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Arrangements of series preserving their convergence or boundedness

Abstract. For a map ρ of $\mathbb{N}$ into itself, consider the induced transformation $\sum_n x_n \rightsquigarrow \sum_n x_{\rho(n)}$ of series in a topological vector space. Then such properties of this transformation as sending convergent series to convergent series, or convergent series to bounded series, or bounded series to bounded series (and a few more) are mutually equivalent. Moreover, they are equivalent to an intrinsic property of ρ which reduces to those found by Agnew and Pleasants (in the case of permutations) and Wituła (in the general case) as necessary and sufficient conditions for the above transformation to preserve convergence of scalar series. In the paper, the scalar case is treated first using simple Banach space methods, and then the result is easily extended to the general setting.

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1. Introduction. Let ρ be an *arrangement* of $\mathbb{N} = \{1, 2, \dots\}$; that is, ρ is a map of $\mathbb{N}$ into itself. Then for every sequence (of scalars or vectors) $\boldsymbol{x} = (x_n)$, and the corresponding series $\Sigma(\boldsymbol{x}) := \sum_n x_n$, their *arrangements induced by* ρ are defined as the sequence $A_\rho(\boldsymbol{x}) = \boldsymbol{x} \circ \rho = (x_{\rho(n)})$, and the series $\Sigma_\rho(\boldsymbol{x}) = \sum_n x_{\rho(n)}$. (When ρ is a *permutation*, i.e. one-to-one and onto, the term *rearrangement* is most often used.) Consider the following properties that the transformation $\Sigma(\boldsymbol{x}) \rightsquigarrow \Sigma_\rho(\boldsymbol{x})$ may or may not have:

(bb) whenever the series $\Sigma(\boldsymbol{x})$ is bounded, then the series $\Sigma_\rho(\boldsymbol{x})$ is bounded;

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(cb) whenever the series $\Sigma(\boldsymbol{x})$ is convergent, then the series $\Sigma_\rho(\boldsymbol{x})$ is bounded;

(cc) whenever the series $\Sigma(\boldsymbol{x})$ is convergent, then the series $\Sigma_\rho(\boldsymbol{x})$ is convergent.

Here, a series is called *bounded* if the sequence of its partial sums is bounded.

A number of authors have been interested in characterizing *permutations* ρ that *preserve convergence* of scalar series; that is, those for which (cc) is valid for scalar series. Let us only mention here Agnew [1] and Pleasants [5], and refer the reader to [8] for a thorough survey and discussion of relevant results, and to [10] for additional information. The characterization obtained in [1] and [5] is particularly natural: *A permutation ρ preserves convergence (and sums) of scalar series iff*

(ap) *there exists an integer k such that for each n the set $\rho(\{1,\dots,n\})$ is the union of at most k disjoint intervals in $\mathbb{N}$.*

The insistance on scalar series is not too important here. In fact, in a remark made at the end of [5] it was noted that all of the above remains valid for series in *arbitrary* topological vector spaces.

By inspecting the proof of Lemma 1 in [5] it can be seen that condition (ap) is both sufficient and necessary not only for (cc) (and a fortiori (cb)), but also for (bb) to hold. Indeed, the argument showing that (ap) implies (bb) is similar to (and even simpler than) that (ap) implies (cc). On the other hand, if (ap) fails, then in this case the proof in [5] produces a convergent series $\Sigma(\boldsymbol{x})$ for which the series $\Sigma_\rho(\boldsymbol{x})$ is not only divergent, but also unbounded. Hence (cb), the weakest of conditions (bb), (cb), and (cc), implies (ap).

In particular, these observations provide an affirmative answer to the following conjecture of T. Šalát concerning series of reals, mentioned in [4]: *If ρ is a permutation, then* (cb) *implies* (cc)*, and both the series in* (cc) *have the same sum.*

An extension of the above characterization to the case of *arrangements* was given by Wituła [10]: *An arrangement ρ preserves convergence of scalar series iff*

(w) *there exist integers q, r such that, for every n, the set $\{1,\dots,n\}$ admits a partition $D_1,\dots,D_p$ with $p \leqslant q$ so that the restrictions $\rho|D_j$ are one-to-one, and each $\rho(D_j)$ is the union of at most r disjoint intervals in $\mathbb{N}$.*

This does not look as transparent as (ap): Indeed, while it is easy to visualize the sort of 'behavior' that a map ρ ought to have in order to satisfy condition (ap), it is certainly not so with condition (w). Thus a more clear version of (w) is desirable.

Similarly as above, an analysis of the arguments in [10] reveals that (w) is sufficient and necessary for an arrangement ρ to have any one of the properties (bb), (cb), or (cc).

In this paper, it is shown how one can apply elementary methods of functional analysis to cope with problems like those mentioned above. First, in Theorem 4.2, we use this approach to prove that, for scalar series and an arbitrary arrangement ρ, properties (bb), (cb) and (cc) are mutually equivalent, and also equivalent to an intrinsic property (m) of ρ. What is interesting here is that the latter comes up in a completely natural way (so that no guess work is needed) and simply reflects boundedness of the involved operators. Moreover, although we prefered to give a

self-contained proof, all of Theorem 4.2 can be deduced from the classic Silverman-Toeplitz theorem once it is observed that the partial sums of 'output' series $\Sigma_\rho(\boldsymbol{x})$ are obtained from those for 'input' series $\Sigma(\boldsymbol{x})$ by applying a matrix transformation (see Remark 4.3). Now, property (m) is analytic in character but, as we show in Proposition 4.6, it can also be restated in a number of ways as a combinatorial type condition on ρ. In particular, (m) can be rephrased as (ap) when ρ is a permutation, or as (w) in the general case. Also, some of the reformulations of (m) provide more transparent replacements for condition (w).

We next give in Theorem 5.1 an extension of Theorem 4.2 to series in *arbitrary* topological vector spaces. In fact, the list of mutually equivalent properties in Theorem 5.1 is even longer and includes the variants of (cb) and (cc) with 'c' (convergent) replaced by 'C' (Cauchy). The reason that such an extension of Theorem 4.2, at first sight somewhat unexpected, does indeed hold can be understood from the following line of arguments: Each of the properties listed in Theorem 5.1, when 'localized' to a one-dimensional subspace, implies one of the equivalent properties for scalar series in Theorem 4.2. These, in turn, are equivalent to (m). Finally (and fortunately), the latter implies, in a rather straightforward manner, each of the properties considered in Theorem 5.1.

In Remarks 4.9 and 5.2 we point out that analogs of Theorems 4.2 and 5.1 are also valid for transformations of series induced by finite 'linear combinations' of arrangements.

We finish with Theorem 7.2 which extends Theorem 4.2 (and its proof) to more general transformations $\sum_n x_n \rightsquigarrow \sum_n T_{\rho(n)}(x_{\rho(n)})$ of series in F-spaces.

Basic notions and standard facts from functional analysis will be used freely; the reader may consult e.g. [2] or [3] whenever necessary. Throughout, we let the abbreviation TVS stand for *Hausdorff topological vector space*(*s*).

2. Spaces of bounded or convergent vector series. Let X be a TVS. We denote by $\omega(X)$ the TVS of all sequences $\boldsymbol{x} = (x_n)$ in X (with the coordinatewise convergence topology), by $l_\infty(X)$ the TVS of all bounded sequences in X (with the topology of uniform convergence on $\mathbb{N}$), and by $c(X)$ the subspace of $l_\infty(X)$ consisting of convergent sequences. Clearly, the operators $s^n : \omega(X) \to X$ defined by

$$s^n(\boldsymbol{x}) := \sum_{i=1}^{n} x_i$$

are linear and continuous. Moreover, the operator

$$\Sigma = (s^n) : \boldsymbol{x} \to \big(s^n(\boldsymbol{x})\big)$$

is a topological automorphism of $\omega(X)$. (It can be viewed as the 'natural' isomorphism between formal series and sequences.) Next, we introduce

$$bs(X) := \Sigma^{-1}\big(l_\infty(X)\big) \quad \text{and} \quad cs(X) := \Sigma^{-1}\big(c(X)\big),$$

the vector spaces consisting of all sequences $\boldsymbol{x}$ in X such that the series $\Sigma(\boldsymbol{x}) := \sum_n x_n$ is bounded or convergent, respectively.

Whenever $bs(X)$ (or any of its subspaces) is treated as a TVS, it is meant that $bs(X)$ is equipped with the topology for which the operator Σ is a topological isomorphism from $bs(X)$ onto $l_\infty(X)$. In other words, it is the weakest vector topology such that the operators $s^k : bs(X) \to X$ $(k \in \mathbb{N})$ are equicontinuous. Obviously, the inclusion $bs(X) \subset \omega(X)$ is continuous, and so is the linear operator

$$s : cs(X) \to X, \quad \text{where} \quad s(\boldsymbol{x}) := \text{the sum of the series } \Sigma(\boldsymbol{x}).$$

Note that when X is sequentially complete, so are $l_\infty(X)$ and $bs(X)$; moreover, $c(X)$ is a closed subspace of $l_\infty(X)$, and $cs(X)$ is a closed subspace of $bs(X)$.

Given a sequence $\mathfrak{X} = (X_n)$ of closed subspaces of X, we write $\omega(\mathfrak{X})$ for the closed subspace of $\omega(X)$ consisting of sequences $\boldsymbol{x}$ such that $x_n \in X_n$ for every n. Furthermore, we let

$$bs(\mathfrak{X}) := bs(X) \cap \omega(\mathfrak{X}) \quad \text{and} \quad cs(\mathfrak{X}) := cs(X) \cap \omega(\mathfrak{X}),$$

and note that these are closed subspaces of $bs(X)$ and $cs(X)$, respectively. Also, the finitely nonzero sequences $(x_1, \dots, x_k, 0, 0, \dots)$ form a dense subspace of $cs(\mathfrak{X})$.

If X is an F-space (a complete metrizable TVS) and its topology is defined by an F-norm $\|\cdot\|$, then also $l_\infty(X)$ and $bs(X)$ are F-spaces, and their topologies are determined by the F-norms $\|\boldsymbol{x}\| = \sup_n \|x_n\|$ and $\|\boldsymbol{x}\| = \sup_n \|s^n(\boldsymbol{x})\|$, respectively. In this case the operator Σ is an isometry from $bs(X)$ onto $l_\infty(X)$. Moreover, the subspaces $cs(X)$, $bs(\mathfrak{X})$ and $cs(\mathfrak{X})$ (for $\mathfrak{X}$ as above) are closed in $bs(X)$, hence they are F-spaces in the induced topologies. Of course, if $X = (X, \|\cdot\|)$ is a Banach space, so are $l_\infty(X)$, $c(X)$, $bs(X)$, and $cs(X)$.

PROPOSITION 2.1 *Let X and Z be* TVS*, and let $T_n : Z \to X$ $(n \in \mathbb{N})$ be continuous linear operators. Then the following are equivalent.*

(a) *The operator $T = (T_n) : z \to \big(T_n(z)\big)$ maps Z into $bs(X)$ and is continuous.*

(b) *The operators $s^n \circ T : Z \to X; z \to \sum_{i=1}^n T_i(z)$ $(n \in \mathbb{N})$ are equicontinuous.*

(c) *The operator $\Sigma \circ T$ maps Z into $l_\infty(X)$ and is continuous.*

Moreover, if Z is an F-space, then these conditions are also equivalent to the following.

(a') *The operator T maps Z into $bs(X)$.*

(b') *The operators $s^n \circ T : Z \to X$ $(n \in \mathbb{N})$ are pointwise bounded on Z.*

(c') *The operator $\Sigma \circ T$ maps Z into $l_\infty(X)$.*

PROOF The equivalence of conditions in each of the above groups is obvious, and (b) is equivalent to (b') when Z is an F-space by the Banach-Steinhaus theorem. ∎

If X is the space of (real or complex) scalars, we denote the spaces considered above by ω, l_∞, c, bs, and cs. (See [2], Sec. IV.2, and Sec. IV.13, Exerc. 11–14.)

We will also need bv, the Banach space of all sequences $\boldsymbol{y} = (y_n) \in \omega$ having finite *variation*

$$v(\boldsymbol{y}) := \sum_{n=1}^{\infty} |y_n - y_{n+1}|,$$

with the norm (equivalent to that in [2, p. 239])

$$\|\boldsymbol{y}\| := v(\boldsymbol{y}) + \lim_n |y_n| = \lim_n v^n(\boldsymbol{y}) = \sup_n v^n(\boldsymbol{y}),$$

where

$$v^n(\boldsymbol{y}) := \sum_{i=1}^{n-1} |y_i - y_{i+1}| + |y_n|,$$

and its closed subspace bv_0 consisting of those $\boldsymbol{y}$ with $\lim_n y_n = 0$. Note the following

FACT 2.2 *The sequence (e_n) of unit vectors is a Schauder basis of both cs and bv_0 and, for each n, the operator $\boldsymbol{x} \to \sum_{i=1}^n x_i e_i$ is a norm one projection from bs (resp., bv) onto the subspace $[e_1, \dots, e_n]$ of cs (resp., bv_0).*

For $\boldsymbol{x}, \boldsymbol{y} \in \omega$, let $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_n := \sum_{i=1}^n x_i y_i$, and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{\infty} x_i y_i$ if the series converges. Since

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_n = \sum_{i=1}^{n-1} (y_i - y_{i+1}) s^i(\boldsymbol{x}) + y_n s^n(\boldsymbol{x}),$$

whence

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle_n| \leqslant \sup_{i \leqslant n} |s^i(\boldsymbol{x})| \cdot v^n(\boldsymbol{y}),$$

it is easily seen that the functional $\langle \cdot, \boldsymbol{y} \rangle_n$ on cs (or bs) has norm $v^n(\boldsymbol{y})$, and the functional $\langle \boldsymbol{x}, \cdot \rangle_n$ on bv (or bv_0) has norm $\sup_{i \leqslant n} |s^i(\boldsymbol{x})|$. Moreover, from the equality for $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_n$ above it follows that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ exists if either $\boldsymbol{x} \in cs$ and $\boldsymbol{y} \in bv$, or $\boldsymbol{x} \in bs$ and $\boldsymbol{y} \in bv_0$ (Abel's and Dirichlet's convergence tests). Combining these observations with the fact stated above, one easily arrives at the following (well known) result.

PROPOSITION 2.3 *The map $\boldsymbol{y} \to \langle \cdot, \boldsymbol{y} \rangle$ is a linear isometry between bv and $(cs)^*$, and a linear isometric embedding of bv_0 into $(bs)^*$.*

The map $\boldsymbol{x} \to \langle \boldsymbol{x}, \cdot \rangle$ is a linear isometry between bs and $(bv_0)^*$.

3. Operators associated with arrangements. Again, let X be a TVS. Also. let ρ be an arrangement of $\mathbb{N}$. It induces linear operators

$$A_\rho : \omega(X) \to \omega(X); \boldsymbol{x} \to \boldsymbol{x} \circ \rho = (x_{\rho(n)}),$$

$$s^n_\rho := s^n \circ A_\rho : \omega(X) \to X; \boldsymbol{x} \to \sum_{i=1}^{n} x_{\rho(i)};$$

$$\Sigma_\rho := \Sigma \circ A_\rho = (s^n_\rho) : \omega(X) \to \omega(X).$$

We denote

$$s_\rho(\boldsymbol{x}) = \text{the sum of the series } \Sigma_\rho(\boldsymbol{x}) = \sum_n x_{\rho(n)} \text{ if it converges.}$$

Next, we define the *multiplicity function* m_ρ and the *partial multiplicity functions* m_ρ^n for ρ by

$$m_\rho(k) = |\rho^{-1}(\{k\})| \qquad \text{and} \qquad m_\rho^n(k) = |\,[1,n] \cap \rho^{-1}(\{k\})| \quad \text{for } n,k \in \mathbb{N},$$

where $[1,n] := \{1,\dots,n\}$. Evidently,

$$m_\rho^n(k) = 0 \quad \text{for } k \notin \rho([1,n]), \qquad \text{and} \qquad m_\rho^n(k) \nearrow m_\rho(k) \quad \text{as } n \to \infty.$$

It will be important for what follows to observe that

$$\begin{aligned} s_\rho^n(\boldsymbol{x}) &= \sum_{k\in\rho([1,n])} m_\rho^n(k)x_k = \sum_{k=1}^{\infty} m_\rho^n(k)x_k \\ &= \sum_{k=1}^{\infty} \big(m_\rho^n(k) - m_\rho^n(k+1)\big) s^k(\boldsymbol{x}). \end{aligned}$$

Hence, if we introduce linear operators

$$b_\rho^n : \omega(X) \to X \quad \text{and} \quad B_\rho := (b_\rho^n) : \omega(X) \to \omega(X)$$

by setting

$$b_\rho^n(\boldsymbol{x}) = \sum_{k=1}^{\infty} \big(m_\rho^n(k) - m_\rho^n(k+1)\big) x_k,$$

then

$$s_\rho^n = b_\rho^n \circ \Sigma, \quad \Sigma_\rho = B_\rho \circ \Sigma, \quad \text{and} \quad B_\rho = \Sigma \circ A_\rho \circ \Sigma^{-1}.$$

PROPOSITION 3.1 *Let X be an F-space, Z a closed subspace of $bs(X)$, and ρ an arrangement of $\mathbb{N}$. If A_ρ maps Z into $bs(X)$ or, equivalently, Σ_ρ maps Z into $l_\infty(X)$ or, equivalently, B_ρ maps the subspace $\Sigma(Z) \subset l_\infty(X)$ into $l_\infty(X)$, then each of these operators is continuous* [*and all have the same norm when X is a Banach space*].

PROOF This follows from Proposition 2.1 applied to $T_n(\boldsymbol{x}) := x_{\rho(n)}$, and the assertion about norms is obvious because $\Sigma : bs(X) \to l_\infty(X)$ is an isometry.

An alternative way to see that e.g. A_ρ is continuous: It is so when both Z and $bs(X)$ are considered with the topology of coordinatewise convergence. Hence the graph of A_ρ is closed for the original topologies of these two spaces. Apply the closed graph theorem. ∎

4. Arrangements of scalar series. Given an arrangement ρ of $\mathbb{N}$, let

$$v_n(\rho) := \sum_{k=1}^{\infty} |m_\rho^n(k) - m_\rho^n(k+1)| \quad \text{and} \quad v(\rho) := \sup_n v_n(\rho).$$

The lemma below follows easily from Propositions 2.3 and 3.1. Alternatively, instead of Proposition 2.3 use the relation $s_\rho^n = b_\rho^n \circ \Sigma$ and the elementary fact that the norm of any functional $\boldsymbol{x} \to \sum_{i=1}^n a_i x_i$ on both c and l_∞ is equal to $\sum_{i=1}^n |a_i|$.

LEMMA 4.1 *Let ρ be an arrangement of $\mathbb{N}$. Then, for every n, the linear functionals s_ρ^n on cs and bs, and b_ρ^n on c and l_∞ have the same norm, $v_n(\rho)$. In consequence, the operator A_ρ maps cs or bs into bs (and is continuous) iff the operator B_ρ maps c or l_∞ into l_∞ (and is continuous) iff $v(\rho) < \infty$, and in each case $\|A_\rho\| = \|B_\rho\| = v(\rho)$.*

THEOREM 4.2 *Let ρ be an arrangement of $\mathbb{N}$. Then for the operator $A_\rho : \omega \to \omega$ the following properties*

(bb) *A_ρ maps bs into bs,*

(cb) *A_ρ maps cs into bs,*

(cc) *A_ρ maps cs into cs*

are mutually equivalent, and also equivalent to the condition

(m) *$v(\rho) < \infty$.*

Moreover, in each of conditions (bb), (cb), *and* (cc) *the operator A_ρ is continuous and its norm is equal to $v(\rho)$. Also, if the above conditions hold, then*

$$s_\rho(\boldsymbol{x}) = \sum_{k=1}^{\infty} m_\rho(k) x_k \qquad \text{for every } \boldsymbol{x} \in cs,$$

PROOF In view of Proposition 3.1, the continuity of the operator A_ρ in (bb), (cb), and (cc) is obvious. Trivially, (bb) $\Longrightarrow$ (cb).

(cb) $\Longrightarrow$ (cc): Take any unit vector e_k in cs. Then $A_\rho(e_k)$ is in bs and its terms are 0's and 1's only. Hence $m_\rho(k) < \infty$ and $A_\rho(e_k) \in cs$. Since the linear span of the e_k's is dense in cs, and A_ρ is continuous, we see that $A_\rho(cs) \subset cs$. Now, the sequence (e_k) is in fact a Schauder basis of cs, and $s_\rho = s \circ A_\rho$ is a continuous linear functional on cs. Hence if $\boldsymbol{x} \in cs$, then

$$s_\rho(\boldsymbol{x}) = s_\rho\Big(\sum_k x_k e_k\Big) = \sum_k s_\rho(e_k) x_k = \sum_k m_\rho(k) x_k.$$

(cc) $\Longrightarrow$ (bb): Since $A_\rho : cs \to cs$ is continuous, the functionals s_ρ^n are uniformly bounded on cs. Take any $\boldsymbol{x} \in bs$. Then the sequence $\boldsymbol{x}_n := (x_1, \dots, x_n, 0, 0, \dots)$ is bounded in cs. In consequence, $M := \sup_{k,n} |s_\rho^k(\boldsymbol{x}_n)| < \infty$. But, for each k, we have $s_\rho^k(\boldsymbol{x}) = s_\rho^k(\boldsymbol{x}_n)$ for large n. Therefore, $|s_\rho^k(\boldsymbol{x})| \leqslant M$ for all k, and so $A_\rho(\boldsymbol{x}) \in bs$.

To complete the proof apply Lemma 4.1. ∎

REMARK 4.3 The operator B_ρ can be viewed as the transformation defined by the (row finite) matrix (b_{nk}), where $b_{nk} = m_\rho^n(k) - m_\rho^n(k+1)$. Having observed this, it is not hard to deduce Theorem 4.2 from the Silverman-Toeplitz theorem (see [2, Sec. II.4, Exerc. 36], [9]). Alternatively, one can apply directly to the transformation $\Sigma(\boldsymbol{x}) \rightsquigarrow \Sigma_\rho(\boldsymbol{x})$ the series-to-series version of the Silverman-Toeplitz theorem of Vermes (see [8, Sec. 10], [2, Sec. II.4, Exerc. 45]). For this approach, see [6].

Most of the rest of this section is devoted to a discussion of condition (m), culminating in Proposition 4.6.

In what follows, by an *interval* we mean an interval in $\mathbb{N}$, and use notation like $[m,n]$ or (m,n) for intervals in this sense. By a *component* of a finite set $F \subset \mathbb{N}$ we understand any maximal interval contained in F, and write (cf. [5])

$$\kappa(F) = \text{ the number of components of } F.$$

Next, given a map $\varphi : N \to \mathbb{N}$, where $N \subset \mathbb{N}$, we define

$$\kappa(\varphi) = \sup_n \kappa\big(\varphi([1,n] \cap N)\big).$$

We will appeal to the following lemma in the proof of Proposition 4.6.

LEMMA 4.4 *Let (F_n) be a nondecreasing sequence of finite subsets of $\mathbb{N}$ such that $k := \sup_n \kappa(F_n) < \infty$. Then the union G of the F_n's is either finite, or else has a finite complement in $\mathbb{N}$.*

In consequence, if a map $\varphi : N \to \mathbb{N}$, where $N \subset \mathbb{N}$, has an infinite range and $\kappa(\varphi) < \infty$, then $\varphi(N)$ has a finite complement in $\mathbb{N}$.

PROOF Suppose that both G and $H := \mathbb{N} \setminus G$ are infinite. Then we can find elements $h_1, \dots, h_{k+1}$ in H and indices $n_1, \dots, n_k$ so that $h_j < \max F_{n_j} < h_{j+1}$ for $j = 1, \dots, k$.

Take any $n \geqslant n_k$. Then from the above it follows that $F_n \cap (h_j, h_{j+1}) \neq \emptyset$ for $j = 1, \dots, k$ and that $h_1, \dots, h_{k+1} \notin F_n$. Hence $\kappa(F_n) = k$. Let $I_1, \dots, I_k$ be the components of F_n arranged so that $\max I_j < \min I_{j+1}$ for $1 \leqslant j < k$. Then $I_j \subset (h_j, h_{j+1})$ for $j = 1, \dots, k$. In consequence, $F_n \subset [1, h_{k+1}]$, which is impossible because $|F_n| \to \infty$. ∎

Let ρ be an arrangement of $\mathbb{N}$. In addition to the quantities $v_n(\rho)$ and $v(\rho)$ introduced at the beginning of this section, we now define a few more. First, let

$$m_n(\rho) := \sup_k m_\rho^n(k), \qquad m(\rho) := \sup_n m_n(\rho) = \sup_k m_\rho(k),$$
$$j_n(\rho) := |\{k : m_\rho^n(k) \neq m_\rho^n(k+1)\}|, \qquad j(\rho) := \sup_n j_n(\rho).$$

Next, given $n \in \mathbb{N}$ and a finite set $F \subset \mathbb{N}$, by an *m_ρ^n-component* of F let us mean any maximal interval contained in F on which m_ρ^n is constant, and let

$$k_n(\rho) := \text{ the number of } m_\rho^n\text{-components of } \rho([1,n]),$$
$$k(\rho) := \sup_n k_n(\rho).$$

Note that while any two distinct components of a set F are *nonadjacent* intervals (i.e., there is always a 'gap' in $\mathbb{N}$ between them), it is not necessary so for the m_ρ^n-components.

Clearly, if ρ is one-to-one, then the components and m_ρ^n-components of a set $F \subset \rho([1,n])$ coincide. Hence in this case $k_n(\rho) = \kappa\big(\rho([1,n])\big)$ and $k(\rho) = \kappa(\rho)$.

Basic relations between these quantities are given in the following.

LEMMA 4.5 *Let ρ be an arrangement of $\mathbb{N}$. Then for $n = 1, 2, \ldots$*

$$\max\{m_n(\rho), j_n(\rho)\} \leqslant v_n(\rho) \leqslant 2m_n(\rho)j_n(\rho) \quad \textit{and} \quad k_n(\rho) \leqslant j_n(\rho) \leqslant 2k_n(\rho),$$

whence

$$\max\{m(\rho), j(\rho)\} \leqslant v(\rho) \leqslant 2m(\rho)j(\rho) \quad \textit{and} \quad k(\rho) \leqslant j(\rho) \leqslant 2k(\rho).$$

PROOF We have for each k and n,

$$m_\rho^n(k) \leqslant \sum_{j=k}^{\infty} |m_\rho^n(j) - m_\rho^n(j+1)| \leqslant v_n(\rho),$$

whence $m_n(\rho) \leqslant v_n(\rho)$. Moreover, since the terms of the series defining $v_n(\rho)$ are integers, it is obvious that $j_n(\rho) \leqslant v_n(\rho)$. The other inequality, $v_n(\rho) \leqslant 2m_n(\rho)j_n(\rho)$, and the rest of the proof, are equally simple. ∎

PROPOSITION 4.6 *For an arrangement ρ of $\mathbb{N}$, the following conditions are equivalent*

(m) $v(\rho) < \infty$.

(m1) $m(\rho) < \infty$ *and* $j(\rho) < \infty$.

(m2) $m(\rho) < \infty$ *and* $k(\rho) < \infty$.

(m3) *There exists a finite partition $\mathbb{N} = N_1 \cup \cdots \cup N_p$ such that, for $j = 1, \ldots, p$, the restriction $\rho|N_j$ is one-to-one and $\kappa(\rho_j|N_j) < \infty$.*

(m4) *There exist sets $M, L \subset \mathbb{N}$ with finite complements in $\mathbb{N}$, and a partition $M_1, \ldots, M_s$ of M such that for $i = 1, \ldots, s$, ρ is a bijection from M_i onto L and $\kappa(\rho|M_i) < \infty$.*

(m5) *There exist integers q, r such that, for every n, the set $[1,n]$ admits a partition $D_1, \ldots, D_p$ with $p \leqslant q$ so that ρ is one-to-one over each D_j, and $\kappa\big(\rho(D_j)\big) \leqslant r$.*

Moreover, if $v(\rho) < \infty$, then $\sum_{k=1}^{\infty} |m_\rho(k) - m_\rho(k+1)| \leqslant v(\rho)$ and, consequently, m_ρ is eventually constant: there exist integers k_0 and m_0 such that $m_\rho(k) = m_0$ for $k > k_0$.

PROOF The mutual equivalence of conditions (m), (m1), and (m2) follows from the relations between $v(\rho)$, $j(\rho)$, and $k(\rho)$ stated in Lemma 4.5.

(m1) $\Longrightarrow$ (m3): For $j = 1, \dots, p := m(\rho)$ let N_j be the set of those $i \in \mathbb{N}$ such that

$$|\{r \in \mathbb{N} : r \leqslant i \ \&\ \rho(r) = \rho(i)\}| = j.$$

Clearly, $\rho|N_j$ is one-to-one. Fix n and let I be a component of $\rho([1,n] \cap N_j)$. Denote by k the right end-point of I. Since $k \in \rho([1,n] \cap N_j)$, there are at least j integers $r \leqslant n$ such that $\rho(r) = k$. Hence $m_\rho^n(k) \geqslant j$. On the other hand, as $(k+1) \notin \rho([1,n] \cap N_j)$, there are at most $(j-1)$ integers $r \leqslant n$ such that $\rho(r) = k+1$. Hence $m_\rho^n(k+1) \leqslant j-1$. It follows that $m_\rho^n(k) \neq m_\rho^n(k+1)$. Thus $\kappa\big(\rho([1,n] \cap N_j)\big) \leqslant j_n(\rho) \leqslant j(\rho)$.

(m3) $\Longrightarrow$ (m4): Assume (m3) and denote by I the set of those indices j for which the set N_j is infinite. By relabelling it can be assumed that $I = \{1, \dots, s\}$. By Lemma 4.4, $\mathbb{N} \setminus \rho(N_i)$ is finite for each $i \in I$. Then the intersection L of the sets $\rho(N_i)$ for $i \in I$, and the sets $M_i := \rho_i^{-1}(L)$ for $i \in I$ are as required in (m4).

(m4) $\Longrightarrow$ (m5): This is obvious.

(m5) $\Longrightarrow$ (m1): From (m5) it is clear that for any n and k one has $m_\rho^n(k) \leqslant q$. Hence $m(\rho) \leqslant q$. Now, fix n and let $D_1, \dots, D_p$ be a partition of $[1,n]$ provided by (m5). Let χ_j denote the characteristic function of the set $\rho(D_j)$. Then $m_\rho^n(k) = \chi_1(k) + \dots + \chi_p(k)$ for every k. If $m_\rho^n(k) \neq m_\rho^n(k+1)$, then $\chi_j(k) \neq \chi_j(k+1)$ for some j. The latter, for any fixed j, can happen at most $2r$ times. Hence $j_n(\rho) \leqslant p \cdot 2r \leqslant 2qr$.

The final assertion follows from the inequalities $v_n(\rho) \leqslant v(\rho)$ letting $n \to \infty$ and taking into account that $m_\rho^n(k) \to m_\rho(k) < \infty$, by (m1) or (m2). ∎

REMARK 4.7 Among the equivalent conditions above, also the following variant of (m2) could be included:

(m$\overline{2}$) *$m(\rho) < \infty$ and there is $r \in \mathbb{N}$ such that, for every interval $I \subset \mathbb{N}$, the number of maximal intervals contained in $\rho(I)$ on which the multiplicity function $m_\rho^I(k) := |I \cap \rho^{-1}(\{k\})|$ is constant does not exceed r.*

To justify this it should suffice to note that if $I = [l+1, n]$, then $m_\rho^I(k) = m_\rho^n(k) - m_\rho^l(k)$, whence $\sum_k |m_\rho^I(k) - m_\rho^I(k+1)| \leqslant 2v(\rho)$.

COROLLARY 4.8 *Let ρ be a permutation of $\mathbb{N}$. Then properties* (bb), (cb), *and* (cc) *as in Theorem* 1 *are mutually equivalent, and equivalent to the condition that $\kappa(\rho) < \infty$. Moreover, if these conditions are satisfied, then the norm of the operator A_ρ from bs (or cs) to bs is related to $\kappa(\rho)$ as follows:*

$$\|A_\rho\| = \begin{cases} 2\kappa(\rho) & \text{if } 1 \notin \rho([1,n]) \text{ and } \kappa\big(\rho([1,n])\big) = \kappa(\rho) \text{ for some } n, \\ 2\kappa(\rho) - 1 & \text{otherwise.} \end{cases}$$

PROOF The first part is a direct consequence of Theorem 4.2 and Proposition 4.6. As for the second part note that, as ρ is one-to-one, $v_n(\rho) = j_n(\rho)$ which in turn is equal to $2\kappa\big(\rho([1,n])\big) - 1$ or $2\kappa\big(\rho([1,n])\big)$ depending on whether or not 1 is in $\rho([1,n])$. From this using Lemma 4.1 the desired conclusion follows easily. ∎

REMARK 4.9 The method used in this section works also for transformations of the form

$$\sum_n x_n \rightsquigarrow \sum_n (a_1 x_{\rho_1(n)} + \cdots + a_s x_{\rho_s(n)}),$$

where $a_1, \ldots, a_s$ are scalars, and $\rho_1, \ldots, \rho_s$ are arrangements of $\mathbb{N}$. To see this denote the formal linear combination $a_1\rho_1 + \cdots + a_s\rho_s$ by $\boldsymbol{\rho}$, let

$$m_{\boldsymbol{\rho}}(k) := a_1 m_{\rho_1}(k) + \cdots + a_s m_{\rho_s}(k) \quad \text{and} \quad m^n_{\boldsymbol{\rho}}(k) := a_1 m^n_{\rho_1}(k) + \cdots + a_s m^n_{\rho_s}(k),$$

and define $v_n(\boldsymbol{\rho})$ and $v(\boldsymbol{\rho})$ formally as before. It is easy to verify that the sums $s^n_{\boldsymbol{\rho}}(\boldsymbol{x}) := \sum_{i=1}^n (a_1 x_{\rho_1(i)} + \cdots + a_s x_{\rho_s(i)})$ can be expressed in terms of the sums $s^k(\boldsymbol{x})$ similarly as before. Then by inspecting the proof of Theorem 4.2 it can be seen that the above transformation maps bounded (or convergent) series to bounded series iff (m) $v(\boldsymbol{\rho}) < \infty$, and that it maps convergent series to convergent series iff (m) $v(\boldsymbol{\rho}) < \infty$ and

(L) the (proper) limit $\lim_n m^n_{\boldsymbol{\rho}}(k) =: \mu_{\boldsymbol{\rho}}(k)$ exists for each k.

Moreover, when this is so and $\boldsymbol{x} \in cs$, then the sum of the 'output' series $\sum_n \big(\sum_j a_j x_{\rho_j(n)}\big)$ is equal to the sum of the series $\sum_k \mu_{\boldsymbol{\rho}}(k) x_k$. The additional condition (L) is satisfied automatically when each ρ_j is of finite multiplicity (i.e., $m_{\rho_j}(k) < \infty$ for all $k \in \mathbb{N}$). Also, if the a_j's are nonnegative, then (m) implies (L). In either case, $\mu_{\boldsymbol{\rho}}(k) = m_{\boldsymbol{\rho}}(k)$.

Furthermore, if the a_j's are integers, then using $|m^n_{\boldsymbol{\rho}}(k)|$ in the definition of $m_n(\boldsymbol{\rho})$, and keeping the definitions of the other quantities formally unchanged, it can readily be checked that the first three conditions of Proposition 4.6 are still equivalent in the present setting. Also, if both (m) and (L) are satisfied, then the $\mu_{\boldsymbol{\rho}}(k)$'s are eventually constant.

5. Arrangements of vector series. For a TVS X, in addition to the spaces $bs(X)$ and $cs(X)$ defined in Section 2, we now let $Cs(X)$ stand for the space of all sequences $\boldsymbol{x} \in \omega(X)$ such that the series $\Sigma(\boldsymbol{x}) = \sum_n x_n$ is Cauchy.

THEOREM 5.1 *Let T be a nonzero continuous linear operator from a* TVS *X to a* TVS *Y, ρ an arrangement of $\mathbb{N}$, and $A_\rho : \omega(X) \to \omega(X)$ the induced operator. Then for the operator*

$$A_{T,\rho} : \omega(X) \to \omega(Y); \boldsymbol{x} \to \big(T(x_{\rho(n)})\big)$$

the following conditions are mutually equivalent.

(m) $v(\rho) < \infty$.

(cc) *$A_{T,\rho}$ maps $cs(X)$ into $cs(X)$.*

(cC) *$A_{T,\rho}$ maps $cs(X)$ into $Cs(X)$.*

(cb) *$A_{T,\rho}$ maps $cs(X)$ into $bs(X)$.*

(CC) *$A_{T,\rho}$ maps $Cs(X)$ into $Cs(X)$.*

(Cb) *$A_{T,\rho}$ maps $Cs(X)$ into $bs(X)$.*

(bb) *$A_{T,\rho}$ maps $bs(X)$ into $bs(X)$.*

Moreover, if the above conditions hold, then

$$s_\rho(\boldsymbol{x}) = \sum_{k=1}^{\infty} m_\rho(k)x_k, \qquad \forall\, \boldsymbol{x} \in cs(X),$$

and the operator A_ρ maps continuously $bs(X)$ into itself.

Proof It is trivial that (cc) $\Longrightarrow$ (cC) $\Longrightarrow$ (cb), (bb) $\Longrightarrow$ (Cb) $\Longrightarrow$ (cb), and (CC) $\Longrightarrow$ (Cb).

(cb) $\Longrightarrow$ (m): Fix $x \in X$ with $y := T(x) \neq 0$. By (cb), for every sequence (a_n) of reals, whenever the series $\sum_n a_n x$ is convergent, then the series $\sum_n a_{\rho(n)} y$ is bounded. It follows that ρ satisfies condition (cb) of Theorem 4.2, whence also condition (m).

We now prove that (m) $\Longrightarrow$ (cc) and, at the same time, establish the final assertions of the theorem. Of course, we may assume here that $X = Y$ and that T is the identity operator. By Proposition 4.6, there exist integers p, k_0, m_0, and r such that

1) $m_\rho(k) \leqslant p$ for all k and $m_\rho(k) = m_0$ for $k > k_0$, and

2) for every n, the number of m_ρ^n-components of $\rho([1,n])$ does not exceed r.

Let $\boldsymbol{x} \in cs(X)$. Clearly, the series $\sum_k m_\rho(k)x_k$ converges in X; let z denote its sum. Take any neighborhood U of zero in X, and next choose a balanced neighborhood V of 0 in X so that $pV + \cdots + pV \subset U$ ($r+1$ summands). Then there exists $\bar{k} > k_0$ such that

$$\sum_{k=j}^{\infty} x_k \in V \quad \text{whenever } j \geqslant \bar{k}; \quad \sum_{k=j}^{l} x_k \in V \quad \text{whenever } l \geqslant j \geqslant \bar{k}.$$

Take any n so large that

$$m_\rho^n(k) = m_\rho(k) \quad \text{for } 1 \leqslant k < \bar{k},$$

and denote $F = \rho([1,n]) \setminus [1, \bar{k}-1]$. Let $I_1, \ldots, I_s$ be the maximal intervals contained in F on which m_ρ^n assumes constant values, say $\mu_1, \ldots, \mu_s$. Note that $s \leqslant r$. Now,

$$\begin{aligned} z - s_\rho^n(\boldsymbol{x}) &= m_0 \sum_{k=\bar{k}}^{\infty} x_k - \sum_{k \in F} m_\rho^n(k)x_k \\ &= m_0 \sum_{k=\bar{k}}^{\infty} x_k - \sum_{j=1}^{s} \mu_j \sum_{k \in I_j} x_k \in pV - \sum_{j=1}^{s} pV \subset U. \end{aligned}$$

This proves that $s_\rho(\boldsymbol{x}) = z$.

We now verify the continuity assertion. Pick any neighborhood U of zero in X, and next choose a symmetric neighborhood V of 0 in X so that $V + \cdots + V \subset U$ ($2r$ summands). Let $\boldsymbol{x} \in bs(X)$ be such that $s^k(\boldsymbol{x}) \in V$ for every k. Note that then $\sum_{i\in I} x_i \in V + V$ for every interval $I \subset \mathbb{N}$. Now, take any $n \in \mathbb{N}$, let $I_1, \ldots, I_s$ ($s \leqslant r$) be the m^n_ρ-components of $\rho([1, n])$, and let $m^n_\rho(I_j) = \{\mu_j\}$ for $j = 1, \ldots, s$. Then

$$s^n_\rho(\boldsymbol{x}) = \sum_{j=1}^{s} \mu_j \sum_{k\in I_s} x_i \in (V + V) + \cdots + (V + V) \subset U,$$

which completes the argument.

Finally, a verification of (m) $\Longrightarrow$ (bb) is similar to that above and, likewise, for (m) $\Longrightarrow$ (CC), using the variant of condition (m2) stated in Remark 4.7. ∎

Remark 5.2 In view of the observations made in Remark 4.9, it should be clear that Theorem 5.1 admits an extension to transformations of the form

$$\sum_n x_n \rightsquigarrow \sum_n T(a_1 x_{\rho_1(n)} + \cdots + a_s x_{\rho_s(n)}).$$

More precisely, for such transformations each of conditions (cb), (Cb), and (bb) is equivalent to condition (m), while each of conditions (cc), (cC), and (CC) is equivalent to the conjunction of conditions (m) and (L). Here conditions (m) and (L) are to be understood as in Remark 4.9.

6. Dual arrangements. If ρ is an arrangement of $\mathbb{N}$ such that $m_\rho(k) < \infty$ for each k, then we define the *dual arrangement operator* $D_\rho : \omega \to \omega$ by

$$(*) \qquad D_\rho(\boldsymbol{y}) = (z_k), \quad \text{where} \quad z_k = \sum_{i\in\rho^{-1}(\{k\})} y_i.$$

Note that if ρ is a permutation, then $D_\rho = A_{\rho^{-1}}$.

We are going to show now that the operators A_ρ and D_ρ, when considered on suitable Banach spaces, are indeed dual to each other. Below, we identify $(cs)^*$ with bv, and $(bv_0)^*$ with bs (see Proposition 2.3).

Proposition 6.1 *Let ρ be an arrangement of $\mathbb{N}$.*

(a) *If A_ρ maps cs into itself, then the dual operator $A^*_\rho : bv \to bv$ coincides with D_ρ on bv so that, in consequence, D_ρ maps bv into itself.*

(b) *If D_ρ maps bv_0 into itself, then the dual operator $D^*_\rho : bs \to bs$ coincides with A_ρ on bs so that, in consequence, A_ρ maps bs into itself.*

Proof (a): Let $\boldsymbol{y} \in bv = (cs)^*$ and take any unit vector e_k in cs. Then $\langle e_k, A^*_\rho(\boldsymbol{y})\rangle = \langle A_\rho(e_k), \boldsymbol{y}\rangle = z_k$ (z_k is given by (∗)). Hence $A^*_\rho(\boldsymbol{y}) = D_\rho(\boldsymbol{y})$ and $D_\rho(\boldsymbol{y}) \in bv$.

(b): By the closed graph theorem, the operator $D_\rho : bv_0 \to bv_0$ is continuous. Let $\boldsymbol{x} \in bs = (bv_0)^*$ and take any unit vector e_k in bv_0. Then $\langle D^*_\rho(\boldsymbol{x}), e_k\rangle = \langle \boldsymbol{x}, D_\rho(e_k)\rangle = \langle \boldsymbol{x}, e_{\rho(k)}\rangle = x_{\rho(k)}$. Hence $D^*_\rho(\boldsymbol{x}) = (x_{\rho(k)}) = A_\rho(\boldsymbol{x})$ and $A_\rho(\boldsymbol{x}) \in bs$. ∎

Note that this gives an alternative explanation of why properties (cc) and (bb) are equivalent. The special case of permutations below is due to Sarigol [7].

COROLLARY 6.2 *Let ρ be an arrangement of $\mathbb{N}$. Then D_ρ maps bv (or bv_0) into itself iff $v(\rho) < \infty$. Hence, if ρ is a permutation, then $A_{\rho^{-1}}$ maps bv (or bv_0) into itself iff $\kappa(\rho) < \infty$.*

We are now going to extend the second part of this corollary to a suitably defined vector version of the space bv. For this purpose, given a permutation ρ, we will need to translate the condition that $\kappa(\rho) < \infty$ into an explicit combinatorial condition for ρ^{-1}. The following result is known, see [8, Thm. 3 and Sec. 7]; we include a proof for the sake of completeness.

PROPOSITION 6.3 *For a permutation ρ the following are equivalent:*

(a) $\kappa(\rho) < \infty$.

(b) *The inverse permutation $\tau = \rho^{-1}$ has the following property: there is $M \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ there are at most M values $k \in \mathbb{N}$ for which $[j, j+1]$ is contained in the interval with the endpoints $\tau(k)$ and $\tau(k+1)$.*

PROOF (a) $\Longrightarrow$ (b): Suppose (b) is false. Then for every M we can find $j \in \mathbb{N}$ and $k_1, \dots, k_{2M} \in \mathbb{N}$ so that $k_i + 1 < k_{i+1}$ and $[j, j+1]$ is contained in the interval with the endpoints $\tau(k_i)$ and $\tau(k_i + 1)$. Let $A := \{i : \tau(k_i) \leqslant j\}$ and $B := \{i : \tau(k_i + 1) \leqslant j\}$. Then A and B form a partition of $[1, 2M]$, hence $|A| \geqslant M$ or $|B| \geqslant M$. Now, $k_i \in \rho([1, j])$ but $k_i + 1 \notin \rho([1, j])$ for $i \in A$, and $k_i + 1 \in \rho([1, j])$ but $k_i \notin \rho([1, j])$ for $i \in B$. It follows that $\kappa\big(\rho([1, j])\big) \geqslant M - 1$. In consequence, $\kappa(\rho) = \infty$; a contradiction.

(b) $\Longrightarrow$ (a): Take any $n \in \mathbb{N}$. Let $C_1, \dots, C_m$ be the components of $\rho([1, n])$ arranged from left to right, and let $k_i := \max C_i$. Then $\tau(k_i) \leqslant n < \tau(k_i + 1)]$ for each i. Hence, by (b), $\kappa\big(\rho([1, n])\big) = m \leqslant M$. In consequence, $\kappa(\rho) \leqslant M < \infty$. ∎

We now extend the definition of the space bv to the vector case. If X is a TVS, then we define $bv(X)$ as the vector space of all $\boldsymbol{x} = (x_n) \in \omega(X)$ such that the series $\sum_n (x_n - x_{n+1})$ is subseries convergent (hence unconditionall Cauchy) in X. Note that then for every bounded sequence (a_j) of integers also the series $\sum_j a_j(x_j - x_{j+1})$ is subseries convergent.

If $X = (X, \|\cdot\|)$ is an F-normed space, then we denote by $abv(X)$ the vector space of all $\boldsymbol{x} \in \omega(X)$ with $\sum_n \|x_n - x_{n+1}\| < \infty$. Note that $abv(X)$ depends on the choice of the F-norm $\|\cdot\|$.

THEOREM 6.4 *Let ρ be a permutation such that $\kappa(\rho) < \infty$ and denote $\tau = \rho^{-1}$. If X is a TVS, then A_τ maps $bv(X)$ into itself; and if X is an F-normed space, then A_τ maps also $abv(X)$ into itself.*

PROOF Let M be as in condition (b) of Proposition 6.3 and, for each $k \in \mathbb{N}$, let

$$l_k = \min\{\tau(k), \tau(k+1)\},\ r_k = \max\{\tau(k), \tau(k+1)\} - 1,\ \varepsilon_k = \operatorname{sign}\big(\tau(k+1) - \tau(k)\big).$$

We have to show that if $\boldsymbol{x} \in bv(X)$ and $K = \{k_1 < k_2 < \dots\}$ is an infinite subset of $\mathbb{N}$, then the series

$$(*) \qquad \sum_{k\in K}(x_{\tau(k)} - x_{\tau(k+1)})$$

converges in X. In fact, denoting

$$J = \bigcup_{k\in K}[l_k, r_k], \quad K_j = \{k \in K : l_k \leqslant j \leqslant r_k\}, \quad c_j = \sum_{k\in K_j}\varepsilon_k \quad (j \in \mathbb{N}),$$

we will show that the series $(*)$ has the same sum as the series

$$(**) \qquad \sum_{j=1}^{\infty} c_j(x_j - x_{j+1}) \equiv \sum_{j\in J} c_j(x_j - x_{j+1});$$

the latter series converges because the c_j's are integers and $|c_j| \leqslant |K_j| \leqslant M$.

In order to prove this, for $j, n \in \mathbb{N}$ let

$$K(n) = \{k_1, \dots, k_n\}, \quad J(n) = \bigcup_{k\in K(n)}[l_k, r_k],$$
$$K_j(n) = K_j \cap K(n), \quad c_j(n) = \sum_{k\in K_j(n)}\varepsilon_k.$$

Then the nth partial sum of the series $(*)$ is

$$y_n := \sum_{k\in K(n)}(x_{\tau(k)} - x_{\tau(k+1)}) = \sum_{k\in K(n)}\varepsilon_k\sum_{j=l_k}^{r_k}(x_j - x_{j+1})$$
$$= \sum_{j\in J(n)}\sum_{k\in K_j(n)}\varepsilon_k(x_j - x_{j+1}) = \sum_{j\in J(n)} c_j(n)(x_j - x_{j+1})$$

Recall that the series $(**)$ is subseries convergent and denote by z its sum. Also, denote

$$I(n) = \{j \in J(n) : c_j(n) = c_j\} \quad \text{and} \quad z_n = \sum_{j\in I(n)} c_j(x_j - x_{j+1}).$$

Now, for each j we have $c_j(n) = c_j$ for sufficiently large n. Hence $I(n) \nearrow J$ and, in consequence, $z_n \to z$. We claim that also $y_n \to z$, or that

$$y_n - z_n = \sum_{j\in H(n)} c_j(n)(x_j - x_{j+1}) \to 0,$$

where $H(n) := J(n) \setminus I(n)$. Note that $\min H(n) \to \infty$ as $n \to \infty$.

To verify this take any neighborhood U of zero in X, and next a balanced neighborhood V of zero in X with $M(V+\dots+V) \subset U$ ($2M$ summands V). Since the series $\sum_j(x_j - x_{j+1})$ is unconditionally Cauchy, there is j_0 such that $\sum_{j\in F}(x_j - x_{j+1}) \in V$

whenever F is a finite subset of $\mathbb{N}$ with $\min F > j_0$. Next, choose $n_0 \in \mathbb{N}$ so that $\min H(n) > j_0$ for $n > n_0$. Now, fix any $n > n_0$ and for each integer r with $0 < |r| \leqslant M$ denote $F_r = \{j \in H(n) : c_j(n) = r\}$. Then

$$y_n - z_n = \sum_{0<|r|\leqslant M} r \sum_{j\in F_r} (x_j - x_{j+1}) \in \sum_{0<|r|\leqslant M} rV,$$

and the latter set is contained in $M(V + \cdots + V) \subset U$ ($2M$ summands V). Thus indeed $y_n - z_n \to 0$.

The proof of the other assertion is similar (and simpler). ∎

Remark 6.5 By a similar argument, or by applying Theorem 6.4 in the completion of X, one shows that if $\boldsymbol{x} \in \omega(X)$ and the series $\sum_n (x_n - x_{n+1})$ is unconditionally Cauchy, so is the series $\sum_k (x_{\tau(k)} - x_{\tau(k+1)})$.

7. Arrangements of series followed by operators. In this section, we show under fairly general conditions that a transformation of the type

$$\sum_n x_n \rightsquigarrow \sum_n T_{\rho(n)}(x_{\rho(n)})$$

preserves convergence of series iff it preserves boundedness of series.

In what follows,

$$\text{(H)} = \begin{cases} X & \text{is an } F\text{-space, and } Y \text{ a sequentially complete TVS;} \\ \mathfrak{X} = (X_n) & \text{is a sequence of closed subspaces of } X; \\ T_n : X_n \to Y \ (n = 1, 2, \dots) & \text{are continuous linear operators; and} \\ \rho & \text{is an arrangement of } \mathbb{N}. \end{cases}$$

In this setting, letting $\mathcal{T} = (T_n)$, we define a linear operator $A_{\mathcal{T},\rho} : \omega(\mathfrak{X}) \to \omega(Y)$ by

$$A_{\mathcal{T},\rho}(\boldsymbol{x}) = \big(T_{\rho(n)}(x_{\rho(n)})\big).$$

Applying the Banach-Steinhaus theorem, one shows the following generalization of Proposition 3.1 (in which Y need not be sequentially complete).

Proposition 7.1 *Under assumptions* (H)*, if Z is a closed subspace of $bs(\mathfrak{X})$ and $A_{\mathcal{T},\rho}$ acts from Z to $bs(Y)$, then $A_{\mathcal{T},\rho} : Z \to bs(Y)$ is a continuous linear operator. In other words, the operators $S_n : Z \to Y$ defined by*

$$S_n(\boldsymbol{x}) = \sum_{i=1}^{n} T_{\rho(i)}(x_{\rho(i)})$$

are equicontinuous.

THEOREM 7.2 *Under assumptions* (H)*, the following are equivalent for the operator* $A_{\mathcal{T},\rho}$ *defined above.*

(bb) $A_{\mathcal{T},\rho}$ *maps* $bs(\mathfrak{X})$ *into* $bs(Y)$.

(cb) $A_{\mathcal{T},\rho}$ *maps* $cs(\mathfrak{X})$ *into* $bs(Y)$.

(cc) $A_{\mathcal{T},\rho}$ *maps* $cs(\mathfrak{X})$ *into* $cs(Y)$.

Moreover, in each of conditions (bb), (cb), *and* (cc), *the operator* $A_{\mathcal{T},\rho}$ *is continuous. Also, if the above conditions hold, then* $m_\rho(k) < \infty$ *whenever* $T_k \neq 0$, *and for each* $\boldsymbol{x} \in cs(\mathfrak{X})$ *the sum of the series* $\sum_n T_{\rho(n)}(x_{\rho(n)})$ *is equal to the sum of the series* $\sum_k m_\rho(k)T_k(x_k)$, *where the summation is over all* k*'s with* $m_\rho(k) < \infty$.

PROOF In view of Proposition 7.1, the continuity of the operator $A_{\mathcal{T},\rho}$ in (bb), (cb), and (cc) is obvious. Trivially, (bb) $\Longrightarrow$ (cb).

(cb) $\Longrightarrow$ (cc): Fix any $k \in \mathbb{N}$ and $x_k \in X_k$, and denote $\hat{x}_k = (0, \dots, 0, x_k, 0, \dots)$, where x_k stands in the kth slot. If $k \notin \rho(\mathbb{N})$ or $T_k = 0$, then $S_n(\hat{x}_k) = 0$ for all n. If $k \in \rho(\mathbb{N})$ and $T_k \neq 0$, then

$$S_n(\hat{x}_k) = m_\rho^n(k)T_k(x_k) \qquad \text{for } n = 1, 2, \dots$$

Since this is a bounded sequence, choosing x_k so that $T_k(x_k) \neq 0$ shows that $m_\rho(k) < \infty$. Therefore,

$$S_n(\hat{x}_k) = m_\rho(k)T_k(x_k) \qquad \text{for all } n \geqslant \max \rho^{-1}(\{k\}).$$

Thus $A_{\mathcal{T},\rho}(\hat{x}_k) \in cs(Y)$ for each of the cases considered. Now, the linear span of the $\hat{x}_k$'s is dense in $cs(\mathfrak{X})$, and $cs(Y)$ is sequentially closed in $bs(Y)$ (because Y is sequentially complete), so $A_{\mathcal{T},\rho}$ maps all of $cs(\mathfrak{X})$ into $cs(Y)$.

We continue this part of the proof by verifying the last assertion of the theorem. Let

$$\sigma(\boldsymbol{x}) = \sum_n T_{\rho(n)}(x_{\rho(n)}) = \lim_n S_n(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in cs(\mathfrak{X}).$$

By what was shown above, $\sigma(\hat{x}_k)$ equals $m_\rho(k)T_k(x_k)$ if $m_\rho(k) < \infty$, and 0 otherwise. Moreover, $\sigma = s \circ A_{\mathcal{T},\rho} : cs(\mathfrak{X}) \to Y$ is a continuous linear operator. Hence, denoting $K = \{k : m_\rho(k) < \infty\}$, we have for each $\boldsymbol{x} = (x_k) \in cs(\mathfrak{X})$,

$$\sigma(\boldsymbol{x}) = \sigma\Big(\sum_{k=1}^{\infty} \hat{x}_k\Big) = \sum_{k=1}^{\infty} \sigma(\hat{x}_k) = \sum_{k \in K} m_\rho(k)T_k(x_k),$$

because $\boldsymbol{x} = \sum_k \hat{x}_k$ (convergence in $cs(\mathfrak{X})$).

(cc) $\Longrightarrow$ (bb): The operators S_n on $cs(\mathfrak{X})$ are equicontinuous, hence also uniformly bounded. That is, whenever B is a bounded subset of $cs(\mathfrak{X})$, so is the union of the sets $S_n(B)$ in Y. Now, let $\boldsymbol{x} = (x_i) \in bs(\mathfrak{X})$. Then the sequence $\boldsymbol{x}_i := (x_1, \dots, x_i, 0, \dots)$ in $cs(\mathfrak{X})$ is bounded, hence the set $\{S_n(\boldsymbol{x}_i) : n, i \in \mathbb{N}\}$ is bounded in Y. Note that for each n we have $S_n(\boldsymbol{x}) = S_n(\boldsymbol{x}_i)$ for $i \geqslant \max\{n, \rho(1), \dots, \rho(n)\}$. It follows that the sequence $\big(S_n(\boldsymbol{x})\big)$ is bounded in Y, which proves (bb). ∎

COROLLARY 7.3 *Let X, Y, and ρ be as in* (H)*, and let (x_n) and (y_n) be sequences in X and Y, respectively. For every sequence of scalars (a_n) consider the series* $(*)$ $\sum_{n=1}^{\infty} a_n x_n$ *in X, and the series* $(**)$ $\sum_{n=1}^{\infty} a_{\rho(n)} y_{\rho(n)}$ *in Y. The following are then equivalent.*

(cc) *Whenever the series $(*)$ converges, so does the series $(**)$.*

(cb) *Whenever the series $(*)$ converges, the series $(**)$ is bounded.*

(bb) *Whenever the series $(*)$ is bounded, so is the series $(**)$.*

PROOF Apply Theorem 7.2 with $X_n := \operatorname{lin}\{x_n\}$ and $T_n(\alpha x_n) := \alpha y_n$. ∎

For the following consequence of Theorem 7.2 we include a simple direct proof.

COROLLARY 7.4 *Let X and $\mathfrak{X} = (X_n)$ be as in* (H)*, and let ρ be a permutation of $\mathbb{N}$. If for every convergent series $\sum_{n=1}^{\infty} x_n$, where $(x_n) \in \omega(\mathfrak{X})$, also the series $\sum_{n=1}^{\infty} x_{\rho(n)}$ is convergent, then both have the same sum.*

PROOF By the assumptions and Proposition 3.1, the operator A_ρ maps continuously $cs(\mathfrak{X})$ into $cs(X)$. Hence also the operator $s \circ A_\rho : cs(\mathfrak{X}) \to X$ is continuous. Since

$$s\big((0,\dots,0,x,0,\dots)\big) = x = s \circ A_\rho\big((0,\dots,0,x,0,\dots)\big),$$

the operators s and $s \circ A_\rho$ coincide on a dense subset of $cs(\mathfrak{X})$. It follows that they are identical on all of $cs(\mathfrak{X})$, which finishes the proof. ∎

REMARK 7.5 In contrast to Sections 4 and 5, there is not much hope, in general, for any 'intrinsic' characterizations of convergence preserving transformations of the type considered in this section. In fact, even the following special case (related to Corollary 7.3) of this problem seems to be open:

Given a Banach space X with a conditional Schauder basis (x_n), characterize permutations ρ such that whenever a series $\sum_n a_n x_n$ converges, so does the series $\sum_n a_{\rho(n)} x_{\rho(n)}$.

As easily seen, ρ has the property required here iff the natural projections from X onto the subspaces $[x_k : k \in \rho([1,n])]$ are uniformly bounded.

Of course, by Theorem 5.1 and Proposition 4.6 (or by applying the preceding observation), condition $\kappa(\rho) < \infty$ ensures this property, but if the basis (x_n) has a complemented unconditional infinite subbasis, then it is far from being necessary. However, for the space $X = cs$ and its unit vector basis (e_n) the answer to the above problem is: $\kappa(\rho) < \infty$. For, if ρ has the required property, then for every $(a_n) \in cs$ one has $s\big(\sum_n a_{\rho(n)} e_{\rho(n)}\big) = \sum_n a_{\rho(n)}$. Hence ρ preserves convergence of scalar series, and so $\kappa(\rho) < \infty$.

The same condition $\kappa(\rho) < \infty$ is also necessary in the case of the basis (e_n) in the Banach space bv_0. To see this, take any $n \in \mathbb{N}$ and let $r = \max \rho([1,n])$. Then $\|e_1 + \dots + e_r\| = 1$, while $\|\sum_{i=1}^{n} e_{\rho(i)}\| \geqslant 2\kappa\big(\rho([1,n])\big) - 1$. By the uniform boundedness of the involved projections, $\kappa(\rho) < \infty$.