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On the Chaplygin method for the Darboux problem

Abstract. In the paper we deal with the Darboux problem for hyperbolic functional differential equations. We give the sufficient conditions for the existence of the sequence $\{z^{(m)}\}$ such that if $\tilde{z}$ is a classical solution of the original problem then $\{z^{(m)}\}$ is uniformly convergent to $\tilde{z}$. The convergence that we get is of the Newton type.

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1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. Let $a_0, b_0 \geq 0$ and $a, b > 0$ be fixed. We define the sets

$$D = [-a_0, 0] \times [-b_0, 0], \quad E = [0, a] \times [0, b],$$

$$G = [-a_0, a] \times [-b_0, b], \quad E_0 = G \setminus \Big((0, a] \times (0, b]\Big).$$

Given a function $z : G \to R$ and a point $(x, y) \in E$, we consider the function $z_{(x,y)} : D \to R$ defined by

$$z_{(x,y)}(\tau, \eta) = z(x + \tau, y + \eta), \ (\tau, \eta) \in D.$$

The function $z_{(x,y)}$ is the restriction of z to the set $[x - a_0, x] \times [y - b_0, y]$ and this restriction is shifted to the set D. Put

$$\Omega = E \times [C(D, R)]^3$$

and suppose that $f : \Omega \to R$, $\varphi : E_0 \to R$ are given functions of variables (x, y, w, u, v) and (x, y) respectively. We deal with the following Darboux problem for the second order partial functional differential equation

$$\partial_{xy} z(x,y) = f\Big(x, y, z_{(x,y)}, (\partial_x z)_{(x,y)}, (\partial_y z)_{(x,y)}\Big) \tag{1}$$

$$z(x,y) = \varphi(x,y), \ (x,y) \in E_0. \tag{2}$$

We denote by $C^1(G,R)$ the class of all functions $z : G \to R$ such that $z \in C(G,R)$, the partial derivatives $\partial_x z$, $\partial_y z$ exist on G and $\partial_x z, \partial_y z \in C(G,R)$.

Definition 1.1 A function $z : G \to R$ will be called the function of class $C^{1,*}(G,R)$ if $z \in C^1(G,R)$, the mixed derivative $\partial_{xy} z$ exists and it is continuous on E.

Definition 1.2 A function $\bar{z} : G \to R$ is called a classical solution of (1), (2) if $\bar{z} \in C^{1,*}(G,R)$, $\bar{z}$ satisfies (1) on E and condition (2) holds.

We transform the Darboux problem into an integral functional equation. We prove that there exists exactly one classical solution of this equation and the solution is a limit of a sequence of successive approximations.

The Chaplygin method proposed in [2] consists in approximating of solutions of ordinary nonlinear differential equations by solutions of suitable linear differential equations. This method has been extended on systems of first order partial semilinear differential equations [6] and second order hyperbolic equations [9]. The paper [4] initiated the theory of the Chaplygin method for first order partial functional differential equations. The ideas presented also in [7] were generalized in [1] on parabolic functional differential equations but linearization only with respect to the nonfunctional argument was allowed here. The existence of two monotone sequences which are uniformly convergent to a classical solution of the functional Darboux problem has been proved in [3].

The following property of the Chaplygin method is proved in [8]: the Chaplygin sequence for an initial value problem of ordinary differential equation and the Newton sequence for the integral equation generated by the original problem are the same. The aim of the paper is to extend the above property of ordinary differential equations on the Darboux problem for hyperbolic functional differential equations.

We define the sequence approximating the classical solution of (1), (2). Suppose that $f \in C(\Omega, R)$ and that the Fréchet derivatives $\partial_w f(x,y,w,u,v)$, $\partial_u f(x,y,w,u,v)$, $\partial_v f(x,y,w,u,v)$ exist on Ω. For a function $z : G \to R$ and for a point $(x,y) \in E$ we write

$$\mathbf{f}[\mathbf{z}](x,y) = f(x, y, z_{(x,y)}, (\partial_x z)_{(x,y)}, (\partial_y z)_{(x,y)}),$$

$$\partial_{\mathbf{w}} \mathbf{f}[\mathbf{z}](x,y) = \partial_w f(x, y, z_{(x,y)}, (\partial_x z)_{(x,y)}, (\partial_y z)_{(x,y)}),$$

$$\partial_{\mathbf{u}} \mathbf{f}[\mathbf{z}](x,y) = \partial_u f(x, y, z_{(x,y)}, (\partial_x z)_{(x,y)}, (\partial_y z)_{(x,y)}),$$

$$\partial_{\mathbf{v}} \mathbf{f}[\mathbf{z}](x,y) = \partial_v f(x, y, z_{(x,y)}, (\partial_x z)_{(x,y)}, (\partial_y z)_{(x,y)}).$$

We consider the sequence $\{z^{(m)}\}$, $z^{(m)} : G \to R$, defined by the relations:

(i) $z^{(0)} \in C^{1,*}(G,R)$ and $z^{(0)}(x,y) = \varphi(x,y)$ on E_0,

(ii) if $z^{(m)} : G \to R$ is a given function then $z^{(m+1)}$ is a solution of the linear Darboux problem

$$\begin{aligned}\partial_{xy} z(x,y) &= \mathbf{f}[\mathbf{z^{(m)}}](x,y) + \partial_{\mathbf{w}}\mathbf{f}[\mathbf{z^{(m)}}](x,y)\Big(z_{(x,y)} - (z^{(m)})_{(x,y)}\Big) \\ &\quad +\partial_{\mathbf{u}}\mathbf{f}[\mathbf{z^{(m)}}](x,y)\Big((\partial_x z)_{(x,y)} - (\partial_x z^{(m)})_{(x,y)}\Big) \\ &\quad +\partial_{\mathbf{v}}\mathbf{f}[\mathbf{z^{(m)}}](x,y)\Big((\partial_y z)_{(x,y)} - (\partial_y z^{(m)})_{(x,y)}\Big)\end{aligned}$$

and

$$z(x,y) = \varphi(x,y),\ (x,y) \in E_0.$$

DEFINITION 1.3 The sequence $\{z^{(m)}\}$ satisfying the above relations is called a Chaplygin sequence of (1), (2).

We prove that under natural assumptions on f the Chaplygin sequence is convergent to a classical solution of (1), (2).

2. Notations and assumptions. We consider the integral functional equation for the problem (1), (2). Write

$$\psi(x,y) = \varphi(x,0) + \varphi(0,y) - \varphi(0,0),\ (x,y) \in E.$$

Then (1), (2) is equivalent to the following equation

$$z(x,y) = \psi(x,y) + \int_0^x \int_0^y \mathbf{f}[\mathbf{z}](t,s)dsdt \tag{3}$$

with the initial condition

$$z(x,y) = \varphi(x,y),\ (x,y) \in E_0. \tag{4}$$

We introduce some notations and assumptions. In the space $C(D,R)$ we use the maximum norm

$$||z|| = \max\{|z(\tau,\eta)| :\ (\tau,\eta) \in D\},\ z \in C(D,R).$$

For $z \in C(G,R)$ and for a constant $\lambda \geq 0$ we put

$$||z||_\lambda = \max\{|z(x,y)|e^{-\lambda(x-a+y-b)} :\ (x,y) \in G\}.$$

Note that for $\lambda = 0$ we obtain the maximum norm in $C(G,R)$. Moreover for every $\lambda \geq 0$ and $z \in C(G,R)$ we have $||z||_0 \leq ||z||_\lambda$.

In the space $C^1(G,R)$ we use the following norm

$$||z||^*_\lambda = ||z||_\lambda + ||\partial_x z||_\lambda + ||\partial_y z||_\lambda,\ z \in C^1(G,R).$$

Assumption H. Suppose that $f \in C(\Omega,R)$ and that the Fréchet derivatives $\partial_w f(x,y,w,u,v)$, $\partial_u f(x,y,w,u,v)$, $\partial_v f(x,y,w,u,v)$ exist and they are continuous on Ω and

1) there is $A > 0$ such that for $(x, y, w, u, v) \in \Omega$

$$||\partial_w f(x, y, w, u, v)|| \le A,\ ||\partial_u f(x, y, w, u, v)|| \le A,\ ||\partial_v f(x, y, w, u, v)|| \le A,$$

2) there is $L > 0$ such that the expressions

$$||\partial_w f(x, y, w, u, v) - \partial_w f(x, y, \bar{w}, \bar{u}, \bar{v})||,\ ||\partial_u f(x, y, w, u, v) - \partial_u f(x, y, \bar{w}, \bar{u}, \bar{v})||,$$

$$||\partial_v f(x, y, w, u, v) - \partial_v f(x, y, \bar{w}, \bar{u}, \bar{v})||$$

are bounded on Ω by $L\Big(||w - \bar{w}|| + ||u - \bar{u}|| + ||v - \bar{v}||\Big)$,

3) a constant $A_0 > 0$ is such that for $z^{(0)}$ defined by (i) and for $(x, y) \in E$

$$\Big|\partial_{xy} z^{(0)}(x, y) - \mathbf{f}[\mathbf{z^{(0)}}](x, y)\Big| \le A_0,$$

4) $\varphi \in C^1(E_0, R)$ and for $(x, y) \in E_0 \cap E$ there exists $\partial_{xy}\varphi(x, y)$ and

$$\partial_{xy}\varphi(x, y) = f\Big(x, y, \varphi_{(x,y)}, (\partial_x\varphi)_{(x,y)}, (\partial_y\varphi)_{(x,y)}\Big).$$

3. The Newton sequence. Write the problem (3), (4) in the form

$$F[z] = 0 \tag{5}$$

where

$$F[z](x, y) = z(x, y) - \psi(x, y) - \int_0^x \int_0^y \mathbf{f}[\mathbf{z}](t, s)dsdt,\ (x, y) \in E,$$

$$F[z](x, y) = z(x, y) - \varphi(x, y),\ (x, y) \in E_0.$$

Consider the Newton sequence for the equation (5). Let $u^{(0)} \in C^{1,*}(G, R)$ and $u^{(0)} = z^{(0)}$. Define

$$u^{(m+1)} = u^{(m)} - (F'[u^{(m)}])^{-1} F[u^{(m)}],\ m = 0, 1, \dots \tag{6}$$

THEOREM 3.1 *Suppose that Assumption H is satisfied. Then*
(i) there exists exactly one solution $\tilde{z} \in C^{1,*}(G, R)$ of (1), (2),
(ii) the sequence $\{u^{(m)}\}$ exists and $u^{(m)} \in C^{1,*}(G, R)$, $m \ge 0$,
(iii) if we assume that

$$A_0(ab + a + b)\frac{4L}{A} \exp\Big\{(2A + \sqrt{4A^2 + 2A})(a + a_0 + b + b_0)\Big\} \le 1 \tag{7}$$

then there exist positive constants η and $\mu \le \frac{1}{2}$ such that

$$||u^{(m)} - \tilde{z}||_0^* \le \frac{1}{2^{m-1}}(2\mu)^{2^{m-1}}\eta,\ m \ge 1.$$

PROOF Let us fix $\lambda > 0$ in the following way

$$\lambda = 2A + \sqrt{4A^2 + 2A}. \tag{8}$$

Then

$$A\Big(\frac{1}{\lambda^2} + \frac{2}{\lambda}\Big) = \frac{1}{2} < 1.$$

In the first part of the proof we show that the problem (3), (4) has a unique solution $\tilde{z} \in C^{1,*}(G, R)$.

For $z \in C^1(G, R)$ we define

$$T[z](x, y) = \psi(x, y) + \int_0^x \int_0^y \mathbf{f}[\mathbf{z}](t, s)dsdt, \ (x, y) \in E$$

$$T[z](x, y) = \varphi(x, y), \ (x, y) \in E_0.$$

From Assumption H it follows that $T[z] \in C^1(G, R)$. We prove that the operator T is a contraction.

First note that if $u \in C(G, R)$ and $(t, s) \in E$ then

$$\begin{aligned} & ||u_{(t,s)}||e^{-\lambda(t-a+s-b)} \\ = \ & \max\{|u(t+\tau, s+\eta)|e^{-\lambda(t+\tau-a+s+\eta-b)}e^{\lambda(\tau+\eta)} : \ (\tau, \eta) \in D\} \\ \leq \ & ||u||_\lambda. \end{aligned}$$

It follows that

$$\Big|(T[z] - T[\bar{z}])(x, y)\Big| \leq \int_0^x \int_0^y \Big|\mathbf{f}[\mathbf{z}](t, s) - \mathbf{f}[\bar{\mathbf{z}}](t, s)\Big|dsdt \leq$$

$$\leq \int_0^x \int_0^y A\Big(||(z - \bar{z})_{(t,s)}|| + ||(\partial_x z - \partial_x \bar{z})_{(t,s)}|| + ||(\partial_y z - \partial_y \bar{z})_{(t,s)}||\Big)dsdt \leq$$

$$\leq \int_0^x \int_0^y A\Big(||z - \bar{z}||_\lambda + ||\partial_x z - \partial_x \bar{z}||_\lambda + ||\partial_y z - \partial_y \bar{z}||_\lambda\Big)e^{\lambda(t-a+s-b)}dsdt \leq$$

$$\leq A||z - \bar{z}||^*_\lambda \frac{1}{\lambda^2} e^{\lambda(x-a+y-b)}$$

and consequently

$$||T[z] - T[\bar{z}]||_\lambda \leq \frac{A}{\lambda^2}||z - \bar{z}||^*_\lambda.$$

According to Assumption H, we have

$$\Big|\partial_x(T[z] - T[\bar{z}])(x, y)\Big| \leq \int_0^y \Big|\mathbf{f}[\mathbf{z}](x, s) - \mathbf{f}[\bar{\mathbf{z}}](x, s)\Big|ds \leq$$

$$\leq A||z - \bar{z}||^*_\lambda \int_0^y e^{\lambda(x-a+s-b)}ds \leq A||z - \bar{z}||^*_\lambda \frac{1}{\lambda} e^{\lambda(x-a+y-b)}$$

which gives

$$||\partial_x(T[z] - T[\bar{z}])||_\lambda \leq \frac{A}{\lambda}||z - \bar{z}||^*_\lambda.$$

In the similary way we obtain

$$||\partial_y(T[z]-T[\bar{z}])||_\lambda \leq \frac{A}{\lambda}||z-\bar{z}||^*_\lambda.$$

The result is

$$||T[z]-T[\bar{z}]||^*_\lambda \leq A(\frac{1}{\lambda^2}+\frac{2}{\lambda})||z-\bar{z}||^*_\lambda. \tag{9}$$

By the Banach fixed point theorem there exists the unique $\tilde{z}\in C^1(G,R)$ such that

$$\tilde{z}=T[\tilde{z}].$$

This proves the assertion (i).

If $z\in C^1(G,R)$ then for $h\in C^1(G,R)$

$$F'[z]h(x,y)=h(x,y)-$$

$$-\int_0^x\int_0^y\Big(\partial_{\mathbf{w}}\mathbf{f}[\mathbf{z}](t,s)h_{(t,s)}+\partial_{\mathbf{u}}\mathbf{f}[\mathbf{z}](t,s)(\partial_x h)_{(t,s)}+\partial_{\mathbf{v}}\mathbf{f}[\mathbf{z}](t,s)(\partial_y h)_{(t,s)}\Big)dsdt$$

for $(x,y)\in E$ and

$$F'[z]h(x,y)=h(x,y)$$

for $(x,y)\in E_0$. Then $F'[z]=I-H[z]$ where

$$H[z](x,y)=$$

$$=\int_0^x\int_0^y\Big(\partial_{\mathbf{w}}\mathbf{f}[\mathbf{z}](t,s)h_{(t,s)}+\partial_{\mathbf{u}}\mathbf{f}[\mathbf{z}](t,s)(\partial_x h)_{(t,s)}+\partial_{\mathbf{v}}\mathbf{f}[\mathbf{z}](t,s)(\partial_y h)_{(t,s)}\Big)dsdt,$$

for $(x,y)\in E$ and

$$H[z]h(x,y)=0,$$

for $(x,y)\in E_0$. Since

$$\Big|H[z]h(x,y)\Big|\leq$$

$$\leq\int_0^x\int_0^y\Big(||\partial_{\mathbf{w}}\mathbf{f}[\mathbf{z}](t,s)||\cdot||h_{(t,s)}||+||\partial_{\mathbf{u}}\mathbf{f}[\mathbf{z}](t,s)||\cdot||(\partial_x h)_{(t,s)}||+$$

$$+||\partial_{\mathbf{v}}\mathbf{f}[\mathbf{z}](t,s)||\cdot||(\partial_y h)_{(t,s)}||\Big)dsdt\leq$$

$$\leq A\int_0^x\int_0^y\Big(||h||_\lambda+||\partial_x h||_\lambda+||\partial_y h||_\lambda\Big)e^{\lambda(t-a+s-b)}dsdt\leq A||h||^*_\lambda\frac{1}{\lambda^2}e^{\lambda(x-a+y-b)}$$

we obtain

$$||H[z]h||_\lambda\leq\frac{A}{\lambda^2}||h||^*_\lambda.$$

We conclude from Assumption H that

$$\Big|\partial_x H[z]h(x,y)\Big|\leq$$

$$\leq \int_0^y \Big(||\partial_{\mathbf{w}} \mathbf{f}[\mathbf{z}](x,s)|| \cdot ||h_{(x,s)}|| + ||\partial_{\mathbf{u}} \mathbf{f}[\mathbf{z}](x,s)|| \cdot ||(\partial_x h)_{(x,s)}|| +$$

$$+ ||\partial_{\mathbf{v}} \mathbf{f}[\mathbf{z}](x,s)|| \cdot ||(\partial_y h)_{(x,s)}|| \Big) ds \leq$$

$$\leq A||h||_\lambda^* \int_0^y e^{\lambda(x-a+s-b)} ds \leq A||h||_\lambda^* \frac{1}{\lambda} e^{\lambda(x-a+y-b)}$$

and consequently

$$||\partial_x H[z]h||_\lambda \leq \frac{A}{\lambda} ||h||_\lambda^*.$$

Analogously we obtain

$$||\partial_y H[z]h||_\lambda \leq \frac{A}{\lambda} ||h||_\lambda^*.$$

The result is

$$||H[z]||_\lambda^* \leq A\Big(\frac{1}{\lambda^2} + \frac{2}{\lambda}\Big) < 1$$

and the constant in the above estimation is the same as the constant in (9). Thus there exists the operator $(F'[z])^{-1} = (I - H[z])^{-1}$ for every $z \in C^1(G,R)$ and the assertion (ii) is proved.

Define

$$(10) \quad \begin{array}{lll} Q = A\Big(\frac{1}{\lambda^2} + \frac{2}{\lambda}\Big), & B = \frac{1}{1-Q}, & K = L\Big(\frac{1}{\lambda^2} + \frac{2}{\lambda}\Big), \\ q = ab + a + b, & \eta = A_0 q B \exp\Big\{\lambda(a + a_0 + b + b_0)\Big\}, & \mu = KB\eta \end{array}$$

It follows from (8) then $Q < 1$. From (7) we deduce that $\mu \leq \frac{1}{2}$.

Write

$$P(t) = \frac{1}{2} K t^2 - \frac{1}{B} t + \frac{\eta}{B}, \quad t^* = \frac{1 - \sqrt{1 - 2\mu}}{\mu} \eta.$$

Consider the sequence $\{t_m\}$ defined by

$$t_0 = 0, \;\; t_{m+1} = t_m - \frac{P(t_m)}{P'(t_m)}, \; m \geq 0.$$

Then $0 \leq t_m < t_{m+1} < t^*$ for $m \geq 0$ and

$$(11) \qquad t^* - t_m \leq 2^{-m+1} (2\mu)^{2^m - 1} \eta, \; m \geq 1.$$

The proof of (11) can be found in [5].

We next show that

$$(12) \qquad ||u^{(m+1)} - u^{(m)}||_\lambda^* \leq t_{m+1} - t_m \text{ for } m \geq 0.$$

First note that for every $z \in C^1(G,R)$ we have

$$||H[z]||_\lambda^* \leq Q < 1$$

and thus

$$||(F'[z])^{-1}||_\lambda^* = ||(I - H[z])^{-1}||_\lambda^* \leq B.$$

Let $z, \bar{z} \in C^1(G,R)$ and $h \in C^1(G,R)$. It follows that

$$\Big|(F'[z]-F'[\bar{z}])h(x,y)\Big| \leq$$

$$\leq L\Big(||z-\bar{z}||_0 + ||\partial_x(z-\bar{z})||_0 + ||\partial_y(z-\bar{z})||_0\Big)$$

$$\times \int_0^x \int_0^y \Big(||h_{(t,s)}|| + ||(\partial_x h)_{(t,s)}|| + ||(\partial_y h)_{(t,s)}||\Big)dsdt \leq$$

$$\leq L||z-\bar{z}||_0^* ||h||_\lambda^* \int_0^x \int_0^y e^{\lambda(t-a+s-b)}dsdt \leq \frac{L}{\lambda^2}||z-\bar{z}||_\lambda^* ||h||_\lambda^* e^{\lambda(x-a+y-b)}$$

and consequently

$$||(F'[z]-F'[\bar{z}])h||_\lambda \leq \frac{L}{\lambda^2}||z-\bar{z}||_\lambda^* ||h||_\lambda^*.$$

In the similar way we can prove that

$$||\partial_x(F'[z]-F'[\bar{z}])h||_\lambda \leq \frac{L}{\lambda}||z-\bar{z}||_\lambda^* ||h||_\lambda^*$$

and

$$||\partial_y(F'[z]-F'[\bar{z}])h||_\lambda \leq \frac{L}{\lambda}||z-\bar{z}||_\lambda^* ||h||_\lambda^*.$$

The result is

$$||F'[z]-F'[\bar{z}]||_\lambda^* \leq K||z-\bar{z}||_\lambda^*.$$

Since

$$F[u^{(0)}](x,y) = u^{(0)}(x,y) - \psi(x,y) - \int_0^x \int_0^y \mathbf{f}[\mathbf{u^{(0)}}](t,s)dsdt =$$

$$= \int_0^x \int_0^y \Big(\partial_{xy}u^{(0)}(t,s) - \mathbf{f}[\mathbf{u^{(0)}}](t,s)\Big)dsdt$$

we obtain

$$||F[u^{(0)}]||_\lambda^* \leq A_0(ab+a+b)\exp\Big\{\lambda(a+a_0+b+b_0)\Big\}.$$

Denote by $\Gamma_0 = (F'[u^{(0)}])^{-1}$. Then $||\Gamma_0||_\lambda^* \leq B$ and

$$||\Gamma_0 F[u^{(0)}]||_\lambda^* \leq \eta.$$

We prove (12) for $m=0$. Since

$$||u^{(1)}-u^{(0)}||_\lambda^* = ||\Gamma_0 F[u^{(0)}]||_\lambda^* \leq \eta$$

and $t_1 - t_0 = -\frac{P(t_0)}{P'(t_0)} = \eta$ we obtain

$$||u^{(1)}-u^{(0)}||_\lambda^* \leq t_1 - t_0.$$

Suppose that $||u^{(k)} - u^{(k-1)}||_\lambda^* \leq t_k - t_{k-1}$ where $k \leq m$ for arbitrary fixed $m - 1 \geq 0$. We will prove that $||u^{(m+1)} - u^{(m)}||_\lambda^* \leq t_{m+1} - t_m$.

It follows from (6) that

$$F[u^{(m-1)}] + F'[u^{(m-1)}](u^{(m)} - u^{(m-1)}) = 0$$

and thus

$$F[u^{(m)}] = (F[u^{(m)}] - F[u^{(m-1)}]) - F'[u^{(m-1)}](u^{(m)} - u^{(m-1)}).$$

We can estimate

$$||F[u^{(m)}]||_\lambda^* \leq \frac{1}{2}K\Big(||u^{(m)} - u^{(m-1)}||_\lambda^*\Big)^2 \leq \frac{1}{2}K(t_m - t_{m-1})^2.$$

We have also

$$P(t_m) = P(t_m) - P(t_{m-1}) + P'(t_{m-1})(t_m - t_{m-1}) =$$

$$= \frac{1}{2}P''(t_{m-1})(t_m - t_{m-1})^2 = \frac{1}{2}K(t_m - t_{m-1})^2.$$

In this way we obtain

$$||F[u^{(m)}]||_\lambda^* \leq P(t_m).$$

Moreover

$$||F'[u^{(m)}] - F'[u^{(0)}]||_\lambda^* \leq K||u^{(m)} - u^{(0)}||_\lambda^* \leq$$

$$\leq K\Big(||u^{(m)} - u^{(m-1)}||_\lambda^* + \ldots + ||u^{(1)} - u^{(0)}||_\lambda^*\Big) \leq$$

$$\leq K\Big((t_m - t_{m-1}) + \ldots + (t_1 - t_0)\Big) = Kt_m.$$

Thus the operator $S = \Gamma_0\Big(F'[u^{(m)}] - F'[u^{(0)}]\Big)$ satisfies

$$||S||_\lambda^* \leq ||\Gamma_0||_\lambda^* ||F'[u^{(m)}] - F'[u^{(0)}]||_\lambda^* \leq BKt_m$$

and

$$BKt_m < BKt^* = 1 - \sqrt{1 - 2\mu} \leq 1.$$

There exists $(I + S)^{-1}$ and

$$||(I + S)^{-1}||_\lambda^* \leq \frac{1}{1 - BKt_m}.$$

Thus

$$||(F'[u^{(m)}])^{-1}||_\lambda^* = ||\Big(\Gamma_0^{-1}(I + \Gamma_0 F'[u^{(m)}] - \Gamma_0 F'[u^{(0)}])\Big)^{-1}||_\lambda^* = ||(I + S)^{-1}\Gamma_0||_\lambda^* \leq$$

$$\leq \frac{1}{1 - BKt_m}B = \frac{1}{\frac{1}{B} - Kt_m} = -\frac{1}{P'(t_m)}.$$

Now we are ready to estimate the norm $||u^{(m+1)} - u^{(m)}||^*_\lambda$. We have

$$||u^{(m+1)} - u^{(m)}||^*_\lambda = ||(F'[u^{(m)}])^{-1}F[u^{(m)}]||^*_\lambda \leq$$

$$\leq ||(F'[u^{(m)}])^{-1}||^*_\lambda ||F[u^{(m)}]||^*_\lambda \leq -\frac{P(t_m)}{P'(t_m)} = t_{m+1} - t_m.$$

In this way (12) is proved.

It follows from

$$||u^{(m+k)} - u^{(m)}||^*_\lambda \leq ||u^{(m+k)} - u^{(m+k-1)}||^*_\lambda + \ldots + ||u^{(m+1)} - u^{(m)}||^*_\lambda \leq$$

$$\leq (t_{m+k} - t_{m+k-1}) + \ldots + (t_{m+1} - t_m) = t_{m+k} - t_m$$

that $\{u^{(m)}\}$ satisfies the Cauchy condition. Thus it is convergent to the unique solution $\tilde{z} \in C^1(G, R)$ of equation (5). Moreover $\tilde{z} \in C^{1.*}(G, R)$ and

$$||\tilde{z} - u^{(m)}||^*_0 \leq ||\tilde{z} - u^{(m)}||^*_\lambda \leq t^* - t_m \leq 2^{-m+1}(2\mu)^{2^m-1}\eta,\ m \geq 1.$$

This completes the proof. ∎

4. Conclusions for the Chaplygin sequence. Now we prove the theorem on convergence of the Chaplygin sequence $\{z^{(m)}\}$.

THEOREM 4.1 *If Assumption H is satisfied then* $\{z^{(m)}\}$ *exists and* $z^{(m)} \in C^{1.*}(G, R)$, $m \geq 0$. *If we assume* (7) *then*

$$||\tilde{z} - z^{(m)}||^*_0 \leq 2^{-m+1}(2\mu)^{2^m-1}\eta,\ m \geq 1$$

where $\tilde{z} \in C^{1.*}(G, R)$ *is a unique solution of* (1), (2) *and the constants* η, μ *are given in* (10).

PROOF From (6) it follows that

$$F'[u^{(m)}]\Big(u^{(m+1)} - u^{(m)}\Big) = -F[u^{(m)}].$$

This gives for $(x, y) \in E$

$$\Big(u^{(m+1)}(x, y) - u^{(m)}(x, y)\Big) -$$

$$- \int_0^x \int_0^y \Big[\partial_{\mathbf{w}}\mathbf{f}[\mathbf{u^{(m)}}](t, s)\Big(u^{(m+1)} - u^{(m)}\Big)_{(t,s)} + \partial_{\mathbf{u}}\mathbf{f}[\mathbf{u^{(m)}}](t, s)\Big(\partial_x u^{(m+1)} - \partial_x u^{(m)}\Big)_{(t,s)} +$$

$$+ \partial_{\mathbf{v}}\mathbf{f}[\mathbf{u^{(m)}}](t, s)\Big(\partial_y u^{(m+1)} - \partial_y u^{(m)}\Big)_{(t,s)}\Big] ds dt =$$

$$= -u^{(m)}(x, y) + \psi(x, y) + \int_0^x \int_0^y \mathbf{f}[\mathbf{u^{(m)}}](t, s) ds dt.$$

Thus

$$u^{(m+1)}(x, y) = \psi(x, y) +$$

$$\int_0^x \int_0^y \Big[\mathbf{f}[\mathbf{u^{(m)}}](t,s) + \partial_{\mathbf{w}}\mathbf{f}[\mathbf{u^{(m)}}](t,s)\Big(u^{(m+1)} - u^{(m)}\Big)_{(t,s)} +$$

$$+\partial_{\mathbf{u}}\mathbf{f}[\mathbf{u^{(m)}}](t,s)\Big(\partial_x u^{(m+1)} - \partial_x u^{(m)}\Big)_{(t,s)} + \partial_{\mathbf{v}}\mathbf{f}[\mathbf{u^{(m)}}](t,s)\Big(\partial_y u^{(m+1)} - \partial_y u^{(m)}\Big)_{(t,s)}\Big] dsdt$$

and

$$\partial_{xy} u^{(m+1)}(x,y) = \mathbf{f}[\mathbf{u^{(m)}}](x,y) + \partial_{\mathbf{w}}\mathbf{f}[\mathbf{u^{(m)}}](x,y)\Big(u^{(m+1)} - u^{(m)}\Big)_{(x,y)} +$$

$$+\partial_{\mathbf{u}}\mathbf{f}[\mathbf{u^{(m)}}](x,y)\Big(\partial_x u^{(m+1)} - \partial_x u^{(m)}\Big)_{(x,y)} + \partial_{\mathbf{v}}\mathbf{f}[\mathbf{u^{(m)}}](x,y)\Big(\partial_y u^{(m+1)} - \partial_y u^{(m)}\Big)_{(x,y)}.$$

In this way we obtain that $\{u^{(m)}\}$ satisfies relations (ii) in definition of the Chaplygin sequence. Since $u^{(0)} = z^{(0)}$ the Newton sequence $\{u^{(m)}\}$ and the Chaplygin sequence $\{z^{(m)}\}$ are identical. The assertion follows from Theorem 3.1. ∎

References

[1] S. Brzychczy, *Chaplygin's method for a system of nonlinear parabolic differential-functional equations* (Russian), Diff. Urav. **22** (1986), 705-708.

[2] S. A. Chaplygin, *Selected Works on Mechanics and Mathematics* (Russian), Moscow 1954.

[3] T. Człapiński, *Iterative methods for the Darboux problem for partial differential-functional equations*, J. Inequal. Appl. **4(2)** (1999), 141-161.

[4] Z. Kamont, *On the Chaplygin method for partial differential-functional equations of the first order*, Ann. Polon. Math. **38** (1980), 313-324.

[5] W. I. Krylov, W. W. Bobkov, P. I. Monastyrnyi, *Numerical Methods* (Russian), vol. 1, Moscow 1976.

[6] W. Mlak, E. Schechter, *On the Chaplygin method for partial differential equations of the first order*, Ann. Polon. Math. **22** (1969), 1-18.

[7] J. Szarski, *Differential Inequalities*, Warszawa 1967.

[8] G. Vidossich, *Chaplygin's method is Newton's method*, J. Math. Anal. Appl. **66(1)** (1978), 188-206.

[9] P. K. Zeragia, *Chaplygin's method for nonlinear hyperbolic differential equations with boundary conditions*, (Georgian), Trudy Tbil. Gos. Univ. **100** (1965), 145-154.

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