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A characterization of $(t_1, ..., t_n)$-Wright affine functions

Abstract. *In 1998 K.Lajkó* [5] *gave a characterization of t−Wright affine functions. Now, we extend this result to* $(t_1, ..., t_n)$*-Wright affine functions of an arbitrary order.*

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1. A characterization of $(t_1, ..., t_n)$-Wright affine functions defined on a whole space. Let $t_1, ..., t_n$, $n \geq 2$, be fixed positive numbers. Throughout the whole paper X will always stand for a linear space over the field $L(t_1, ..., t_n)$ generated by $t_1, ..., t_n$ (i.e. the smallest field containing the set $\{t_1, ..., t_n\}$) and Y will denote a commutative group. The difference operator $\Delta_h \colon Y^X \to Y^X$ with the span $h \in X$ (where X and Y are commutative groups) is defined by the equality

$$\Delta_h f(x) := f(x + h) - f(x), \qquad x \in X.$$

The superposition of several difference opertors $\Delta_{h_1}, ..., \Delta_{h_p}$, $p \in \mathbb{N}$ (the set of all positive integers), will be denote shortly by $\Delta_{h_1,...,h_p} := \Delta_{h_1} \circ ... \circ \Delta_{h_p}$. If $h_1 = h_2 = ... = h_p = h$ then we will write Δ_h^p instead of $\Delta_{h,...,h}$. It is known that the operator $\Delta_{h_1,...,h_p}$ is symmetric with respect to the spans $h_1, ..., h_p$ and, moreover, it is linear i.e. $\Delta_h(\alpha f + \beta g) = \alpha \Delta_h f + \beta \Delta_h g$, for all $f, g \in Y^X$ and $\alpha, \beta \in \mathbb{Z}$ (the set of all integers). A function $f \colon X \to Y$ is called polynomial function of $k - th$ order, $k \in \mathbb{N}$, if

$$\Delta_h^{k+1} f(x) = 0,$$

for every $x, h \in X$.

The main result concerning polynomial functions reads as follows.

THEOREM 1.1 ([1],[4]) *Let Y be a commutative group satisfying the condition: for each $y \in Y$ the equation $(k!)x = y$ has unique solution $x = \frac{y}{(k!)}$, and let $f\colon X \to Y$ be a function. Then f is polynomial function of $k-th$ order, $k \in \mathbb{N}$, whenever there exist i-additive and symmetric functions $a_i\colon X^i \to Y$, $i = 1, 2, ..., k$, and a constant $a_0 \in Y$ such that*

$$f(x) = a_0 + \sum_{i=1}^{k} a_i(x^i), \qquad x \in X,$$

where $a(x^i)$ denotes the diagonalization of a_i i.e. $a_i(x^i) = a_i\underbrace{(x, ..., x)}_{i-times}$, $i = 1, 2, ..., k$.

Let us first quote the following results.

THEOREM 1.2 ([1],[4]) *Let $f\colon X \to Y$ be an arbitrary function. Then for every $k \in \mathbb{N}$ and $j = 1, ..., k$ we have*

$$\Delta_{h_1,...,h_k} f(x) = \sum_{\varepsilon_1,...,\varepsilon_k=0}^{1} (-1)^{k-(\varepsilon_1+...+\varepsilon_k)} f(x + \varepsilon_1 h_1 + ... + \varepsilon_k h_k), \quad x, h_j \in X.$$

THEOREM 1.3 ([1],[4]) *Let $a_i\colon X^i \to Y$, $i \geq 1$, be an i-additive and symmetric function. Then*

$$\Delta_{h_1,...,h_p} a_i(x^i) = \begin{cases} 0, & p > i \\ (i!)a_i(h_1, ..., h_i), & p = i \end{cases}$$

Let $a_k\colon X^k \to Y$ be a k-additive and symmetric function. In the case when $k_1, ..., k_m$ are positive integer such that $k_1 + ... + k_m = k$ and $z_1, ..., z_m \in X$ we will write $a_k(z_1^{k_1}, ..., z_m^{k_m})$ instead of $a_k(\underbrace{z_1, ..., z_1}_{k_1-times}, ..., \underbrace{z_m, ..., z_m}_{k_m-times})$.

A nice survey of results and references on polynomial functions may be found in M.Kuczma's monograph [4].

We will start with the following Lemma.

LEMMA 1.4 *Let $a_k\colon X^k \to Y$ be a k-additive and symmetric function, $k \geq m \geq 1$. Then for arbitrary $\lambda_1, ..., \lambda_m \in L(t_1, ..., t_n)$ and for all $x, z \in X$ the following equality*

$$\Delta_{\lambda_1 z,...,\lambda_m z} a_k(x^k) =$$

$$= \sum_{j_1=1}^{k} ... \sum_{j_m=1}^{k-\sum_{p=1}^{m-1} j_p} \binom{k}{j_1} ... \binom{k-\sum_{p=1}^{m-1} j_p}{j_m} a_k(x^{k-\sum_{p=1}^{m} j_p}, (\lambda_m z)^{j_m}, ..., (\lambda_1 z)^{j_1})$$

holds true.

Proof Induction with respect to m. For $m = 1$ we have (cf.[4], p.393)

$$\Delta_{\lambda_1 z} a_k(x^k) = a_k((x+\lambda_1 z)^k) - a_k(x^k) = \sum_{j_1=1}^{k} \binom{k}{j_1} a_k(x^{k-j_1}, (\lambda_1 z)^{j_1}).$$

Assume our assertion for a positive integer $m < k$ and take arbitrary $x, z \in X$ and $\lambda_1, ..., \lambda_{m+1} \in L(t_1, ..., t_n)$. Then
$\Delta_{\lambda_1 z,...,\lambda_{m+1} z} a_k(x^k) =$
$\sum_{j_1=1}^{k} \cdots \sum_{j_m=1}^{k-\sum_{p=1}^{m-1} j_p} \binom{k}{j_1} ... \binom{k-\sum_{p=1}^{m-1} j_p}{j_m} \Delta_{\lambda_{m+1} z} a_k(x^{k-\sum_{p=1}^{m} j_p}, (\lambda_m z)^{j_m}, ..., (\lambda_1 z)^{j_1})$
$= \sum_{j_1=1}^{k} \cdots \sum_{j_m=1}^{k-\sum_{p=1}^{m-1} j_p} \binom{k}{j_1} ... \binom{k-\sum_{p=1}^{m-1} j_p}{j_m} \sum_{j_{m+1}}^{k-\sum_{p=1}^{m} j_p} \binom{k-\sum_{p=1}^{m} j_p}{j_{m+1}}$
$a_k(x^{k-\sum_{p=1}^{m+1} j_p}, (\lambda_{m+1} z)^{j_{m+1}}, ..., (\lambda_1 z)^{j_1})$ This ends the proof of Lemma 1.4. ∎

As an immediate consequence of Lemma 1.4 we obtain the following Corollary.

Corollary 1.5 *Let $a_k \colon X^k \to Y$, $k \geq n$, $k \in \mathbb{N}$, be a k-additive and symmetric function. Let $t_1, ..., t_n$ be fixed positive numbers and let $z \in X$ be arbitrary fixed. Then for every $x \in X$ we have*

$$\Delta_{t_1 z,...,t_n z} a_k(x^k) = \begin{cases} s_k a_k(x^{k-n}, t_n z, ..., t_1 z) + b_{k-n-1}(x, z), & k > n \\ n! a_n(t_n z, ..., t_1 z), & k = n \end{cases}$$

where $s_k = \binom{k}{1} ... \binom{k-n+1}{1}$ is a constant, and b_{k-n-1} is a polynomial function of variable x at most $(k-n-1)$-th order.

Let us introduce the following notations. Put

$$A := \{1, 2, ..., 2^n - 1\}$$

and let $A_1, ..., A_n$ be subsets of A with the properties:

(i) $r, s \in \mathbb{N},\ 1 \leq r < s \leq n \ \Rightarrow \bigwedge_{\alpha \in A_r} \bigwedge_{\beta \in A_s} (\alpha < \beta)$,
(ii) $\bigcup_{j=1}^{n} A_j = A$,
(iii) $card(A_j) = \binom{n}{j},\ j = 1, ..., n$.

For every $k \in \{1, ..., n\}$ we define B_k as the set of all sequences $(\delta_1, ..., \delta_n),\ \delta_i \in$

$\{0,1\}$, $i = 1, ..., n$, such that $\delta_1 + ... + \delta_n = k$, i.e. $B_k = \{(\delta_1, ..., \delta_n) \in \{0,1\}^n;\ \sum_{i=1}^n \delta_i = k\}$. Note that the cardinality of the sets A_k and B_k are equal, $k = 1, ..., n$. Let ϕ_k be a fixed bijection of B_k onto A_k, $k = 1, ..., n$.

Now we are able to present our second Lemma.

LEMMA 1.6 *For arbitrary function $f\colon X \to Y$ we have*

$$\Delta_z^{2^n-1} f(x) =$$

$$\sum_{\varepsilon_1,...,\varepsilon_{2^n-1}=0}^{1} (-1)^{2^n+n-1-(\varepsilon_1+...+\varepsilon_{2^n-1})} \Delta_{t_1 y_{\varepsilon_1,...,\varepsilon_{2^n-1}},...,t_n y_{\varepsilon_1,...,\varepsilon_{2^n-1}}} f(x+(\varepsilon_1+...+\varepsilon_{2^n-1})z),$$

where

$$y_{\varepsilon_1,...,\varepsilon_{2^n-1}} := -\sum_{k=1}^{n} \sum_{(\delta_1,...,\delta_n)\in B_k} \frac{\varepsilon_{\phi_k((\delta_1,...,\delta_n))}}{\delta_1 t_1 + ... + \delta_n t_n} z.$$

PROOF According to Theorem 1.2 we get

$$\begin{aligned} P &:= \textstyle\sum_{\varepsilon_1,...,\varepsilon_{2^n-1}=0}^{1} (-1)^{2^n+n-1-(\varepsilon_1+...+\varepsilon_{2^n-1})} \Delta_{t_1 y_{\varepsilon_1,...,\varepsilon_{2^n-1}},...,t_n y_{\varepsilon_1,...,\varepsilon_{2^n-1}}} f(x+ \\ &+ (\varepsilon_1 + ... + \varepsilon_{2^n-1})z) = \\ &= \textstyle\sum_{\varepsilon_1,...,\varepsilon_{2^n-1}=0}^{1} (-1)^{2^n+n-1-(\varepsilon_1+...+\varepsilon_{2^n-1})} \sum_{\beta_1,...,\beta_n=0}^{1} (-1)^{n-(\beta_1+...+\beta_n)} f(x+ \\ &+ (\varepsilon_1 + ... + \varepsilon_{2^n-1})z + (\beta_1 t_1 + ... + \beta_n t_n) y_{\varepsilon_1,...,\varepsilon_{2^n-1}}) = \\ &= \textstyle\sum_{\beta_1,...,\beta_n=0}^{1} (-1)^{\beta_1+...+\beta_n} \sum_{\varepsilon_1,...,\varepsilon_{2^n-1}=0}^{1} (-1)^{2^n-1-(\varepsilon_1+...+\varepsilon_{2^n-1})} f\big(x+ \\ &+ (\varepsilon_1 + ... + \varepsilon_{2^n-1})z + (\beta_1 t_1 + ... + \beta_n t_n) y_{\varepsilon_1,...,\varepsilon_{2^n-1}}\big) \end{aligned}$$

For arbitrary fixed $\beta_1, ..., \beta_n \in \{0,1\}$ we can rewrite the argument of f in the following way

$$x + (\varepsilon_1 + ... + \varepsilon_{2^n-1})z + (\beta_1 t_1 + ... + \beta_n t_n) y_{\varepsilon_1,...,\varepsilon_{2^n-1}} =$$

$$= x + (\varepsilon_1 + ... + \varepsilon_{2^n-1})z - \sum_{k=1}^{n} \sum_{(\delta_1,...,\delta_n)\in B_k} \varepsilon_{\phi_k((\delta_1,...,\delta_n))} \Big(\frac{\sum_{i=1}^n \beta_i t_i}{\sum_{i=1}^n \delta_i t_i}\Big) z =$$

$$= x + \sum_{k=1}^{n} \sum_{(\delta_1,...,\delta_n)\in B_k} \varepsilon_{\phi_k((\delta_1,...,\delta_n))} \Big(1 - \frac{\sum_{i=1}^n \beta_i t_i}{\sum_{i=1}^n \delta_i t_i}\Big) z.$$

This yields us to the equality

$$P = \sum_{\beta_1,...,\beta n=0}^{1} (-1)^{\beta_1+...+\beta_n} \sum_{\varepsilon_1,...,\varepsilon_{2^n-1}=0}^{1} (-1)^{2^n-1-(\varepsilon_1+...+\varepsilon_{2^n-1})} f(x+$$

$$+\sum_{k=1}^{n}\sum_{(\delta_1,\ldots\delta_n)\in B_k}\varepsilon_{\phi_k((\delta_1,\ldots,\delta_n))}(1-\frac{\sum_{i=1}^{n}\beta_i t_i}{\sum_{i=1}^{n}\delta_i t_i})z) =$$

$$=\sum_{\beta_1,\ldots,\beta_n=0}^{1}(-1)^{\beta_1+\ldots+\beta_n}\Delta_{h_1,\ldots,h_{2^n-1}}f(x),$$

where $\{h_1,...,h_{2^n-1}\}=\{(1-\frac{\sum_{i=1}^{n}\beta_i t_i}{\sum_{i=1}^{n}\delta_i t_i})z: \quad (\delta_1,...,\delta_n)\in\bigcup_{k=1}^{n}B_k\}$.

If $\sum_{i=1}^{n}\beta_i t_i\neq 0$ (which is equivalent that $(\beta_1,...,\beta_n)\neq(0,...,0)$), then there exists a $k\in\{1,...,n\}$ and a sequence $(\delta_1,...,\delta_n)\in B_k$ such that $\sum_{i=1}^{n}\beta_i t_i=\sum_{i=1}^{n}\delta_i t_i$. Therefore there exists a $j\in\{1,...,2^{n-1}\}$ such that $h_j=0$. In such a case $\Delta_{h_1,\ldots,h_{2^n-1}}f(x)=0$. Then (cf.Theorem 1.2) we have

$$\begin{aligned}P &= \sum_{\varepsilon_1,\ldots,\varepsilon_{2^n-1}=0}^{1}(-1)^{2^n-1-(\varepsilon_1+\ldots+\varepsilon_{2^n-1})}f(x+\sum_{k=1}^{n}\sum_{(\delta_1,\ldots,\delta_n)\in B_k}\varepsilon_{\phi_k((\delta_1,\ldots,\delta_n))}z)\\ &= \Delta_z^{2^n-1}f(x).\end{aligned}$$

The proof of Lemma 1.6 is finished. ∎

Following [3], where the definition of $(t_1,...,t_n)$-Wright convex function is given, we introduce the following definition.

DEFINITION 1.7 A function $f\colon X\to Y$ is said to be $(t_1,...,t_n)$-Wright affine if the equality

$$\Delta_{t_1 z,\ldots,t_n z}f(x)=0, \tag{1}$$

holds true for all $x,z\in X$.

Our main result reads as follows.

THEOREM 1.8 *If the target group Y is divisible, then a function $f\colon X\to Y$ is a $(t_1,...,t_n)$-Wright affine if and only if there are i-additive and symmetric functions $a_i\colon X^i\to Y$, $i=1,...,2^n-2$ and a constant $a_0\in Y$ such that*

$$f(x)=a_0+\sum_{i=1}^{2^n-2}a_i(x^i), \qquad x\in X \tag{2}$$

and

$$a_{2^n-2}(x^{2^n-2-n},t_1 z,...,t_n z)=0, \qquad x,z\in X. \tag{3}$$

PROOF By the definition of $(t_1, ..., t_n)$-Wright affinity of f, Lemma 1.6 and Theorem 1.1 we obtain the representation (2). Fix a $z \in X$ and take an arbitrary $x \in X$. Using again the definition of $(t_1, ..., t_n)$-Wright affinity and Theorem 1.3 we obtain

$$0 = \sum_{k=n}^{2^n-2} \Delta_{t_1 z,...,t_n z} a_k(x^k).$$

Hence and on account of Corollary 1.5 we get

$$0 = \sum_{k=n}^{2^n-2} [s_k a_k(x^{k-n}, t_n z, ..., t_1 z) + b_{k-n-1}(x, z)] = \tag{4}$$

$$= s_{2^n-2} a_{2^n-2}(x^{2^n-2-n}, t_n z, ..., t_1 z) + B_{2^n-3-n}(x, z),$$

for each $x \in X$ where s_{2^n-2} is a constant and B_{2^n-3-n} is a polynomial function of variable x at most $(2^n - 3 - n)$-th order. Therefore

$$-s_{2^n-2} a_{2^n-2}(x^{2^n-2-n}, t_n z, ..., t_1 z) = B_{2^n-3-n}(x, z) \qquad x \in X.$$

The left-hand side of the above equality is a monomial of variable x of $(2^n - 2 - n)$-th order then also the right-hand side has the same form. Howewer, because Y is a divisible group and the right-hand side is a polynomial at most $(2^n - 3 - n)$-th order, whence the both sides are equal to zero. Therefore

$$a_{2^n-2}(x^{2^n-2-n}, t_n z, ..., t_1 z) = 0, \qquad x \in X. \tag{5}$$

Since $z \in X$ was arbitrary chosen the condition (3) is a consequense of (5), and the proof of the Theorem 1.8 is completed. ∎

REMARK 1.9 The representation (2) in the above Theorem can be also obtained using the methods presented in L.Székelyhidi's book [7] concerning linear functional equations. The author believes that the proof given here (Lemma 1.6) may present an interest of its own.

2. A characterization of $(t_1, ..., t_n)$-Wright-affine functions defined on a convex domain . We recall that a subset $D \subset X$ is called $L(t_1, ..., t_n)$-convex if with any two points $x, y \in D$ the segment $\{\alpha x + (1-\alpha)y : \alpha \in L(t_1, ..., t_n) \cap (0, 1)\}$ is contained in D.

A point $x_0 \in X$ is termed to be algebraically internal for a set $A \subset X$ if

$$\bigwedge_{y \in X \setminus \{0\}} \bigvee_{\varepsilon_y > 0} \quad x_0 + \lambda y \in A, \qquad \lambda \in (-\varepsilon_y, \varepsilon_y) \cap L(t_1, ..., t_n).$$

The set of all algebraically internal points of A will be denoted by $algint_{L(t_1,...,t_n)} A$.

DEFINITION 2.1 Let $D \subset X$ be a $L(t_1, ..., t_n)$-convex set. A function $f\colon D \to Y$ is said to be $(t_1, ..., t_n)$-Wright affine if the equality

$$\Delta_{t_1 z, ..., t_n z} f(x) = 0 \tag{6}$$

holds true for all $(x, z) \in D \times X$ such that $x + (t_1 z + ... + t_n z) \in D$.

By (6) and Theorem 1.2 it follows that

$$\sum_{(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n} (-1)^{n-(\varepsilon_1 + ... + \varepsilon_n)} f(x + (\varepsilon_1 t_1 + ... + \varepsilon_n t_n) z) = 0, \tag{7}$$

for all $x \in D$ and $z \in X$ such that $x + (t_1 + ... + t_n) z \in D$, or equivalently, by putting $z := \frac{y-x}{t_1 + ... + t_n}$

(8)
$$\sum_{(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n} (-1)^{n-(\varepsilon_1 + ... + \varepsilon_n)} f\Big((1 - \frac{\varepsilon_1 t_1 + ... + \varepsilon_n t_n}{t_1 + ... + t_n}) x + \frac{\varepsilon_1 t_1 + ... + \varepsilon_n t_n}{t_1 + ... + t_n} y \Big) = 0,$$

for all $x, y \in D$.

By (8) we see that the natural domain of $(t_1, ..., t_n)$-Wright affine function are $L(t_1, ..., t_n)$-convex sets. Then the following question arises: has any solution $f\colon D \to Y$ of equation (8), where D is $L(t_1, ..., t_n)$-convex subset of X, the form (2), (3) with the functions $a_k\colon X^k \to Y$, $k \in \{1, ..., n\}$ restricted to D ? We will show that under some additional conditions the answer is yes.

The idea of proofs of all results in the present section is inspired by methods contained in [2].
Now, for an arbitrary set $D \subset X$ and $T := (t_1, ..., t_n)$, where $t_1, ..., t_n \in (0, \infty)$ are fixed number, we define the set

$$H_T(D) := \tag{9}$$

$$= \{x \in X \ : \bigvee_{y \in D} (1 - \alpha_{\varepsilon_1, ..., \varepsilon_n}) x + \alpha_{\varepsilon_1, ..., \varepsilon_n} y \in D \ , \ (\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n \setminus \{(0, ..., 0)\}\}$$

where $\alpha_{\varepsilon_1, ..., \varepsilon_n} = \frac{\varepsilon_1 t_1 + ... + \varepsilon_n t_n}{t_1 + ... + t_n}$.

LEMMA 2.2 *Let $D \subset X$ be a nonempty $L(t_1, ..., t_n)$-convex set. Then the set $H_T(D)$ defined by (9) is $L(t_1, ..., t_n)$-convex. More precisely, for any $x, y \in H_T(D)$ and $u, v \in D$ such that for all $(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n$*

$$(1 - \alpha_{\varepsilon_1, ..., \varepsilon_n}) x + \alpha_{\varepsilon_1, ..., \varepsilon_n} u \in D \quad \text{and} \quad (1 - \alpha_{\varepsilon_1, ..., \varepsilon_n}) y + \alpha_{\varepsilon_1, ..., \varepsilon_n} v \in D \ ,$$

we have

$$\alpha_{\varepsilon_1,...,\varepsilon_n}\big(\lambda u + (1-\lambda)v\big) + (1-\alpha_{\varepsilon_1,...,\varepsilon_n})\big(\lambda x + (1-\lambda)y\big) \in D$$

for all $(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n$ *and* $\lambda \in (0,1) \cap L(t_1, ..., t_n)$. *Moreover, the inclusion* $D \subset H_T(D)$ *holds true.*

The proof of the above Lemma is obvious.

LEMMA 2.3 *Let* $D \subset X$ *be a nonempty* $L(t_1, ..., t_n)$*-convex set. Then for every* $(t_1, ..., t_n)$*- Wright affine function* $f\colon D \to Y$ *there exists exactly one* $(t_1, ..., t_n)$*-Wright affine function* $F\colon H_T(D) \to Y$ *such that* $F\,|_D = f$.

PROOF Fix an $x \in H_T(D)$. There exists a $y \in D$ such that

$$s_{\varepsilon_1,...,\varepsilon_n} := (1-\alpha_{\varepsilon_1,...,\varepsilon_n})x + \alpha_{\varepsilon_1,...,\varepsilon_n} y \in D\,, \quad \text{for all } (\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n \backslash \{(0, ..., 0)\}.$$

Put

$$F(x) := \sum_{\substack{(\varepsilon_1,...,\varepsilon_n)\in\{0,1\}^n \\ \varepsilon_1+...+\varepsilon_n \neq 0}} (-1)^{\varepsilon_1+...+\varepsilon_n - 1} f(s_{\varepsilon_1,...,\varepsilon_n}). \tag{10}$$

Using the same arguments as in [2] and Lemma 2.2 it is easy to check that the function F given by the formula (10) is well-defined $(t_1, ..., t_n)$-Wright affine function such that $F\,|_D = f$ and, moreover, this extension is unique. ∎

Let us introduce the following class of sets

$$A_T(X) := \{S \subset X \ : \ S \text{ is } L(t_1, ..., t_n)\text{-convex and } \bigcup_{k=1}^{\infty} H_T^k(S) = X \ \}$$

where H_T^k denotes the $k-th$ iterate of the operation H_T defined by (9).

With the aid of Lemma 2.3 (used step by step) using Theorem 1.8 one can show that the following theorem holds true.

THEOREM 2.4 *Let* Y *be a commutative, divisible group. A function* $f\colon D \to Y$ *is a* $(t_1, ..., t_n)$ *-Wright affine, where the set* $D \in A_T(X)$*, if and only if, there are i-additive and symmetric functions* $a_i\colon X^i \to Y$ *,*$i = 1, ..., 2^n - 2$ *and a constant* $a_0 \in Y$ *such that*

$$f(x) = a_0 + \sum_{i=1}^{2^n-2} a_i(x^i)\ , \quad x \in D,$$

and

$$a_{2^n-2}(x^{2^n-2-n}, t_n z, ..., t_1 z) = 0\ , \quad x, z \in X.$$

THEOREM 2.5 *If $D \subset X$ is a $L(t_1, ..., t_n)$-convex set such that $algint_{L(t_1,...,t_n)} D \neq \emptyset$ then $D \in A_T(X)$.*

PROOF Without loss of generality we may assume that $0 \in algint_{L(t_1,...,t_n)} D$ and D itself is symmetric with respect to zero. We will show that

$$\bigcup_{k=1}^{\infty} H_T^k(D) = X \ .$$

Let $p_D \colon X \to \mathbb{R}$ be the Minkowski's functional in linear space X over the field $L(t_1, ..., t_n)$ i.e. a function given by the formula

$$p_D(x) := \inf\{t \in (0, \infty) \cap L(t_1, ..., t_n) \ : \ \frac{x}{t} \in D\}.$$

It is easy to check that the ordered pair (X, p_D) yields a semi-normed space over the field $L(t_1, ..., t_n)$ and then the unit ball $B := \{x \in X \ : \ p_D(x) < 1 \ \}$ is contained in D.

First we will show that for every natural number k one has

$$B_k := \{x \in X \ : \ p_D(x) < 1 + ka \ \} \subset H_T^k(B), \tag{11}$$

where $a := \{\min \frac{t_i}{t_1+...+t_n} \ : \ 1 \ \leq i \leq n\}$.

Relation (11) is fulfilled for $k = 1$ since for any $x \in B_1$, $(\varepsilon_1, ..., \varepsilon_n) \in \{0, 1\}^n \setminus \{(0, ..., 0)\}$ and $x_{\varepsilon_1,...,\varepsilon_n} := (1 - \alpha_{\varepsilon_1,...,\varepsilon_n})x + \alpha_{\varepsilon_1,...,\varepsilon_n} 0$, one has

$$p_D(x_{\varepsilon_1,...,\varepsilon_n}) = (1-\alpha_{\varepsilon_1,...,\varepsilon_n})p_D(x) < (1-\alpha_{\varepsilon_1,...,\varepsilon_n})(1+a) \leq (1-a)(1+a) = 1-a^2 < 1 \ ,$$

i.e. $x_{\varepsilon_1,...,\varepsilon_n} \in B$, for all $(\varepsilon_1, ..., \varepsilon_n) \in \{0, 1\}^n \setminus \{(0, ..., 0)\}$ which implies that x belongs to $H_T(B) = H_T^1(B)$ because $0 \in B$.

Now, assume that (11) valid for some number $k \in \mathbb{N}$. By monotonicity of the operation H_T it suffices to show that

$$B_{k+1} \subset H_T(B_k).$$

For all $(\varepsilon_1, ..., \varepsilon_n) \in \{0, 1\}^n \setminus \{(0, ..., 0)\}$ we have

$$p_D(x_{\varepsilon_1,...,\varepsilon_n}) = (1 - \alpha_{\varepsilon_1,...,\varepsilon_n})p_D(x) < (1 - \alpha_{\varepsilon_1,...,\varepsilon_n})(1 + (k + 1)a) \leq$$

$$\leq (1 + a)(1 + (k + 1)a) = 1 + ka - (1 + k)a^2 < 1 + ka$$

and because $0 \in B_k$ then we obtain that $x \in H_T(B_k)$, which finishes the inductive proof of (11). Obviously

$$X = \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{k=1}^{\infty} H_T^k(B) \subset \bigcup_{k=1}^{\infty} H_T^k(D) \subset X \ .$$

and the proof of our Theorem is finished. ∎

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