1}a_iD^{\alpha_i}\right)x(t)=f(t,x(\varphi(t))),\quad \text{a.e. on } (0,1),\quad \left(\sum_{i=1}^{n}a_iI^{1-\alpha_i}x\right)(0)=0,$$

where $a_1,a_2,\cdots,a_{n-1},$ are constans, $a_n=-1,\quad 0<\alpha_1<\alpha_2<\cdots<\alpha_n<1$ and D^{α_i} denotes the standard Riemann-Liouville fractional derivative. Our investigation is based on reducing the problem (1) to the Volterra integral equation

$$(2)\qquad x(t)=\sum_{i=1}^{n-1}a_iI^{\alpha_n-\alpha_i}x(t)+I^{\alpha_n}f(t,x(\varphi(t))),\quad \text{a.e. on } (0,1).$$

The notation $I^{1-\alpha}x(0)$ means that the limits taken at almost all points of the right-sided neighborhood $(0,\epsilon),\epsilon>0$ of 0.

The problems (1) and (2) have been intensively studied by many authors (see [3], [6], [10], [11], [12], [14], [18] and others). But most of the above investigations were not complete, however, most researchers have obtain results not for the initial value problems, but for the corresponding Volterra integral equations. Moreover, some of the results obtained contain errors in the proof of equivalence of the initial value problems and the Volterra integral equations. In this regard, see survey paper by Kilbas and Trujillo [16]. Our paper is a continuation of mentioned papers.
Our goal here is to prove the existence of nondecreasing solutions to the problems (1) and (2) under a linear growth condition on the nonlinearity. The proof is based on the standard fixed point theorem due to Emmanuele.

2. Preliminaries. Denote by $L_1(0,1)$, the class of all Lebesgue integrable functions on the interval (0,1) with the usual norm $\|\cdot\|$. Define fractional integral operator by

$$I^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)\,ds, \quad t \in [0,1].$$

Using the known relation between Beta- and Gamma functions, a well known calculation with the Fubini-Tonelli theorem shows that $I^{\alpha}I^{\beta}x = I^{\alpha+\beta}x$ for each $x \in L_1[0,1]$ and each $\alpha, \beta > 0$. In particular, I^n is the n-th iterate of usual integral operator, and so I^{α} may indeed be considered as a corresponding fractional integral. The fractional (Riemann-Liouville) differential operator corresponding to I^{α} is defined by $D^{\alpha}x(t) := DI^{1-\alpha}x(t), \alpha \in (0,1)$, where D denotes the usual differential operator.

The following lemma is folklore (*cf.* [11], [17] and [21]).

PROPOSITION 2.1 *Let $\alpha, \beta \in \mathbb{R}^+, f \in L_1(0,1)$ and $n = 1,2,3, \cdots$. Then we have:*

1. *$I^{\alpha} : L_1(0,1) \to L_1(0,1)$ is continuous operator and $I^{\alpha}\ I^{\beta}f(t) = I^{\alpha+\beta}f(t)$,*
2. *I^{α} maps the nondecreasing functions into functions of the same type.*
3. *$D^{\alpha}I^{\alpha}f(t) = f(t)$. If the fractional derivative $D^{\beta}f$ is integrable, then*

$$I^{\alpha}D^{\beta}f(t) = I^{\alpha-\beta}f(t) - \left[I^{1-\beta}f(t)\right]_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \beta \le \alpha < 1.$$

For more remarks concerning the fractional calculus, we refer to ([17], [19] and [21]).

Now denote by $S = S(0,1)$ the set of all Lebesgue measurable function acting from (0,1) into $\mathbb{R}$. Let us furnish this set with the metric

$$\rho_s(x,y) := \inf[a + meas\{s : |x(s) - y(s)| \ge a\} : a > 0],$$

where the symbol $measD$ stands for the Lebesgue measure of the set D. Then $S(0,1)$ becomes a complete metric space if we identify functions which are equal a.e. on (0,1) [7]. The complete description of compactness in measure (i.e. the compactness

in the space $S(0,1)$ was given by Fréchet [7].We will not quote this criterion because it has rather complicated form. Based on this criterion we have the following result [4].

THEOREM 2.2 *Let X be a bounded subset of $L_1(0,1)$ consisting of functions which are a.e. nondecreasing on the interval (0,1). Then X is compact in measure.*

In what follows we shall need the following fundamental result [13].

THEOREM 2.3 (EMMANUELE'S FIXED POINTTHEOREM) *Let Q be a nonempty, closed, bounded, compact in measure and convex subset of a real Banach space $L_1(0,1)$ and $T : Q \longrightarrow Q$ be $\beta -$ condensing i.e., it is continuous and have the property that there is a constant $\xi \in (0,1)$ such that $\beta(TX) \leq \xi\beta(X)$ for any subset X of Q, where $\beta(X)$ is the measure of weak noncompactness* (De Blasi [9]) *given by the formula* [1],

$$\beta(X) := \lim_{\epsilon \longrightarrow 0} \sup_{x \in X} \left(\sup_{D} \left[\int_D |x(t)|dt : D \subset (0,1), measD \leq \epsilon \right] \right).$$

Then T has at least one fixed point in Q.

3. Equivalence of the Cauchy type problem and the Volterra integral equation. In this section, we proceed to show that the Cauchy problem (1) and the Volterra integral equation (2) are equivalent in the sense that, if $x \in L_1(0,1)$ satisfies one of these relations, then it also satisfies the other. We prove such a result by assuming that function $f \in L_1(0,1)$. To facilitate our discussion, let us first state the following assumptions:

1. Suppose G be an open subset of $\mathbb{R}$ and let $f : (0,1) \times G \longrightarrow \mathbb{R}$ be a function with the following properties:
   (a) $f(t,\cdot)$ is continuous for each $t \in (0,1)$,
   (b) $f(\cdot,x)$ is measurable for each $x \in G$,
   (c) there exist two real functions $a(t), b(t)$ such that:
   $$|f(t,x)| \leq a(t) + b(t)|x| \quad \text{for each } t \in (0,1) \text{ and } x \in G,$$
   where $a \in L^1(0,1)$ and b is measurable and bounded,
2. $\varphi : (0,1) \longrightarrow (0,1)$ is nondecreasing, absolutely continuous and there is a constant $M > 0$ such that $\varphi^{'} \geq M$ *a.e. on* $(0,1)$.

Thus, we are in a position to formulate and prove the following

LEMMA 3.1 *Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Assume that the assumptions* 1. *and* 2. *are satisfied. If $x \in L_1(0,1)$, then x satisfies a.e. the problem* (1) *if and only if x satisfies the integral equation* (2).

PROOF On the one hand, we prove the necessity. Let $x \in L_1(0,1)$ satisfy a.e. the problem (1). Since f satisfies Carathéodory conditions (a),(b) and since φ satisfies the assumption 2., $f(\cdot, x(\varphi(\cdot)))$ is measurable and from (c) we have

$$\begin{aligned}
\int_0^1 |f(s,x(\phi(s))|\ ds &\leq \int_0^1 \{|a(s)| + |b(s)|\ |x(\phi(s))|\}\ ds \\
&\leq \|a\| + \frac{\sup|b(t)|}{M}\int_0^1 |x(\varphi(s))||\varphi'(s)|\ ds \\
&\leq \|a\| + \frac{\sup|b(t)|}{M}\int_{\varphi(0)}^{\varphi(1)} |x(u)|\ du \\
&\leq \|a\| + \frac{\sup|b(t)|}{M}\|x\|. \qquad (3)
\end{aligned}$$

Thus, $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$ and consequently, equation (1) means that there exists a.e. on (0,1) the fractional derivatives $D^{\alpha_i}x \in L_1(0,1), i = 1,2,\cdots$, and hence [21] the function $I^{1-\alpha_i}x$ is absolutely continuous on [0,1] for every i. From Proposition 2.1 we deduce

$$\begin{aligned}
I^{\alpha_n}(D^{\alpha_n} &- \sum_{i=1}^{n-1} a_i D^{\alpha_i})x(t) = I^{\alpha_n}D^{\alpha_n}x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n}D^{\alpha_i}x(t) \\
&= x(t) - \frac{t^{\alpha_n-1}}{\Gamma(\alpha_n)}I^{1-\alpha_n}x(0) - \sum_{i=1}^{n-1} a_i\left\{I^{\alpha_n-\alpha_i}x(t) - \frac{t^{\alpha_n-1}}{\Gamma(\alpha_n)}I^{1-\alpha_i}x(0)\right\}.
\end{aligned}$$

In the view of the initial condition of the problem (1) we obtain

$$I^{\alpha_n}\left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right)x(t) = x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i}x(t).$$

Since, $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$, Proposition 2.1 result in $I^{\alpha_n}f \in L_1(0,1)$ a.e. on (0,1). Applying the operator I^{α_n} on both sides of (1) we have

$$I^{\alpha_n}D^{\alpha_n}x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i}x(t) = I^{\alpha_n}f(t, x(\varphi(t))),$$

therefore, x is also a solution of the integral equation (2).

On the other hand, let $x \in L_1(0,1)$ satisfies the integral equation (2) a.e. on (0,1). Applying the operator D^{α_n} on both sides of (2) and using Proposition 2.1 we obtain

$$D^{\alpha_n}x(t) = \sum_{i=1}^{n-1} a_i D I^{1-\alpha_n}I^{\alpha_n-\alpha_i}x(t) + f(t, x(\varphi(t))) = \sum_{i=1}^{n-1} a_i D^{\alpha_i}x(t) + f(t, x(\varphi(t))).$$

From here, we arrive at the equation (1). Now, we show that the initial condition of the problem (1) also hold. For this we apply the operator $I^{1-\alpha_n}$ on both sides of

(2). Then we obtain

$$I^{1-\alpha_n}x(t) = \sum_{i=1}^{n-1} a_i I^{1-\alpha_i}x(t) + \int_0^t f(s, x(\varphi(s)))\,ds,$$

that is

$$I^{1-\alpha_n}x(t) - \sum_{i=1}^{n-1} a_i I^{1-\alpha_i}x(t) = \int_0^t f(s, x(\varphi(s)))\,ds.$$

Keeping in mind the absolute continuity of $I^{1-\alpha_i}x$ and taking the limit as $t \to 0^+$, we obtain the initial condition of the problem (1). Thus the sufficiency is proved, which completes the proof. ∎

4. Main result. In this section, we present our main result by proving the existence of solutions of equation (1) in $L_1(0,1)$. This result is contained in the following theorem.

THEOREM 4.1 *Assume, in addition to the assumptions of Lemma* 3.1*, that*

3. *for each nondecreasing function x, the map $t \mapsto f(t, x(\varphi(t)))$ is nondecreasing,*

4.
$$\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(t)|}{M\Gamma(\alpha_n+1)} < 1.$$

Then equation (1) *has at least one monotonic solution.*

PROOF Let us define the operator T as

$$(Tx)(t) := \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i}x(t) + I^{\alpha_n}f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1). \tag{4}$$

We show that T transforms $L_1(0,1)$ into itself and it is continuous. First note that, as in the proof of Lemma 3.1, for each $x \in L_1(0,1)$ also $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$ that is, the operator T well-defined. Further, f is continuous in x (assumption 1.) and I^α maps $L_1(0,1)$ continuously into itself (Proposition 2.1), so $x \mapsto I^\alpha f(t, x(\varphi(t)))$ is continuous in x. Since x is an arbitrary element in $L_1(0,1)$, then T well-defined and maps $L_1(0,1)$ continuously into $L_1(0,1)$. Let x be an arbitrary element in the ball B_r (of radius r and center at θ). According to our assumptions 1. and 2. we have

$$\begin{aligned}\|Tx\| &= \int_0^1 |Tx(t)|dt \\ &\le \int_0^1 \int_0^t \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)}|x(s)| + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)}|f(s, x(\varphi(s)))|\right] ds\ dt.\end{aligned}$$

By interchanging the order of integration, we get

$$\begin{aligned}
\|Tx\| &\leq \int_0^1 \int_s^1 \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)} |x(s)| + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} |f(s,x(\varphi(s)))| \right] dt\, ds \\
&\leq \int_0^1 \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} |x(s)| + \frac{1}{\Gamma(1+\alpha_n)} \{|a(s)| + |b(s)||x(\varphi(s))|\} \right] ds \\
&\leq \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \int_0^1 |x(\varphi(s))||\varphi'(s)|\, ds \\
&\leq \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \int_{\varphi(0)}^{\varphi(1)} |x(u)|\, du
\end{aligned}$$

Therefore

$$\|Tx\| \leq \frac{\|a\|}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \right] \|x\|. \tag{5}$$

Above inequality proves that the operator T maps B_r into itself, where

$$r = \frac{\dfrac{\|a\|}{\Gamma(1+\alpha_n)}}{1 - \left[\displaystyle\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \right]}.$$

Now, let Q_r denotes the subset of B_r consisting of all functions being a.e. nondecreasing on (0,1). Obviously Q_r is nonempty, closed, bounded, compact in measure and convex. We claim $T : Q_r \to Q_r$ is $\beta -$ condensing operator. Once our claim is established, the Emmanuele's fixed point theorem guarantees that T has a fixed point in Q_r, i.e. the problem (2) (hence the problem (1)) has a solution in Q_r. It remains to prove this claim. Firstly, for any $x \in Q_r$, assumption 3. and Proposition 2.1 result that Tx is a.e. nondecreasing on (0,1). Thus the operator T maps Q_r into itself. Now, let X be a nonempty subset of Q_r. Fix $\epsilon > 0$, and take a measurable subset $D \subset (0,1)$ such that the measure of $D = measD \leq \epsilon$. For arbitrary $x \in X$ we deduce that

$$\|Tx\|_D \leq \frac{\|a\|_D}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)} \right] \|x\|_D.$$

Keeping in mind the formula expressing the measure of weak noncompactness β mentioned before (*cf.* [1] and [5]) and taking into account that

$$\lim_{\epsilon \to 0} \sup_{x \in X} \left(\sup_D [\|a\|_D : D \subset (0,1), measD \leq \epsilon \ ] \right) = 0,$$

we have ([4] and [13])

$$\beta(TX) \leq \xi \beta(X),$$

where

$$\xi = \left(\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(t)|}{M\Gamma(\alpha_n+1)} \right) < 1.$$

Thus T has at least one fixed point in Q_r in the view of Theorem 2.3. Hence there exists at least one nondecreasing solution $x \in L_1(0,1)$ of (2). In the view of Lemma 3.1, equation (1) has a nondecreasing solution $x \in L_1(0,1)$. This completes the proof. ∎

Example 4.2 Consider the following nonlinear inhomogenous fractional differential equation of order $\alpha \in (0,1)$:

$$D^{\alpha}x(t) = \lambda t^{\beta}[x(t)]^m + Lt^{\gamma}, \quad a.e.\ on\ (0,1), \quad I^{1-\alpha}x(0) = 0, \tag{6}$$

where λ, L, β, m and γ are real numbers, $(\lambda \neq 0)$. Assume that $\beta > 0, m \in (0,1)$ and $\gamma = \dfrac{\beta + m\alpha}{1-m}$. We establish the conditions for the existence of monotonic solution of the equation (6). For this, we have to choose the domain G and check when the conditions of Theorem 4.1 with

$$f(t,x) = \lambda t^{\beta}x^m + Lt^{\gamma},$$

are satisfied.

Let ω and $k > 0$ be constants such that $k < \dfrac{\Gamma(1+\alpha)}{|\lambda|}$, and $\omega \leq \dfrac{\beta}{1-m}$. We choose the following domain

$$D := \left\{ (t,x) \in \mathbb{R}^2 : \ t \in (0,1), 0 < x < k^{\frac{1}{m-1}}\ t^{\omega} \right\}.$$

This means that the domain G for x is unbounded for $\omega < 0$ and bounded for $\omega \geq 0$. To prove that f satisfies the condition $1(c)$, we note that, for $(t,x) \in D$

$$\begin{aligned} |f(t,x)| &\leq |\lambda| t^{\beta}|x|^m + |L|t^{\gamma} \\ &\leq |\lambda| t^{\beta}|x|^{m-1}|x| + |L|t^{\gamma} \\ &\leq |\lambda| t^{\beta} k t^{\omega(m-1)}|x| + |L|t^{\gamma} \\ &\leq |\lambda| k t^{\beta+\omega(m-1)}|x| + |L|t^{\gamma}. \end{aligned}$$

Since, $\beta + \omega(m-1) \geq 0$, the function f thus has the linear growth condition

$$|f(t,x)| \leq |L|t^{\gamma} + |\lambda|k|x|,$$

Therefore the assumption 4. of Theorem 4.1 holds with

$$M = 1, a_1 = a_2 = \cdots = a_{n-1} = 0, \quad \frac{|\lambda|k}{\Gamma(1+\alpha)} = \frac{\sup|b(t)|}{\Gamma(1+\alpha)} < 1.$$

According to Theorem 4.1, the differential equation (6) has at least a nondecreasing solution $x \in L_1(0,1)$. In fact, one can easily verified that the solution given by

$$x(t) = \mu t^{\left(\frac{\beta+\alpha}{1-m}\right)},$$

where μ is the the positive solution of the transcendental equation (see [15])

$$\Gamma\left(\frac{\beta+\alpha}{1-m}+1-\alpha\right)[\lambda\mu^m+L]-\Gamma\left(\frac{\beta+\alpha}{1-m}+1\right)\mu=0.$$

References

[1] J. Appell and E. De Pascale, *Su alcuni parametri connesi con la misura di noncompacttezza di Hausdoff in spazzi functioni misurabili.* Boll. Un. Matm. Ital. **3B(6)** (1984), 497-515.

[2] T.M. Atanackovic and B. Stankovic, *On a system of differential equations with fractional derivatives in rod theory* J. Phys. **A 37** (2004), 1241-1250.

[3] A. Babakhani and V. Daftardar-Gejji, *E*xistence of positive solutions of nonlinear fractional differential equations, J. Math. Anal. Appl. 278 (2003), 434-442.

[4] J. Banaś, *On the superposition operator and integrable solutions of some functional equations,* Nonlinear Anal. Vol. **12** (1988), 777-784.

[5] J. Banaś and K. Goebel, *Measure of noncompactness in Banach spaces,* Lecture Notes in Pure and Appl. Math., Vol. **60**, Dekker, New York 1980.

[6] M. Cichoń, A.M.A. El-Sayed and H.A.H. Salem, *Existence theorem for nonlinear functional integral equations of fractional orders,* Commentationes Math. **41** (2001), 59-67.

[7] N. Dunford and J.T. Schwartz *Linear Operators Part I.* Interscience Publ. John Wiley, New York 1964.

[8] L. Debnath, *Recents applications of fractional calculus to scinence and engineering,* Int. J. Math. Appl. Sci. **54** (2003), 3413-3442.

[9] F.S. De Blasi, *On a property of the unit sphere in a Banach space,* Bull. Math. Soc. Sci. Math. R.S. Roum. (N. S.) **21** (1977), 259-262.

[10] A.M.A. El-Sayed, *Linear differential equations of fractional order*, Appl. Math. Comput. Vol. **55** (1993), 1-12.

[11] A.M.A. El-Sayed, *Nonlinear functional differential equations of arbitrary order*, Nonlinear Anal. Vol. **33(2)** (1998), 181-186.

[12] A.M.A. El-Sayed and A.G. Ibrahim, *Set-valued integral equations of arbitrary (fractional) order,* Appl. Math. Comput. **118** (2001), 113-121.

[13] G. Emmanuele, *Measure of weak noncompactness and fixed point theorem*, Bull. Math. Soc. Sci. Math. R.S. Roum. **25** (1981), 253-258.

[14] S.B. Hadid, *Local and global existence theorems on differential equations of non-integer order*, Journal of Fractional Calculus **7** (1995), 101-105.

[15] N.K. Karapetyants, A. Kilbas and M. Saig, *On the solution of nonlinear Volterra convolution equation with power nonlinearity*, J. Int. Equat. Appl., **84** (1996), 429-445.

[16] A.A. Kilbas and J.J. Trujillo, *Differential equations of fractional order: methods, results problems, I* Appl. Anal. **78(1-2)** (2001), 153-192.

[17] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York 1993.

[18] Z. Luszczki and B. Rzepecki, *An existence theorem for ordinary differential equations of order* $\alpha \in (0, 1]$, Demonste. Math. **20(3-4)** (1987),471-475.

[19] I. Podlubny, *Fractional Differential Equations*. Acad. Press, San Diego-New York-London 1999.

[20] Yu. Rossikhin and M.V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev. **50(1)** (1997), 15-67.

[21] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publisher 1993.

[22] B.J. West, M. Bologna and P. Grigolini (Eds.) *Physics of Fractal Operators*, Springer Verlag, New York 2003.

[23] K. Wiener, *Solutions of systems of differential equations with derivatives of noninteger order,* Wiss. Z. Martin-Luther university Hall-Wittenberg Math. Natur. Reihe, **38 (5)** (1989) 13-19.

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