ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLVII (1) (2007), 77-83

Danuta Borowska, Jerzy Grzybowski

Minkowski difference and Sallee elements in an ordered semigroup

Abstract. In the manner of Pallaschke and Urbański ([5], chapter 3) we generalize the notions of the Minkowski difference and Sallee sets to a semigroup. Sallee set (see [7], definition of the family $\mathcal{S}$ on p. 2) is a compact convex subset A of a topological vector space X such that for all subsets B the Minkowski difference $A \dot{-} B$ of the sets A and B is a summand of A. The family of Sallee sets characterizes the Minkowski subtraction, which is important to the arithmetic of compact convex sets (see [5]). Sallee polytopes are related to monotypic polytopes (see [4]). We generalize properties of Minkowski difference and Sallee sets to semigroup and investigate the families of Sallee elements in several specific semigroups.

1991 Mathematics Subject Classification: 20M10, 52A07.

Key words and phrases: Minkowski difference, Sallee elements, semigroups.

1. Properties of Minkowski difference in a semigroup. In this section we are going to generalize the Minkowski difference to semigroups. In the following we assume that $(S, +, \leq)$ is an ordered commutative semigroup with unit element 0 satisfying the order cancellation law, that is for all $s, t, u \in S$ the relation "$\leq$" is a partial ordering in S and

(i) $(s + t) + u = s + (t + u)$,

(ii) $s + t = t + s$,

(iii) $s + 0 = s$,

(iv) $s + u \leq t + u$ if and only if $s \leq t$.

By $s \dot{-} t = \max\{u \mid t + u \leq s\}$ we denote the Minkowski difference of $s, t \in S$ (see [6]).
In general $s \dot{-} t$ does not exist. However, the following properties of Minkowski difference hold true:

PROPOSITION 1.1 (SEE [6]) *For all $s, t, u \in S$ holds:*

(i) *If $(s \dot{-} t)$ exists then $(s \dot{-} t) + t \leq s$.*

(ii) *If $s = t + u$ then $s \dot{-} t = u$.*

(iii) *If $t \leq s$ and $t \dot{-} u$, $s \dot{-} u$ exist then $t \dot{-} u \leq s \dot{-} u$.*

(iv) *If $u \leq t$ and $s \dot{-} t$, $s \dot{-} u$ exist then $s \dot{-} t \leq s \dot{-} u$.*

(v) *If $s \dot{-} t$ exists then $(s + u) \dot{-} (t + u) = s \dot{-} t$.*

(vi) *If $s \dot{-} t$, $t \dot{-} u$, $s \dot{-} u$ exist then $(s \dot{-} t) + (t \dot{-} u) \leq s \dot{-} u$.*

Proposition 1.1 follows immediately from the definition of the Minkowski difference.

DEFINITION 1.2 The element $t \in S$ is a *summand* of $s \in S$ if $s = t + u$ for some $u \in S$. We understand by $s \dot{-} (s \dot{-} t)$ the *$s-$hull* of t. We say that t is a *quasisummand* of s if the $s-$hull of t is a summand of s.

In general the $s-$hull of t does not exist. We give more properties of the Minkowski difference in the following proposition.

PROPOSITION 1.3 *For all $s, t, u, w \in S$*

(i) *If $s \dot{-} (s \dot{-} t)$ exists then $t \leq s \dot{-} (s \dot{-} t)$.*

(ii) *If $s \dot{-} (s \dot{-} t)$ exists then $s \dot{-} (s \dot{-} (s \dot{-} t)) = s \dot{-} t$.*

(iii) *If $t \dot{-} w$ and $(t + u) \dot{-} w$ exist then $(t \dot{-} w) + u \leq (t + u) \dot{-} w$.*

(iv) *If $t + u = s$ and $s \dot{-} (s \dot{-} w)$ and $t \dot{-} (t \dot{-} w)$ exist then $s \dot{-} (s \dot{-} w) \leq t \dot{-} (t \dot{-} w)$.*

PROOF (i) By Proposition 1.1(i) we have $(s \dot{-} t) + t \leq s$. Then we apply the definition of $s \dot{-} (s \dot{-} t)$.

(ii) By Proposition 1.3(i) we have $t \leq s \dot{-} (s \dot{-} t)$. Now for any $u \in S$ such that $(s \dot{-} (s \dot{-} t)) + u \leq s$ we obtain $t + u \leq s$ and $u \leq (s \dot{-} t)$. On the other hand from Proposition 1.1(i) follows that $(s \dot{-} (s \dot{-} t)) + u \leq s$, where $u = s \dot{-} t$. Hence $s \dot{-} (s \dot{-} (s \dot{-} t))$ exists and it is equal to $(s \dot{-} t)$. Proposition 1.3(ii) is proved for the semigroup of compact convex sets in [1].

(iii) By Proposition 1.1(i)

$$(t \dot{-} w) + w \leq t.$$

Then $(t \dot{-} w) + w + u \leq t + u$. By definition of "$\dot{-}$" we obtain $(t \dot{-} w) + u \leq (t + u) \dot{-} w$.

(iv) By Proposition 1.1(v) $t \dot{-} (t \dot{-} w) = (t + u) \dot{-} ((t \dot{-} w) + u) = s \dot{-} ((t \dot{-} w) + u)$. By the previous property $(t \dot{-} w) + u \leq s \dot{-} w$. By Proposition 1.1(iv) $s \dot{-} ((t \dot{-} w) + u) \geq s \dot{-} (s \dot{-} w)$. ■

2. Sallee elements in a semigroup.

Definition 2.1 We say that s is a *Sallee element* of S if the existence of $s \dot{-} t$ implies that $s \dot{-} t$ is a summand of s.

Proposition 2.2 (Properties of Sallee elements).
For all s, t, $u \in S$ holds:

(i) *If s is a Sallee element and $s \dot{-} t$ exists then $s \dot{-} (s \dot{-} t)$ exists and $s = (s \dot{-} (s \dot{-} t)) + (s \dot{-} t)$.*

(ii) *If t is a summand of Sallee element s and $t \dot{-} u$ exists then $t \dot{-} u$ is a summand of s.*

(iii) *The element $s \in S$ is a Sallee element if the existence of $s \dot{-} t$ implies that t is a quasisummand of s.*

Proof (i) From the definition of Sallee element follows that $s = (s \dot{-} t) + w$ for some $w \in S$. By Proposition 1.1(ii) $w = (s \dot{-} (s \dot{-} t))$.

(ii) By Proposition 1.1(v) $t \dot{-} u = (t + (s \dot{-} t)) \dot{-} (u + (s \dot{-} t)) = s \dot{-} (u + (s \dot{-} t))$. By the definition of Sallee element $t \dot{-} u$ is a summand of s.

(iii) Let s be a Sallee element and assume that $s \dot{-} t$ exists. Then by Proposition 2.2(i) $s \dot{-} (s \dot{-} t)$ exists and $(s \dot{-} t) + (s \dot{-} (s \dot{-} t)) = s$. Hence the $s-$hull of t exists and t is a quasisummand of s. ■

Remark 2.3 For Sallee element $s \in S$ the existence of $s \dot{-} t$ is equivalent to t being a quasisummand of s.

3. Semigroups of nonnegative integers. The triplet $(\mathbb{Z}_+, +, \leq)$ is an ordered commutative semigroup with 0 satisfying the cancellation law.

THEOREM 3.1 *For the semigroup* $(\mathbb{Z}_+, +, \leq)$ *the family* $\mathcal{S}$ *of all Sallee elements is equal to* $\mathbb{Z}_+$.

PROOF $s \in \mathbb{Z}_+$. Then $s \dot{-} t$ exists and is equal to $s - t$ if and only if $t \leq s$. Also every element t such that $t \leq s$ is a summand of s. Then s is a Sallee element in $\mathbb{Z}_+$. ∎

REMARK 3.2 In the semigroup $(S, +, \leq)$, where $S = \mathbb{Z}_+ \setminus \{1, 2, 5\}$ the family of all Sallee elements of S is equal to $S \setminus \{9\}$. Notice, that $9 \dot{-} 4 = 4$ is not a summand of 9.

4. Multiplicative semigroup of integers. The triplet $(\mathbb{N}, \cdot, \leq)$ is an ordered commutative semigroup with unit element 1 satisfying the cancellation law. In our considerations we replace the notion of "summand" with "divisor" difference "$s \dot{-} t$" with quotient "$\lfloor \frac{s}{t} \rfloor$" where $\lfloor \frac{s}{t} \rfloor = \max\{u \in \mathbb{N} \mid ut \leq s\}$. In particular, $\lfloor \frac{s}{t} \rfloor$ exists only if $t \leq s$. The following proposition holds true:

REMARK 4.1 Let $s \in \mathbb{N}$ and for any $t \in \mathbb{N}$ the condition $t \leq s$ implies that t is a divisor of s. Then $s \in \{1, 2\}$.

The following theorem gives a little bit suprising description of Sallee elements in $\mathbb{N}$.

THEOREM 4.2 *In the semigroup* $(\mathbb{N}, \cdot, \leq)$ *the family* S *of all Sallee elements of* $\mathbb{N}$ *is equal to* $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

The family of all Sallee elements is equal to the family of all divisors of 24. To prove the theorem we need the following two lemmas.

LEMMA 4.3 *Let* p *be a Sallee element in* $\mathbb{N}$. *If* $m \in \mathbb{N}$ *and* $m^2 \leq p$ *then* m *is a divisor of* p.

PROOF We have $p = km + l$ where $0 \leq l < m \leq k$. Then $\lfloor \frac{p}{m} \rfloor = k, \frac{p}{m} = \frac{km+l}{m} = k + \frac{l}{m}, \frac{l}{m} < 1$. Hence $\lfloor \frac{p}{\lfloor \frac{p}{m} \rfloor} \rfloor = \lfloor m + \frac{l}{k} \rfloor = m$. Therefore, by the definition of Sallee element, both k and m are divisors of p. ∎

LEMMA 4.4 *Let* $n \geq 5$. *Then the least common multiple of* 1, ... n, (LCM(1, ..., n)) *is geater or equal to* $(n + 1)^2$.

PROOF Let $p \geq (2^n)^2$. From the theorem of Tschebyschev for any $m \geq 1$ there exists prime number belonging to the interval $(2^m, 2^{m+1})$. Then LCM(1, ..., 2^n) $> 2^{n-1} \cdot 2^{n-2} \cdot \ldots \cdot 2 \cdot 2^n = 2^{\frac{n(n+1)}{2}}$. Notice that $\frac{n(n+1)}{2} \geq 2n + 2$ for $n \geq 4$. If $n \geq 4$

then LCM(1, ..., 2^n) $> 2^{2n+2} = (2^{n+1})^2$. If $n \geq 16 = 2^4$ then $n \in [2^k, 2^{k+1})$ for some $k \geq 4$. Then LCM(1, ..., n) $\geq$ LCM(1, ..., 2^k) $> (2^{k+1})^2 \geq (n+1)^2$. Notice also that LCM(1, ..., 5) $= 60 \geq (6+1)^2 > (5+1)^2$ and LCM(1, ..., 7) $= 420 \geq (15+1)^2$. Therefore, our lemma holds true for all $n \geq 5$. ∎

Proof of Theorem 4.2 Let a be a Sallee element such that $n^2 \leq a < (n+1)^2$ for some $n \geq 5$. By Lemma 4.3 the number LCM(1, ..., n) is a divisor of a. By Lemma 4.4 we have $a \geq$ LCM(1, ... n) $\geq (n+1)^2$ which leads to contradiction. Hence all Sallee elements are less than 25. Notice that $\lfloor \frac{s}{2} \rfloor$ is not a divisor of s for $s = 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$. Neither $\lfloor \frac{s}{3} \rfloor$ is a divisor of s for $s = 10, 14, 16, 20, 22$. Moreover $\lfloor \frac{18}{4} \rfloor$ is not a divisor of 18. We leave to the reader to check that all the divisors of the number 24 are Sallee elements. ∎

Remark 4.5 If $S = \ln\mathbb{N} = \{\ln n \mid n \in \mathbb{N}\}$ then the ordered semigroup $(S, +, \leq)$ contains unit element 0 and satsfies the order cancellation law. Then the family of Sallee elements of S is equal to
$\{0, \ln 2, \ln 3, \ln 4, \ln 6, \ln 8, \ln 12, \ln 24\}$.

5. Multiplicative semigroup of multiples of the numbers 2 and 3.

Theorem 5.1 *Let* $\mathcal{S} = \{2^m \cdot 3^n \mid n, m \in \mathbb{Z}_+\} \subset \mathbb{N}$. *The ordered semigroup* $(S, \cdot, \leq)$ *with unit element* 1 *and fulfills the order cancellation law. Then the family of Sallee elements of* $\mathcal{S}$ *is infinite. Also the family of elements that are not Sallee elements is infinite.*

Theorem 5.1 follows immediately from the following propositions.

Proposition 5.2 *Let* S *be the semigroup defined in* Theorem 5.1. *Let* $n, m \in \mathbb{Z}_+$ *and* $2^n < 3^m < 2^{n+1}$ *then* $2^n \cdot 3^m$ *and* $2^{n+1} \cdot 3^m$ *are Sallee elements.*

Proof Let $a \in S$ and $a \leq \sqrt{2^n \cdot 3^m}$. For some $k, l \in \mathbb{Z}_+$ $a = 2^k \cdot 3^l$. Then $2^k \leq \sqrt{2^n \cdot 3^m} < \sqrt{2^n \cdot 2^{n+1}} = \sqrt{2} \cdot 2^n$. Hence $k \leq n$. On the other hand , $3^l \leq \sqrt{2^n \cdot 3^m} < \sqrt{3^m \cdot 3^m} = 3^m$ and $l \leq m$. Therefore a is a divisor of $2^n \cdot 3^m$. Now, let $\sqrt{2^n \cdot 3^m} < a \leq 2^n \cdot 3^m$ then $b = \max\{c \in S \mid a \cdot c \leq 2^n \cdot 3^m\}$ exists and $b \leq \frac{2^n \cdot 3^m}{a} < \sqrt{2^n \cdot 3^m}$. Then b is a divisor of $2^n \cdot 3^m$. The proof that $2^{n+1} \cdot 3^m$ is a Sallee element is analogous. ∎

Proposition 5.3 *Let* S *be a semigroup defined in* Theorem 5.1. *Let* $a = 2^n$, $n \geq 5$. *Then* a *is not a Sallee element of* S.

Proof Let $a = 2^n$ and $n \geq 5$. We know that $2^{n-2} = 2^{n-5} \cdot 2^3 < 2^{n-5} \cdot 3^2 < \frac{2^n}{3} < 2^{n-1}$. Then $2^{n-2} < \max\{c \in S \mid 3 \cdot c \leq 2^n\} < 2^{n-1}$. Hence $\max\{c \in S \mid 3 \cdot c \leq 2^n\} = 2^k \cdot 3^l$ where $l \geq 1$ and $\max\{c \in S \mid 3 \cdot c \leq 2^n\}$ is not a divisor of a. ∎

REMARK 5.4 Let $S = n + m\alpha \mid n, m \in \mathbb{Z}_+$, where $\alpha > 0$ is an irrational number. Then $(S, +, \leq)$ is an ordered semigroup with infinite families of Sallee and non-Sallee elements. The proof of this fact is similar to the proof of Theorem 5.1.

6. Semigroup of compact convex sets. Let X be a Hausdorff topological vector space. By $\mathcal{K}(X)$ we denote the family of all nonempty compact convex subset of X. The *Minkowski sum* of A and $B \in \mathcal{K}(X)$ is defined by

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

The triplet $(\mathcal{K}(X), +\ \subset)$ is an ordered commutative semigroup with singleton $\{0\}$ as zero satisfying the law of cancellation [9].
We have
$A \dot{-} B = \max\{C \in \mathcal{K}(X) \mid B + C \subset A\} = \{x \in X : x + B \subset A\} = \bigcap_{b \in B}(A - b)$ ([2]). For more properties of Minkowski subtraction we refer to [1], [5].
For the sets A, $B \in \mathcal{K}(X)$ the *A-hull* of B is equal to

$$(A \dot{-} (A \dot{-} B)) = \bigcap \left\{ A + x \mid B \subset A + x, x \in X \right\}.$$

If aff $A \cap$ aff $B = \{p\}$, then we write $A \oplus B$ instead $A + B$ and we call it a *direct sum* of A and B.
In the case of $\mathcal{K}(\mathbb{R}^2)$ the family of all Sallee sets is equal $\mathcal{K}(\mathbb{R}^2)$ (see [8]). If S is the semigroup of three-dimensional convex polyhedra (including polygons, segments and singletons)the family of all Sallee sets consists of all prisms, wedges, dull wedges, skew cubes, polygons, segments and singletons (see [4]). The family of all Sallee sets in the semigroup centrally symmetric polytopes in $\mathbb{R}^n$ consists of all direct sums of two dimensional centrally symmetric polygons and segments (see [7], Theorem 3). The family of all Sallee sets in $\mathcal{K}(\mathbb{R}^3)$ contains also all elipsoids (see [3]), non-polyhedral wedges, parts of the Euklidean ball. We still are not able to characterize all Sallee sets in $\mathcal{K}(\mathbb{R}^3)$.

REFERENCES

[1] D. Borowska, H. Przybycień and R. Urbański: *On Summands Properties and Minkowski subtraction.* Journal for Analysis and its Applications Volume **26(2)** (2007), 247-260.

[2] J. Grzybowski, H. Przybycień and R. Urbański: *On Summands of Closed Bounded Convex Sets.* Journal for Analysis and its Applications Volume **21(4)** (2002), 845-850.

[3] H. Maehara: *Convex bodies forming pairs of constant width.* Journal of Geometry vol. **22** (1984), 101-107.

[4] P. McMullen, R. Schneider, G. C. Shepard: *Monotypic polytopes and their intersection properties.* Geometriae Dedicata **3** (1974) 99-129.

[5] D. Pallaschke, R. Urbański: *Pairs of Compact Convex Sets, Fractional Arithmetic with Convex Sets.* Serie Mathematics and Its Applications, vol.548, Kluwer Academic Publisher, Dortrecht-Boston-London, 2002.

[6] H. Przybycień, Doctoral Thesis: *Certain Properties of closed bounded convex sets.* Adam Mickiewicz University Poznań 2003 (in Polish).

[7] G. T. Sallee: *Pairs of sets of constant relative width.* Journal of Geometry vol. **29** (1987) 1-11.

[8] R. Schneider: *Convex Bodies: The Brunn-Minkowski Theory* (Encyklopedia of Mathematics and its Applications: Vol 44). Cambridge Univ. Press 1993.

[9] R. Urbański: *A generalization of the Minkowski Rådström-Hörmander theorem,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **24** (1976) 709-715.

Danuta Borowska
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań
E-mail: dborow@amu.edu.pl

Jerzy Grzybowski
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań
E-mail: jgrz@amu.edu.pl

(Received: 25.10.2006)