ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLVII (1) (2007), 109-116

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Lefschetz Fixed Point Theorems for Approximative Type Maps in Fréchet Spaces

Abstract. In this paper using the projective limit approach we present new Lefschetz fixed point theorems for approximable type maps defined on PRANR's.

2000 Mathematics Subject Classification: 47H10.

Key words and phrases: Approximable maps, Lefchetz fixed point theory.

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1. Introduction. In 1991 Górniewicz, Granas and Kryszewski [3] presented fixed point theorems for multimaps with nonconvex values which were based on approximating such maps with single valued maps. In this paper we present a new fixed point theorem of Lefschetz type for these approximable maps (indeed the theory can easily be extended to decomposable maps). These maps will be defined on PRANR's and these sets are natural in applications in the Fréchet space setting since they include pseudo-open sets. Our theory is based on results in Banach spaces and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N=\{1,2,...\}$).

For the remainder of this section we present some definitions and well known results [1, 2]. Consider vector spaces over a field K. Let E be a vector space and $f: E\to E$ an endomorphism. Now let $N(f)=\{x\in E:\ f^{(n)}(x)=0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E}=E\setminus N(f)$. Since $f(N(f))\subseteq N(f)$ we have the induced endomorphism $\tilde{f}:\tilde{E}\to\tilde{E}$. We call f admissible if $dim\,\tilde{E}<\infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f)=tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f=\{f_q\}: E\to E$ be an endomorphism of degree zero of a graded vector space $E=\{E_q\}$. We call f the Leray endomorphism if (i) all f_q are admissible

and (ii) almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q Tr\,(f_q).$$

Let H be the singular homology functor (with coefficients in the field K) from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional singular homology group of X. For a continuous map $f : X \to Y$, $H(f)$ is the induced linear map $f_\star = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

A continuous map $f : X \to X$ is called a Lefschetz map provided $f_\star : H(X) \to H(X)$ is a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_\star)$. We know if f and g are homotopic ($f \sim g$) and if f is a Lefschetz map, then g is a Lefschetz map with $\Lambda(g) = \Lambda(f)$.

A metrizable space Y is an absolute neighborhood retract (ANR) provided each continuous map $f : D \to Y$, defined on a closed subset D of a metrizable space X, is extendable over some neighborhood of D in X to a map into Y.

Next we present the maps we will discuss in this paper. A compact nonempty space A is called an R_δ set provided there exists a decreasing sequence $\{A_n\}$ of compact absolute retracts such that $A = \cap_{n\geq 1} A_n$. Now let X and Y be metric spaces and $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say that $F : X \to K(Y)$ is a R_δ map if F is upper semicontinuous and for each $x \in X$, $F(x)$ is a R_δ set. Also we will consider more general maps. Let A be a compact subset of a metric space X. A is called ∞–proximally connected in X if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $n = 1, 2, ...$ and any map $g : \partial \triangle^n \to N_\delta(A)$ there exists a map $g' : \triangle^n \to N_\epsilon(A)$ such that $g(x) = g'(x)$ for $x \in \partial \triangle^n$; here $\triangle^n$ is the n–dimensional standard simplex and $N_\epsilon(A) = \{x \in X : dist\,(x, A) < \epsilon\}$. Let X and Y be two metric spaces and $F : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y). We say $F \in J(X, Y)$ if F is upper semicontinuous (u.s.c.) with nonempty, compact, ∞–proximally connected values; see [3 pp 467] for examples of J maps. In particular if $F : X \to K(Y)$ is a R_δ map and X an ANR space then $F \in J(X, Y)$. Let $F : X \to K(Y)$ and $\epsilon > 0$. A continuous mapping $f : X \to Y$ is an ϵ-graph approximation of F (and we write $f \in a(F, \epsilon)$) if for each $x \in X$, $f(x) \in B(F(B(x, \epsilon)), \epsilon)$ where $F(B(x, \epsilon)) = \cup_{y\in B(x,\epsilon)} F(y)$ is the image of $B(x, \epsilon)$ under F. Now let

$$\begin{aligned} A_0(X, Y) \quad = \quad & \{F : X \to K(Y) : F \text{ is u.s.c. and for every} \\ & \epsilon > 0 \text{ there is a } f \in a(F, \epsilon)\}. \end{aligned}$$

We say $F \in A(X, Y)$ if $F \in A_0(X, Y)$ and for each $\delta > 0$ there is an $\epsilon_0 > 0$ such that for every ϵ $(0 < \epsilon < \epsilon_0)$ if $f, g : X \to Y$ are ϵ-graph approximations of F, then there exists a continuous homotopy $h : X \times [0, 1] \to Y$ joining f and g such that $h_t \in a(F, \delta)$ for each $t \in [0, 1]$.

Let X be a compact ANR space and let $F \in A_0(X,X) \equiv A_0(X)$. For each $\epsilon > 0$ there exists a continuous single valued mapping $f \in a(F,\epsilon)$. We let

$$\Lambda_\epsilon(F) = \{\Lambda(f) : \ f \in a(F,\epsilon)\}$$

where $\Lambda(f)$ denotes the Lefschetz number of f. We define the Lefschetz set $\mathbf{\Lambda}(F)$ of $F \in A_0(X)$ by

$$\mathbf{\Lambda}(F) = \cap\{\Lambda_\epsilon(F) : \ \epsilon > 0\}.$$

If $F \in A(X,X) \equiv A(X)$ then $\mathbf{\Lambda}(F)$ is a singleton and will be called the Lefschetz number of F and will be denoted by $\Lambda(F)$.

The following Lefschetz fixed point results were established in [2 pp121, 122] (see also [3]).

THEOREM 1.1 *Let X be a compact ANR and $F \in A(X)$. If $\Lambda(F) \neq 0$ then F has a fixed point.*

THEOREM 1.2 *Let X be a compact ANR and $F \in A_0(X)$. If $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.*

Also the following result was established in [3 pp 473].

THEOREM 1.3 *Let X be a compact ANR space and Y a (metric) space. If $F \in J(X,Y)$ then $F \in A(X,Y)$.*

Of course, as one would expect, one can extend the Lefschetz number to decomposable maps (see [5]). As before X and Y are metric spaces. By a decomposable map we mean a pair (F,D) consisting of a set valued map $F : X \to 2^Y$ and a diagram $D : X \overset{G}{\to} Z \overset{h}{\to} Y$, where $Z \in ANR$, $G : X \to 2^Z$ is an R_δ map and $h : Z \to Y$ a single valued continuous map, such that $F = h \circ G$. Let X be a compact ANR, $Z \in ANR$ and let (F,D) be a decomposable map with $D : X \overset{G}{\to} Z \overset{h}{\to} X$. We define the Lefschetz number

$$\Lambda(F,D) = \lim_{\epsilon \to 0^+} \Lambda(h \circ f_\epsilon)$$

where, for $\epsilon > 0$, $f_\epsilon \in a(F,\epsilon)$.

The following Lefschetz fixed point result was established in [5 pp 1798].

THEOREM 1.4 *Let X be a compact ANR and $(F,D) : X \to X$ a decomposable map. If $\Lambda(F,D) \neq 0$ then F has a fixed point.*

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha, \beta \in I$ such that $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall\, \alpha, \beta \in I, \ \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [4 pp. 439] $\cap_{\alpha\in I} E_\alpha$.)

2. Fixed point theory in Fréchet spaces.. Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \ldots\ldots \quad \text{for every} \quad x \in E. \tag{1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{3}$$

REMARK 2.1 (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii) Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq \ldots\ldots \text{ and for each } n \in N,\ |x|_n \le |x|_{n+1}\ \forall\ x \in E_{n+1}. \tag{4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X :\ j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \quad \text{for every} \quad n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

Let E and E_n be as described above.

DEFINITION 2.2 A set $A \subseteq E$ is said to be PRANR if for each $n \in N$, $A_n \equiv j_n\,\mu_n\,(A)$ is an ANR.

EXAMPLE 2.3 Let A be pseudo-open. Then A is a PRANR.

To see this fix $n \in N$. We now show

$$A_n \text{ is a open subset of } E_n.$$

First notice $A_n \subseteq \overline{A_n} \setminus \partial A_n$ since if $y \in A_n$ then there exists $x \in A$ with $y = j_n\mu_n(x)$ and this together with $A = pseudo - int\,A$ yields $j_n\mu_n(x) \in \overline{A_n} \setminus \partial A_n$ i.e. $y \in \overline{A_n} \setminus \partial A_n$. In addition notice

$$\overline{A_n} \setminus \partial A_n = (int\,A_n \cup \partial A_n) \setminus \partial A_n = int\,A_n \setminus \partial A_n = int\,A_n$$

since $int\,A_n \cap \partial A_n = \emptyset$. Consequently

$$A_n \subseteq \overline{A_n} \setminus \partial A_n = int\,A_n, \quad \text{so} \quad A_n = int\,A_n.$$

As a result A_n is open in E_n. Thus A_n is an ANR, so A is a PRANR.

Our first results are for Volterra type operators.

THEOREM 2.4 *Let E and E_n be as described above, $C \subseteq E$ is a compact PRANR and $F : C \to 2^E$ and for each $n \in N$ assume $F : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F \in A(C_n) = A(C_n, C_n) \tag{5}$$

$$\text{for each } n \in N,\ \Lambda_{C_n}(F) \neq 0 \tag{6}$$

and

$$\begin{cases} \text{for each } n \in \{2,3,....\} \text{ if } y \in C_n \text{ solves } y \in F\,y \text{ in } E_n \\ \text{then } y \in C_k \text{ for } k \in \{1,...,n-1\}. \end{cases} \tag{7}$$

Then F has a fixed point in E.

PROOF Fix $n \in N$. Notice $C_n = j_n\,\mu_n\,(C)$ is compact. Theorem 1.1 guarantees that there exists $y_n \in C_n$ with $y_n \in F\,y_n$. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in C_1$ and $y_k \in C_1$ for $k \in N\backslash\{1\}$ from (7). As a result $y_n \in C_1$ for $n \in N$ and now since C_1 is compact there is a subsequence $N_1^\star$ of N and a $z_1 \in C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in C_2$ for $n \in N_1$ together with the fact that C_2 is compact implies that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (4) that

$z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ....., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : C_k \to 2^{E_k}$ is upper semicontinuous (note $y_n \in C_k$ for $n \in N_k$) implies $y \in F\, y$ in E_k. We can do this for each $k \in N$ so we have $y \in F\, y$ in E. ∎

Essentially the same reasoning as in Theorem 2.4 yields the following result.

THEOREM 2.5 *Let E and E_n be as described above, $C \subseteq E$ is a compact PRANR and $F : C \to 2^E$ and for each $n \in N$ assume $F : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F \in A_0(C_n) = A_0(C_n, C_n) \tag{8}$$

and

$$\text{for each } n \in N,\ \mathbf{\Lambda}_{C_n}(F) \neq \{0\}. \tag{9}$$

Also assume (7) holds. Then F has a fixed point in E.

Our next results were motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (13).

THEOREM 2.6 *Let E and E_n be as described in the beginning of Section 2, $C \subseteq E$ is a compact PRANR and $F : C \to 2^E$. Also for each $n \in N$ assume there exists $F_n : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\begin{cases} \text{for each } n \in \{2, 3, ...\} \text{ if } y \in C_n \text{ solves } y \in F_n\, y \text{ in } E_n \\ \text{then } y \in C_k \text{ for } k \in \{1, ..., n-1\}. \end{cases} \tag{10}$$

$$\text{for each } n \in N,\ F_n \in A(C_n, C_n) = A(C_n) \tag{11}$$

$$\text{for each } n \in N,\ \Lambda_{C_n}(F_n) \neq 0 \tag{12}$$

and

$$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in C_n \text{ and } y_n \in F_n\, y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in F\, w \text{ in } E. \end{cases} \tag{13}$$

Then F has a fixed point in E.

PROOF Fix $n \in N$. Theorem 1.1 guarantees that there exists $y_n \in C_n$ with $y_n \in F_n y_n$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Note from (10) that $y_n \in C_1$ and now since C_1 is compact there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Also there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ....., \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (13) implies $y \in F y$ in E. ∎

REMARK 2.7 Note if $C_1 \supseteq C_2 \supseteq ...$ then clearly (10) holds.

Similarly we have the following result.

THEOREM 2.8 *Let E and E_n be as described in the beginning of Section 2, $C \subseteq E$ is a compact PRANR and $F : C \to 2^E$. For each $n \in N$ assume there exists $F_n : C_n \to 2^{E_n}$ and also assume (10) holds. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in A_0(C_n, C_n) = A_0(C_n) \tag{14}$$

and

$$\text{for each } n \in N,\ \mathbf{\Lambda}_{C_n}(F_n) \neq \{0\}. \tag{15}$$

Finally assume (13) *holds. Then F has a fixed point in E.*

REMARK 2.9 Of course there are analogues of Theorem 2.4 and Theorem 2.6 for decompsable maps.

REFERENCES

[1] R.P. Agarwal and D. O'Regan, *A note on the Lefschetz fixed point theorem for admissible spaces*, Bull. Korean Math. Soc., **42** (2005), 307–313.

[2] L. Górniewicz, *Topological fixed point theory of multivalued mappings*, Kluwer Academic Publishers, Dordrecht, 1999.

[3] L. Górniewicz, A. Granas and W. Kryszewski, *On the homotopy method in the fixed point index of multi-valued mappings of compact absolute neighborhood retracts*, Jour. Math. Anal. Appl., **161** (1991), 457–473.

[4] L.V. Kantorovich and G.P. Akilov, *Functional analysis in normed spaces*, Pergamon Press, Oxford, 1964.

[5] W. Kryszewski and S. Plaskacz, *Periodic solutions to impulsive inclusions with constraints*, Nonlinear Analysis, **65** (2006), 1794–1804.

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(Received: 09.03.2007)