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On new multiple extension of Hilbert's integral inequality (2)

Abstract. Via Laplace transformation some new generalization of Hardy-Hilbert's integral inequality is given.

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1. Introduction.

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \text{ and } 0 < \int_0^\infty g^2(t)\,dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \pi \left(\int_0^\infty f^2(t)\,dt \int_0^\infty g^2(t)\,dt \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy ([1]) as follows.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \text{ and } \int_0^\infty g^q(t)\,dt < \infty,$$

then

$$(2) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < \frac{\pi}{sin(\pi/p)} \left(\int_0^\infty f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)\,dt\right)^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [4]).

B. Yang gave the following extensions of (2) as follows :

THEOREM 1.1 ([6]) *If $\lambda > 2 - min\{p, g\}$ and $f, g \geq 0$ satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \text{ and } \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

then

$$(3) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{1-\lambda} g^q(t)\,dt\right)^{\frac{1}{q}},$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible (B is the beta function).

The object of this paper is to give the following generalizations :

2. Main Results. The following is a generalization of [3, Proposition 319]

THEOREM 2.1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g, \phi, \psi \geq 0$, $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, ϕ^p and ψ^p are in $P(0, \infty)$, $K(f, g)$ is positive in $(0, \infty) \times (0, \infty)$ and homogeneous of degree -1 with*

$$(4) \qquad \int_0^\infty f^{-1/p} K(f, 1) f'\,du = k.$$

Then

$$(5) \qquad \int_0^\infty \int_0^\infty K(f, g)\phi(f)\psi(g)\,df\,dg < k\|\phi\|_p \|\psi\|_q$$

and

$$(6) \qquad \left\| \int_0^\infty K(f, g)\phi(f)\,df \right\|_p < k\|\phi\|_p.$$

The constant k is the best possible.

PROOF

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(f,g)\phi(f)\psi(g)\,df\,dg = \int_0^\infty \int_0^\infty K(f,g)\phi(f)\psi(g)f^{'}g^{'}\,du\,d\nu \\
= &\int_0^\infty \int_0^\infty K^{\frac{1}{p}}(f,g)\phi(f)(f^{'}g^{'})^{\frac{1}{p}}\left(\frac{f}{g}\right)^{\frac{1}{pq}} K^{\frac{1}{q}}(f,g)\psi(g)(f^{'}g^{'})^{\frac{1}{p}}\left(\frac{f}{g}\right)^{-\frac{1}{pq}} du\,d\nu \\
\leq &\left(\int_0^\infty \int_0^\infty K(f,g)\phi^p(f)f^{'}g^{'}\left(\frac{f}{g}\right)^{\frac{1}{q}} du\,d\nu\right)^{\frac{1}{p}} \\
&\times\left(\int_0^\infty \int_0^\infty K(f,g)\psi^q(g)f^{'}g^{'}\left(\frac{f}{g}\right)^{\frac{1}{p}} du\,d\nu\right)^{\frac{1}{q}} \\
= &M^{\frac{1}{p}}N^{\frac{1}{q}}, say.
\end{aligned}$$

First, we consider

$$Q = \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty \left(\frac{f}{g}\right)^{-\frac{1}{p}} K(f,g)f^{'}\,du.$$

By putting $f/g = h$, we have

$$\begin{aligned}
Q &= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}K(gh,g)gh^{'}\,du \\
&= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}g^{-1}K(h,1)gh^{'}\,du \\
&= \int_0^\infty \psi^q(g)g^{'}\,d\nu \int_0^\infty h^{-\frac{1}{p}}K(h,1)h^{'}\,du \\
&= k\|\psi\|_q.
\end{aligned}$$

$$\begin{aligned} P &= \int_0^\infty \phi^p(f) f^{'}\, du \int_0^\infty \left(\frac{f}{g}\right)^{\frac{1}{q}} K(f,g) g^{'}\, d\nu \\ &= \int_0^\infty \phi^p(f) f^{'}\, du \int_0^\infty h^{\frac{1}{q}} K(gh,g) f h^{-2} h^{'}\, d\nu \\ &= \int_0^\infty \phi^p(f) f^{'}\, du \int_0^\infty h^{\frac{1}{q}} g^{-1} K(h,1) g h^{-1}\, d\nu \\ &= \int_0^\infty \phi^p(f) f^{'}\, du \int_0^\infty h^{-\frac{1}{p}} K(h,1) h^{'}\, d\nu \\ &= k\|\phi\|_p. \end{aligned}$$

Concerning the other part, by virtue of the first part, we have

$$\begin{aligned} &\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df\right)^p dg \\ = &\int_0^\infty \int_0^\infty K(f,g)\phi(f) \left(\int_0^\infty K(f,g)\phi(f)\right)^{p-1} df\, dg \\ < &\; k \left(\int_0^\infty \phi^p(f)\, df\right)^{1/p} \left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df\right)^{q(p-1)} dg\right)^{1/q} \\ = &\left(\int_0^\infty \phi^p(f)\, df\right)^{1/p} \left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df\right)^{p} dg\right)^{1/q} \end{aligned}$$

which implies

$$\left(\int_0^\infty \left(\int_0^\infty K(f,g)\phi(f)\, df\right)^p dg\right)^{1/q} < k \left(\int_0^\infty \phi^p(f)\, df\right)^{1/p}$$

This completes the proof of the theorem. ∎

Corollary 2.2 *Let $p>1$, $\alpha+1/p>0$,$f\geq 0$,$f(0)=0$, $f(\infty)=\infty$, $f^{p-p\alpha-2}\phi^p(f)$ is $P(0,\infty)$ Φ is the Laplace transform of ϕ, then*

$$\|h^\alpha \Phi(h)\|_p < \Gamma(\alpha+1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p. \tag{7}$$

PROOF In Theorem 2.1 , by putting

$$K(f,g) = e^{-f/g} f^{\alpha-1+2/p} g^{-\alpha-2/p}$$

and replacing $\phi(f)$ by $f^{-\alpha+1-2/p}\phi(f)$, we obtain

$$\|g^{-\alpha-2/p}\Phi(1/g)\|_p = \Gamma(\alpha+1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p,$$

then writing $g = 1/h$, to get

$$\|h^{\alpha}\Phi(h)\|_p < \Gamma(\alpha+1/p)\|f^{1-\alpha-2/p}\phi(f)\|_p.$$

Now we are coming to the main result :

THEOREM 2.3 *Let* $\sum_{i=1}^{n} 1/p_i = 1$, $p_i > 1$, $\lambda > 1 + 1/p_i$, $f_i \geq 0$, $f_i(0) = 0$, $f_i(\infty) = \infty$, $f^{p_i-\lambda-1}\phi_i^{p_i}(f)$ *are in* $P_i(0,\infty)$, $(i = 1, 2, \ldots, n)$. *Then we have*

$$(8) \quad \int_0^{\infty} \cdots \int_0^{\infty} \frac{\phi_1(f_1)\ldots\phi_n(f_n}{(f_1+\cdots+f_n)^{\lambda}} df_1 \ldots df_n < \prod_{i=1}^{n} \frac{\Gamma(\lambda-1-1/p_i)}{\Gamma\lambda} \left\| f^{1-\frac{\lambda+1}{p_i}} \phi_i(f) \right\|_{p_i}.$$

PROOF We first consider

$$\begin{aligned}
& \int_0^{\infty} f^{\alpha}\phi_1(f)\ldots\phi_n(f)\, df \\
= & \int_0^{\infty} f^{\alpha} \int_0^{\infty} e^{-ff_1}\Phi_1(f_1)\, df_1 \ldots \int_0^{\infty} e^{-ff_n}\Phi_n(f_n)\, df_n\, df \\
= & \int_0^{\infty} \cdots \int_0^{\infty} \phi_1(f_1)\ldots\phi_n(f_n)\, df_1 \ldots df_n \int_0^{\infty} f^{\alpha} e^{-f(f_1+\cdots+f_n)}\, df \\
= & \int_0^{\infty} \cdots \int_0^{\infty} \phi_1(f_1)\ldots\phi_n(f_n)\, df_1 \ldots df_n \int_0^{\infty} [f(f_1+\cdots+f_n)]^{\alpha} \\
& \times e^{-f(f_1+\cdots+f_n)}(f_1+\cdots+f_n)\, df \\
= & \Gamma(\alpha+1) \int_0^{\infty} \cdots \int_0^{\infty} \frac{\phi_1(f_1)\ldots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\alpha+1}}\, df_1 \ldots df_n.
\end{aligned}$$

Therefore, by virtue of Corollary 2.2, we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\dots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\alpha+1}}\,df_1\dots df_n \\
= \;& \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f^\alpha \Phi_1(f)\dots\Phi_n(f)\,df \\
= \;& \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f^{\alpha/p_1}\Phi_1(f)\dots f^{\alpha/p_n}\Phi_n(f)\,df \\
\le \;& \frac{1}{\Gamma(\alpha+1)} \left\| f^{\alpha/p_1}\Phi_1(f)\right\| \times\cdots\times \left\| f^{\alpha/p_n}\Phi_n(f)\right\| \\
= \;& \frac{1}{\Gamma(\alpha+1)} \prod_{i=1}^n \left\| f^{\alpha/p_i}\Phi_i(f)\right\|_{p_i} \\
< \;& \frac{1}{\Gamma(\alpha+1)} \prod_{i=1}^n \frac{\Gamma(\alpha+1/p_i)}{\Gamma(\lambda)} \left\| f^{1-\frac{\lambda+1}{p_i}}\Phi_i(f)\right\|_{p_i}
\end{aligned}$$

consequently , we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{\phi_1(f_1)\dots\phi_n(f_n)}{(f_1+\cdots+f_n)^{\lambda}}\,df_1\dots df_n \\
< \;& \frac{1}{\Gamma(\alpha+1)} \prod_{i=1}^n \frac{\Gamma(\lambda-1-1/p_i)}{\Gamma(\lambda)} \left\| f^{1-\frac{\lambda+1}{p_i}}\phi_i(f)\right\|_{p_i}.
\end{aligned}$$

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COROLLARY 2.4 *On putting* $f_i(x)=x$, $\phi_i=f_i$, $i=1,\dots,n$ *in Theorem* 2.3 *we obtain*

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)\dots f_n(x_n)}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1\dots dx_n \\
< \;& \frac{1}{\Gamma(\alpha+1)} \prod_{i=1}^n \frac{\Gamma(\lambda-1-1/p_i)}{\Gamma(\lambda)} \left\| x^{1-\frac{\lambda+1}{p_i}} f_i(x)\right\|_{p_i}.
\end{aligned}$$

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