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Maximal Inequalities for a Best Approximation Operator in Orlicz Spaces

Abstract. In this paper we study a maximal operator $\mathcal{M}f$ related with the best φ approximation by constants for a function $f \in L^{\varphi'}_{loc}(\mathbb{R}^n)$, where we denote by φ' for the derivative function of the C^1 convex function φ. We get a necessary and sufficient condition which assure strong inequalities of the type $\int_{\mathbb{R}^n} \theta(\mathcal{M}|f|)\,dx \leqslant K \int_{\mathbb{R}^n} \theta(|f|)\,dx$, where K is a constant independent of f. Some pointwise and mean convergence results are obtained. In the particular case $\varphi(t) = t^{p+1}$ we obtain several equivalent conditions on the functions θ that assures strong inequalities of this type.

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1. Introduction and Notations. Let Φ be the class of all nondecreasing functions φ defined for all real number $t \geqslant 0$, with $\varphi(0^+) = 0$, $\varphi(t) \to \infty$ as $t \to \infty$, and $\varphi(t) > 0$ for $t > 0$. Observe that we do not assume continuity for the functions in Φ. This class of functions is quite similar to that treated in [10] though they assume that the functions φ are increasing. When the functions in Φ are also continuous functions they are called Orlicz functions and they were considered by Musielak and Orlicz in [18]. According to [2] we say that a function φ satisfies the Δ_2 condition if there exists $k > 0$ such that $\varphi(2t) \leqslant k\varphi(t)$, for all $t > 0$, and in this case we write $\varphi \in \Delta_2$.

Given a function $\varphi \in \Phi$ and a bounded measurable set $\Omega \subset \mathbb{R}^n$ we set $L^{\varphi}(\Omega)$ for the class of all Lebesgue measurable functions defined on Ω such that $\int_\Omega \varphi(\lambda|f(x)|)\,dx$ is finite for some $\lambda > 0$, where dx denotes the Lebesgue measure on $\mathbb{R}^n$. For $\varphi \in \Phi \cap \Delta_2$, the set $L^{\varphi}(\Omega)$ coincides with the class of all Lebesgue measurable functions defined on $\mathbb{R}^n$ such that $\int_\Omega \varphi(|f(x)|)\,dx$ is finite.

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If the function $\varphi \in \Phi$ is a convex function which satisfies a Δ_2 condition we considered in [7] the multivalued operator $\mu_\varphi(f)(\Omega)$ of all the best approximation by constants to the function $f \in L^\varphi(\Omega)$. That is, a real number c is a best approximation of f if and only if

$$\int_\Omega \varphi(|f(x)-c|)\,dx \leqslant \int_\Omega \varphi(|f(x)-r|)\,dx,$$

for every $r \in \mathbb{R}$. It is easy to see that $\mu_\varphi(f)(\Omega)$ is a non empty set and if φ is a strictly convex function this set has only one element. For the particular case $\Omega = B_\varepsilon(x)$, where $B_\varepsilon(x)$ is a ball centered at $x \in \mathbb{R}^n$ with radius ε, we set $\mu_\varepsilon(f)(x)$ for $\mu_\varphi(f)(\Omega)$.

Further, when φ is a C^1 function, we consider the space $L^{\varphi'}(\Omega)$, where φ' is the derivative of the function φ, and we extend this operator in a continuous way, to functions f in $L^{\varphi'}(\Omega)$. Note that in general $L^{\varphi'}(\Omega)$ is a strictly bigger space than $L^\varphi(\Omega)$. We refer to [4] for an extension of this operator in a more general set up. We will use the notation $\tilde{\mu}_\varphi(f)(\Omega)$ for the extended best approximation operator, given in the next definition.

From now on, the C^1 condition on the function φ will be assumed. We extended in [7] the definition of the multivalued operator $\mu_\varphi(f)(\Omega)$ to functions $f \in L^{\varphi'}(\Omega)$, as follows.

DEFINITION 1.1 Let φ be a convex fuction in $\Phi \cap \Delta_2$ and let f be in $L^{\varphi'}(\Omega)$. We say that a constant c is an extended best approximation of f on Ω if it is a solution of the following inequalities

$$(a)\ \int_{\{f>c\}\cap\Omega} \varphi'(|f(y)-c|)\,dy \leqslant \int_{\{f\leqslant c\}\cap\Omega} \varphi'(|f(y)-c|)\,dy.$$

$$(b)\ \int_{\{f<c\}\cap\Omega} \varphi'(|f(y)-c|)\,dy \leqslant \int_{\{f\geqslant c\}\cap\Omega} \varphi'(|f(y)-c|)\,dy.$$

We write $\tilde{\mu}_\varphi(f)(\Omega)$ for the set of all these constants c.

In [7] we have proved that $\tilde{\mu}_\varphi(f)(\Omega) \neq \emptyset$. The inequalities (a) and (b) in Definition 1.1 characterizes the elements in $\mu_\varphi(f)(\Omega)$ if $f \in L^\varphi$ then $\mu_\varphi(f)(\Omega) = \tilde{\mu}_\varphi(f)(\Omega)$, for $f \in L^\varphi$. The extension of the best approximation operator was studied for the first time in [12] for $\varphi(t) = t^p,\ p > 1$, for a rather general approximation class of functions. In [6] we considered the extension of the best approximation operator for a general approximation class on Orlicz spaces L^φ. Other results involving the extension of operators, where the approximation class are the constant functions, can be found in [15] and [16].

If Ω is the ball $B_\varepsilon(x)$ we write $f_\varepsilon(x)$ for any $c \in \tilde{\mu}_\varphi(f)(\Omega)$, and the notation $\tilde{\mu}_\varepsilon(f)(x)$ will be used for the set of all these constants. We refer to [7] for some equivalences of the Definition 1.1.

One of the main results of the paper are the inequalities given in Theorem 2.1. In the special case that f is a non-negative function the first inequality of the theorem says that $\varphi'(f_\varepsilon(x))$ is equivalent to consider the averages $\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(f(y))\,dy$.

If $\varphi'(0) = 0$, this type of equivalence and some version of the classical Lebesgue Differentiation Theorem, will imply the following pointwise convergence result

$$\sup\{|f_\varepsilon(x) - f(x)| : f_\varepsilon(x) \in \tilde{\mu}_\varepsilon(f)(x)\} \to 0, \ as\ \varepsilon \to 0. \tag{1}$$

The result stated above can be obtained even in the case $\varphi'(0) > 0$. This fact is not a consequence of Theorem 2.1 and a proof will be given at the end of Section 2.

As we pointed out in [7], page 33, if φ' is a strictly increasing function and $\varphi'(0) = 0$ the set $\tilde{\mu}_\varepsilon(f)(x)$ has a unique element $f_\varepsilon(x)$ which is a continuous function on x. With this additional hypothesis on φ' we study the mean convergence

$$\int_{\mathbb{R}^n} \theta(|f(y) - f_\varepsilon(y)|)\, dy \to 0, \ \ as\ \varepsilon \to 0, \tag{2}$$

for some $\theta \in \Phi$.

The only case where the mean convergence (2) can be easily obtained is when L^φ is the space L^2. In this particular case the extended best approximation operator $f_\varepsilon(x)$ is given by the average $\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} f(y)\, dy$, and using standard arguments we have that (2) holds provided $f \in L^\theta(\mathbb{R}^n)$ and θ is a convex function in $\Phi \cap \Delta_2$. For a general function φ to deal with the mean convergence problem (2) it is necessary to introduce the following maximal operator

$$\mathcal{M}f(x) = \sup_{\varepsilon>0}\{|f_\varepsilon(x)| : f_\varepsilon(x) \in \tilde{\mu}_\varepsilon(f)(x)\}. \tag{3}$$

This operator $\mathcal{M}f$ depends on the function φ, therefore sometimes we will denote it as $\mathcal{M}_\varphi f$. The very well known Hardy Littlewood maximal operator is obtained considering $\varphi(t) = t^2$. In this case $\mathcal{M}_{t^2}(|f|)$ will be denoted by $M(f)(x)$ and

$$M(f)(x) = \sup_{\varepsilon>0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |f(y)|\, dy,$$

where $f \in L^1_{loc}(\mathbb{R}^n)$.

We point out that the measurability of the operator $\mathcal{M}_\varphi f$ is unknown and it can be proved only in some specific cases. For example, the Hardy Littlewood maximal operator or $\varphi(t) = t^p, \ \ p > 1$. This is also the situation if φ' is a continuous strictly increasing function and $\varphi'(0) = 0$, since in this case we have that $f_\varepsilon(x)$ is a continuous function on x.

We characterize, in Theorem 2.3, those functions θ such that

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}(|f|))\, dx \leqslant C \int_{\mathbb{R}^n} \theta(|f|)\, dx, \tag{4}$$

for all $f \in L^{\varphi'}_{loc}(\mathbb{R}^n)$. As an application of (4) and using (1) we can get the mean convergence result (2). We also obtain, for some cases, the norm convergence, see Theorem 2.6.

Also we can see, in the proof of Theorem 2.3, that the operators $\mathcal{M}(|f|)$ and the Hardy-Littlewood $M(f)$ are related by

$$\frac{1}{2}\varphi'^{-1}(\frac{1}{C}M(\varphi'(|f|))(x)) \leqslant \mathcal{M}(|f|)(x) \leqslant \varphi'^{-1}(CM(\varphi')(|f|)(x)), \tag{5}$$

where we denote by φ'^{-1} the generalized inverse function of φ' which is given by

$$\varphi'^{-1}(t) = \sup_{\varphi'(s)\leqslant t} s.$$

For the particular case $\varphi(t) = t^{p+1}$, with $p > 0$ a direct calculations in (5) gives

$$\frac{1}{K}(M(|f|^p)^{\frac{1}{p}}(x) \leqslant \mathcal{M}_{t^{p+1}}(|f|)(x) \leqslant K(M(|f|^p)^{\frac{1}{p}}(x),$$

where the constant K is independent of f. Thus sufficient and necessary condition in order to get strong inequalities for the operator

$$M_p(f)(x) = \Big(\sup_{\varepsilon>0} \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} |f(y)|^p\, dy\Big)^{\frac{1}{p}}$$

can be obtained applying Theorem 2.3. This result is given in Corollary 2.8.

Observe that the operator M_p is related with the $p-$averages of a function and it is well known in the literature and often used in the case $p \geqslant 1$, see for example [21] and more recently [3].

We need to introduce the following

DEFINITION 1.2 A function $\varphi \in \Phi$ satisfies the ∇_2 condition if there exists a constant $\alpha > 1$ such that

$$\varphi(t) < \frac{1}{2\alpha}\varphi(\alpha t), \;\; t > 0. \tag{6}$$

In [11] and [20] the condition (6) on the function φ appears as follows

$$\varphi(t) \leqslant \frac{1}{2\alpha}\varphi(\alpha t), \tag{7}$$

for some $\alpha > 1$ and for all $t > 0$. We will see later, Proposition 3.3, that for functions $\varphi \in \Phi$ the conditions (7) and (6) are equivalent.

To obtain a strong inequality for the operator M_p we need, by Corollary 2.8, that the function $\theta(t^{1/p})$ satisfies the ∇_2 condition. The last section of this paper is devoted to study the above condition and related ones. Also one of them is used in the proof of Theorem 2.5.

2. Main Results. Now we state the first result of this paper.

THEOREM 2.1 *Let $\varphi \in \Phi \cap \Delta_2$ be a C^1 convex function, $f \in L^{\varphi'}_{loc}(\mathbb{R}^n)$ and select $f_\varepsilon(x)$ in $\tilde{\mu}_\varepsilon(f)(x)$, where $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have the following estimates.*

$$\frac{1}{C}\varphi'(|f_\varepsilon(x)|) \leqslant \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(|f(y)|)\, dy \leqslant C\varphi'(|f|_\varepsilon(x)). \tag{8}$$

$$\frac{1}{C}\varphi'(|f_\varepsilon(x) - f(x)|) \leqslant \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(|f(y) - f(x)|)dy, \tag{9}$$

We have used $|B_\varepsilon|$ for the Lebesgue measure of the ball B_ε.

PROOF We will prove the right hand side of (8). Without lost of generality we may assume $f \geqslant 0$, then $f_\varepsilon(x) \geqslant 0$. In fact by a) of Definition 1.1 we have a contradiction if we assume $c < 0$. Then we have

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(f(y))\,dy = \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)<f\}} \varphi'\big((f(y)-f_\varepsilon(x))+f_\varepsilon(x)\big)\,dy+ \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\geqslant f\}} \varphi'(f(y))\,dy.$$

Note that for $\varphi \in \Phi \cap \Delta_2$ we have

$$\varphi'(a+b) \leqslant \frac{k^2}{2}(\varphi'(a)+\varphi'(b)),\ a,b>0, \tag{10}$$

since $\varphi(2x) \leqslant k\varphi(x)$ implies that $\varphi'(2x) \leqslant \frac{k^2}{2}\varphi'(x)$.

Using (10), the above expression is bounded by

$$\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)<f\}} \varphi'\big(f(y)-f_\varepsilon(x)\big)\,dy + \frac{k^2}{2}\varphi'(f_\varepsilon(x))+ \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\geqslant f\}} \varphi'(f(y))\,dy. \tag{11}$$

Now by the part (a) of Definition 1.1 we have that (11) is bounded by

$$\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\geqslant f\}} \varphi'\big(f(y)-f_\varepsilon(x)\big)\,dy + \frac{k^2}{2}\varphi'(f_\varepsilon(x))+ \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\geqslant f\}} \varphi'(f(y))\,dy, \tag{12}$$

and assuming $K \geqslant \sqrt{2}$ we get that the equation (12) is bounded by

$$\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\geqslant f\}} \big(\varphi'(f_\varepsilon(x)-f(y))+\varphi'(f(y))\big)\,dy+ \frac{k^2}{2}\varphi'(f_\varepsilon(x)). \tag{13}$$

Since in the above integral $f_\varepsilon(x)-f(y) \geqslant 0$ and $f(y) \geqslant 0$, we have $\varphi'(f_\varepsilon(x)-f(y))+\varphi'(f(y)) \leqslant 2\varphi'\big((f_\varepsilon(x)-f(y))+f(y)\big)$. Then the equation (13) is bounded by $\frac{3}{2}k^2\varphi'(f_\varepsilon(x))$. Thus

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(f(y))\,dy \leqslant c\varphi'(f_\varepsilon(x)),$$

with $c = \frac{3}{2}k^2$.

The left hand side of inequality (8) was proved in [7]. For the sake of completeness we include it here.

We show first that given $f_\varepsilon(x) \in \tilde{\mu}_\varepsilon(f)(x)$ there exists $c_\varepsilon \in \tilde{\mu}_\varepsilon(|f|)(x)$ such that $|f_\varepsilon(x)| \leqslant c_\varepsilon$. Since $|f| \geqslant f \geqslant -|f|$ and the fact that the extended best approximation operator is a monotone operator , see Lemma 12 of [7], there exist $a_\varepsilon \geqslant 0$ and $b_\varepsilon \geqslant 0$ with $-a_\varepsilon \in \tilde{\mu}_\varepsilon(-|f|)(x)$ and $b_\varepsilon \in \tilde{\mu}_\varepsilon(|f|)(x)$ such that $-a_\varepsilon \leqslant f_\varepsilon(x) \leqslant b_\varepsilon$. But now $a_\varepsilon \in \tilde{\mu}_\varepsilon(|f|)(x)$, and $c_\varepsilon = max(a_\varepsilon, b_\varepsilon) \in \tilde{\mu}_\varepsilon(|f|)(x)$, since this set is a closed interval. Observe that we have proved above that $\mathcal{M}f \leqslant \mathcal{M}(|f|)$, see (3). Now we may assume $f \geqslant 0$.

Now $\varphi'(f_\varepsilon(x))$ can be written as

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} \varphi'(f_\varepsilon(x))\,dy = \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'((f_\varepsilon(x)-f(y))+f(y))\,dy+ \frac{1}{|B_\varepsilon|}\int_{|B_\varepsilon|\cap\{f_\varepsilon(x)\leqslant f\}} \varphi'(f_\varepsilon(x))\,dy,$$

and using (10) the above expression can be estimated by

$$\begin{aligned} &\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'(f_\varepsilon(x)-f(y))\,dy+ \\ &\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'(f(y))\,dy + \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)\leqslant f\}} \varphi'(f_\varepsilon(x))\,dy. \end{aligned} \tag{14}$$

Since b) of Definition 1.1 we have

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'(f_\varepsilon(x)-f(y))\,dy \leqslant \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)\leqslant f\}} \varphi'(f(y)-f_\varepsilon(x))\,dy.$$

Then assuming $\frac{k^2}{2} > 1$ we can estimate (14) by

$$\begin{aligned} &\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)\leqslant f\}} \varphi'(f(y)-f_\varepsilon(x))+\varphi'(f_\varepsilon(x))\,dy+ \\ &\frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'(f(y))\,dy. \end{aligned} \tag{15}$$

Thus (15) is bounded by

$$k^2\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)\leqslant f\}} \varphi'(f(y))\,dy + \frac{k^2}{2}\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon\cap\{f_\varepsilon(x)>f\}} \varphi'(f(y))\,dy,$$

which is bounded by

$$k^2\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} \varphi'(f(y))\,dy.$$

To prove part (9), observe that if $f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)$, then $f_\varepsilon(x) - f(x) \in \mu^\varepsilon_{\varphi'}(f - f(x))$. Thus applying part (8) to the function $f - f(x)$, the proof is completed. ∎

The constant C in the above theorem is $\frac{3}{2}k^2$ where k is the constant given for the Δ_2 condition of φ, thus this constant C is independent of the dimension n.

The left hand side of the inequality (8) and inequality (9) of the Theorem 2.1 were proved in [7], Theorem 3, where the unnecessary hypothesis $\varphi'(0) = 0$ was asked but not used in the proof.

We will use the following result given in [10].

THEOREM 2.2 *A function $\psi \in \Phi$ satisfies the ∇_2 condition if and only if*

$$\int_{\mathbb{R}^n} \psi(Mf(x))\,dx \leqslant C \int_{\mathbb{R}^n} \psi(C|f(x)|)\,dx, \quad f \in L^1_{loc}(\mathbb{R}^n), \tag{16}$$

with the constant C independent of f.

Although this result was set there for an increasing function ψ it also holds for a nondecreasing function ψ.

We denote, according to [11], by ${\varphi'}^{-1}$ for the generalized inverse function of the monotone function φ'.

Now we give a necessary and sufficient condition in order to get a continuity property for the maximal operator $\mathcal{M}|f|$.

THEOREM 2.3 *Let $\varphi \in \Phi \cap \Delta_2$ be a C^1 convex function and let φ' be such that $A\varphi'(t) \leqslant \varphi'(Kt)$, $t \geqslant 0$, for some constants $K, A > 1$. Now for a function $\theta \in \Phi \cap \Delta_2$ we have that the function $\theta \circ {\varphi'}^{-1}$ satisfies a ∇_2 condition if and only if there exists a constant C independent of f such that*

$$\int_{\mathbb{R}^n} \theta((\mathcal{M}|f|)(x))\,dx \leqslant C \int_{\mathbb{R}^n} \theta(|f(x)|)\,dx, \quad f \in L^{\varphi'}_{loc}(\mathbb{R}^n). \tag{17}$$

PROOF First we compare the maximal operator $\mathcal{M}(|f|)$ with the Hardy-Littlewood Maximal Operator $M(f)$ by proving

$$\frac{1}{2}{\varphi'}^{-1}\Big(\frac{1}{C}M(\varphi'(|f|))(x)\Big) \leqslant \mathcal{M}(|f|)(x) \leqslant {\varphi'}^{-1}(CM(\varphi')(|f|)(x)). \tag{18}$$

Since ${\varphi'}^{-1}$ is the generalized inverse of φ' we have $x \leqslant {\varphi'}^{-1}(\varphi')(x)$. Thus, by Theorem 2.1, we have

$$|f|_\varepsilon(x) \leqslant {\varphi'}^{-1}\big(C\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} \varphi'(|f(y)|)\,dy\big). \tag{19}$$

Then $\mathcal{M}(|f|)(x) \leqslant {\varphi'}^{-1}(CM(\varphi')(|f|)(x))$.

Now by Theorem 2.1 we also have

$$\frac{1}{C}M(\varphi'(|f|))(x) \leqslant \varphi'(\mathcal{M}(|f|))(x). \tag{20}$$

Note that if we assume, for simplicity, $K = 2$ in the hypothesis of this theorem, we have $\varphi'(x) < A\varphi'(x) \leqslant \varphi'(2x)$, and then clearly $\varepsilon = \varphi'(x) - \varphi'(\frac{x}{2}) > 0$, for

$\varepsilon > 0$. It is known that $\varphi'^{-1}(\varphi'(x) - \varepsilon) \leqslant x$, for every $\varepsilon < \varphi'(x)$. Thus we have $\varphi'^{-1}(\varphi'(x)) \leqslant 2x$. Now, using inequality (20) we have $\varphi'^{-1}(\frac{1}{C}M(\varphi'(|f|))(x)) \leqslant 2\mathcal{M}(|f|)(x)$.

Now we assume that the maximal function $\mathcal{M}(|f|)$ satisfies (17). Since $\theta \in \Delta_2$ we also have

$$\int_{\mathbb{R}^n} \theta(2\mathcal{M}|f|(x))\,dx \leqslant K_1 \int_{\mathbb{R}^n} \theta(K_1|f(x)|)\,dx, \quad f \in L^{\varphi'}_{loc}(\mathbb{R}^n), \tag{21}$$

for some constant K_1 independent of f. Now using (18) and the fact that φ' is also a Δ_2 function we have

$$\int_{\mathbb{R}^n} \theta(\varphi'^{-1}(\frac{1}{C}M(\varphi'(|f(x)|)))))\,dx \leqslant K_1 \int_{\mathbb{R}^n} \theta(\varphi'^{-1}(\varphi'(K_1|f(x)|))))\,dx \leqslant$$

$$K_2 \int_{\mathbb{R}^n} \theta(\varphi'^{-1}K_2(\varphi'(|f(x)|))))\,dx, \quad f \in L^{\varphi'}_{loc}(\mathbb{R}^n),$$

with K_2 independent of f. Thus for $\psi = \theta \circ \varphi'^{-1}$ we have

$$\int_{\mathbb{R}^n} \psi(M(g)(x))\,dx \leqslant K_3 \int_{\mathbb{R}^n} \psi(K_3 g(x))\,dx, \tag{22}$$

where g is obtained as $\frac{1}{C}\varphi'(|f|)$ for any $f \in L^{\varphi'}_{loc}(\mathbb{R}^n)$. In order to get (22) for all $g \in L^1_{loc}(\mathbb{R}^n)$ set $f = \varphi'^{-1}(Cg)$ and use the fact that $\varphi'(\varphi'^{-1}(x)) = x$. Now by Theorem 2.2 we have that $\theta \circ \varphi'^{-1} \in \nabla_2$.

On the other hand if $\theta \circ \varphi'^{-1} = \psi \in \nabla_2$ by Theorem 2.2 we get

$$\int_{\mathbb{R}^n} \psi(M(g)(x))\,dx \leqslant C \int_{\mathbb{R}^n} \psi(Cg(x))\,dx.$$

Now by (18) we have that $\mathcal{M}(|f|)(x) \leqslant \varphi'^{-1}(CM(\varphi'(|f(x)|)))$, and we may use the same constant C in both inequalities. Thus

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}(|f|(x)))\ dx \leqslant \int_{\mathbb{R}^n} \theta(\varphi'^{-1}(CM(\varphi'(|f(x)|))))\,dx \leqslant$$

$$C \int_{\mathbb{R}^n} \psi(C^2\varphi'(|f(x)|))\,dx \leqslant C_1 \int_{\mathbb{R}^n} \psi(C_1\varphi'(|f(x)|))\,dx,$$

and using the hypothesis on φ' we have $C_1\varphi'(x) \leqslant \varphi'(C_2 x)$, where $C_2 = 2^l$ for l such that $A^l > C_1$, we have

$$C_1 \int_{\mathbb{R}^n} \psi(C_1\varphi'(|f(x)|))\,dx \leqslant C_2 \int_{\mathbb{R}^n} \psi(\varphi'(C_2|f(x)|))\,dx.$$

Now recalling that $\varphi'^{-1}\varphi'(x) \leqslant 2x$, we have

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}(|f|(x)))\,dx \leqslant C_3 \int_{\mathbb{R}^n} \theta(|f(x)|)\,dx.$$

The condition $A\varphi'(t) \leqslant \varphi'(Kt),\ t \geqslant 0$, for some constants $K, A > 1$ implies that $\varphi'(0+) = 0$ and $\varphi'(x) \to \infty$ as $x \to \infty$. Thus the generalized inverse function φ'^{-1} is real valued and even more $\varphi'^{-1} \in \Phi$.

Note that if φ' is any function in Φ that satisfies $t^p \leqslant \varphi'(t) \leqslant ct^p$, then $\varphi'(Kt) \geqslant A\varphi'(t),\ t \geqslant 0$ for any K such that $A = \frac{K^p}{c} > 1$. Thus the above Theorem allow us to consider $\varphi' \in \Phi$ which is not an strictly increasing function. In this case the set $\tilde{\mu}_\varepsilon(f)(x)$ may have more than one element.

REMARK 2.4 Observe that using Theorem 2.3 and (1) we have (2), that is

$$\int_{\mathbb{R}^n} \theta(|f(y) - f_\varepsilon(y)|)\, dy \to 0,\ \ as\ \varepsilon \to 0,$$

for the function θ given in Theorem 2.3.

THEOREM 2.5 *Let $\varphi \in \Phi \cap \Delta_2$ be a C^1 convex function and let φ' be such that $A\varphi'(t) \leqslant \varphi'(Kt),\ t \geqslant 0$, for some constants $K, A > 1$. Then for all $f \in L^\varphi(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \varphi((\mathcal{M}_\varphi|f|)(x))\, dx \leqslant C \int_{\mathbb{R}^n} \varphi(|f(x)|)\, dx, \tag{23}$$

where the constant C is independent of f.

PROOF By Theorem 2.3 we only need to show that $\eta = \varphi \circ \varphi'^{-1}$ is a ∇_2 function assuming that φ is a Δ_2 function. Using b) of Proposition 3.3 we need to show that there exist $\beta > 1$ and $K > 1$ such that

$$\frac{\eta(t_1)}{t_1^\beta} \leqslant K^\beta \frac{\eta(Kt_2)}{t_2^\beta},\ if\ 0 < t_1 < t_2. \tag{24}$$

We introduce the complementary function of φ setting

$$\psi(x) = \int_0^x \varphi'^{-1}(t)\, dt. \tag{25}$$

Then

$$\psi(x) \leqslant x\varphi'^{-1}(x) \leqslant \psi(2x). \tag{26}$$

And also

$$\varphi(x) \leqslant x\varphi'(x) \leqslant \varphi(2x). \tag{27}$$

Setting $x = \varphi'^{-1}(y)$ in (27) and using that $\varphi'(\varphi'^{-1}(y)) = y$ we have

$$\varphi(\varphi'^{-1}(y)) \leqslant y\varphi'^{-1}(y) \leqslant \varphi(2(\varphi'^{-1}(y))). \tag{28}$$

By (27) and (26) we have

$$\varphi(\varphi'^{-1}(y)) \leqslant y\varphi'^{-1}(y) \leqslant \psi(2y). \tag{29}$$

Now by (28) and (26)

$$\psi(y) \leqslant y\varphi'^{-1}(y) \leqslant \varphi(2(\varphi'^{-1}(y))). \tag{30}$$

Now we will find the constants β and K in (24). By (29) we have

$$\frac{\varphi(\varphi'^{-1}(t_1))}{t_1^\beta} \leqslant \frac{\psi(2t_1)}{t_1^\beta}. \tag{31}$$

Now we use b) of Proposition 3.3 for the function ψ which we know is a ∇_2 function since φ is a Δ_2 function, see [20]. Thus there exist constants K and β such that

$$\frac{\psi(2t_1)}{t_1^\beta} \leqslant (2K)^\beta \frac{\psi(Kt_2)}{t_2^\beta}, \;\; 0 < t_1 < t_2. \tag{32}$$

By (30) we have

$$\frac{\psi(2t_1)}{t_1^\beta} \leqslant (2K)^\beta \frac{\varphi(2(\varphi'^{-1}(Kt_2))}{t_2^\beta} \leqslant (K_1)^\beta \frac{\varphi((\varphi'^{-1}(Kt_2))}{t_2^\beta},$$

where the last inequality follows since φ is a Δ_2 function. Finally, by (31) for a suitable constant K_2, we get

$$\frac{\varphi((\varphi'^{-1}(t_1))}{t_1^\beta} \leqslant K_2^\beta \frac{\varphi((\varphi'^{-1}(K_2t_2))}{t_2^\beta},$$

and the Theorem follows. ∎

In [7] we have also obtained (23) with different conditions on the function φ. In that paper the ∇_2 condition was asked on φ and in the above Theorem we need that $A\varphi'(t) \leqslant \varphi'(Kt), \;\; t \geqslant 0$, for some constants $K, A > 1$. We do not know how these two conditions are related.

According to [11], a convex function $\varphi \in \Phi$ is called an N-function if its right derivative φ' is also in Φ, and the function $\varphi \in \Phi$ is named a Young function if it is a convex function.

If φ is a Young function and $f \in L^\varphi$ we denote by $\|f\|_\varphi$ the Luxemburg norm of f, which is defined by

$$\|f\|_\varphi = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leqslant 1\}.$$

The following result is a direct consequence of (1) and Theorem 2.5. We observe that the hypothesis of Theorem 2.5 implies $\varphi'(0) = 0$, then we have (1) by (9) in Theorem 2.1. The pointwise convergence of $f_\varepsilon(x)$ to $f(x)$ has been considered in many situations since a long time ago, for example see [22] or [5]. On the other hand the norm convergence is well known only for the case $\varphi(t) = t^2$. For other powers and even for more general functions φ the first result we know is settled in [7].

THEOREM 2.6 *Let $\varphi \in \Phi \cap \Delta_2$ be a C^1 strictly convex function and let φ' be such that $A\varphi'(t) \leqslant \varphi'(Kt),\ t \geqslant 0$, for some constants $K, A > 1$. Let $f \in L^{\varphi}(\mathbb{R}^n)$ and $f_\varepsilon(x)$ the unique element of $\mu_\varepsilon(f)(x)$, then $\|f_\varepsilon - f\|_\varphi \to 0$ as $\varepsilon \to 0$.*

We note that in the above theorem the convergence $\|f_\varepsilon - f\|_\varphi \to 0$ as $\varepsilon \to 0$ can be replaced by $\|f_\varepsilon - f\|_\theta \to 0$, where θ is a convex function satisfying the hypothesis of Theorem 2.3.

In the next Remark we use T for a positive homogeneous operator defined on $L^\theta(\mathbb{R}^n)$ and taking values on the space of measurable functions from $\mathbb{R}^n$ into $\overline{\mathbb{R}}$. An example of operator T can be the operator M_p that appears in Corollary 2.8.

REMARK 2.7 Let θ be a Young function such that

$$\int_{\mathbb{R}^n} \theta(|T(f)(x)|)\,dx \leqslant K \int_{\mathbb{R}^n} \theta(K|f(x)|)\,dx. \tag{33}$$

for every $f \in L^\theta$ and a constant K independent of f. Then there exists a constant C such that for every $f \in L^\theta$ we have

$$\|T(f)\|_\theta \leqslant C\|f\|_\theta. \tag{34}$$

More precisely $C = \max(1, K^2)$.

PROOF Assume $K \geqslant 1$ and use $\theta(\frac{1}{K}x) \leqslant \frac{1}{K}\theta(x)$. Thus

$$\int_{\mathbb{R}^n} \theta(\frac{|T(f)(x)|}{K^2\|f\|_\theta})\,dx \leqslant K \int_{\mathbb{R}^n} \theta(\frac{K|f(x)|}{K^2\|f\|_\theta})\,dx = K \int_{\mathbb{R}^n} \theta(\frac{|f(x)|}{K\|f\|_\theta})\,dx$$

$$\leqslant \int_{\mathbb{R}^n} \theta(\frac{|f(x)|}{\|f\|_\theta})\,dx \leqslant 1.$$

Then $C = K^2$. If $K < 1$ we use the same argument assuming (33) with $K = 1$.■

We observe that assuming inequality (16) for any function $f \in L^\psi(\mathbb{R}^n)$, where ψ is a Δ_2 Young function, we can use Remark 2.7 to obtain

$$\|Mf\|_\psi \leqslant C\|f\|_\psi, \tag{35}$$

for $f \in L^\psi(\mathbb{R}^n)$, and where the constant C is independent of f. As a consequence of the Lorentz-Shimogaki theorem see for example [1], the inequality (35) holds if and only if the superior Boyd index $\overline{\alpha}_\varphi < 1$. A Gallardo's result says that given an N function ψ, equation (35) holds if and only if $\psi \in \nabla_2$, see [8]. Henceforth the ∇_2 condition for an N function ψ is equivalent to $\overline{\alpha}_\varphi < 1$. For an explicit expression of the Boyd index $\overline{\alpha}_\varphi$ we refer to [1] and [13]. Both results are given for a convex function ψ while Theorem 2.2 requires only a function $\psi \in \Phi$. In this paper we deal mostly with inequalities of the type (16).

COROLLARY 2.8 *Let θ be a function in Φ and $p > 0$. Then*

$$\int_{\mathbb{R}^n} \theta(M_p(f)(x))\,dx \leqslant K \int_{\mathbb{R}^n} \theta(K|f(x)|)\,dx \tag{36}$$

for all $f \in L^p_{loc}(\mathbb{R}^n)$, if and only if $\theta(t^{1/p}) \in \nabla_2$.

Corollary 2.8 is an immediate consequence of Theorem 2.2 and gives an improved version, in a particular case, of Theorem 2.2 in [3]. Here we consider the operator M_p for $p > 0$ and we impose the conditions on θ not on θ' as it was considered in [3]. Also we will see, for a Young function θ, that inequality (36) implies

$$\|M_p(f)\|_\theta \leqslant C\|f\|_\theta, \tag{37}$$

for every $f \in L^\theta$ and some constant C independent of f, see Remark 2.7

Similar equivalences to those of Corollary 2.8 for inequalities of the type

$$\int_{|x|\leqslant 1} \theta(M_p(f)(x))\,dx \leqslant K\Big(1 + \int_{|x|\leqslant 1} \theta(K|f(x)|)\,dx\Big), \tag{38}$$

are given in [3] and also in [9] ($p = 1$).

The fact $\theta(t^{1/p}) \in \nabla_2$, used in Corollary 2.8, is considered in detail in Section 3, where several equivalent conditions are given, see Lemma 3.7, 3.11 and 3.12.

Now we use the remaining of the section to prove the convergence result stated in (1) for the case $\varphi'(0) > 0$. This result was proved in Theorem 9 of [7] where the two last lines of the proof are not correct.

For a function f such that $|\{f \neq 0\}| < \infty$ and $\int_{\{f\neq 0\}} \varphi'(|f|)\,dx < \infty$ we consider the next operator.

$$\Gamma f(x) = \limsup_{\varepsilon\to 0}(\sup\{|f_\varepsilon(x) - f(x)| : f_\varepsilon(x) \in \tilde{\mu}_\varepsilon(f)(x)\}).$$

Our purpose is to prove that $\Gamma f(x) = 0$, almost every $x \in \mathbb{R}^n$. We observe that given a step function s, then for almost everywhere x, there exists an $\varepsilon(x)$ such that for every ε, $0 < \varepsilon < \varepsilon(x)$, we have $f_\varepsilon(x) = (f - s)_\varepsilon(x) + s(x)$. Here we have used that for a constant c it holds $(f + c)_\varepsilon(x) = f_\varepsilon(x) + c$. Now we estimate Γf by

$$|\{\Gamma f > t\}|^* \leqslant |\{\mathcal{M}(f - s) > t/2\}|^* + |\{|f - s| > t/2\}|. \tag{39}$$

The following weak type inequality was obtained in Theorem 8 of [7],

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}|^* \leqslant \frac{C}{\varphi'(0)} \int_{\{|f|>t\}} \varphi'(|f(y)|)\,dy \tag{40}$$

where the constant C is independent of f and the $*$ means the outer Lebesgue measure. Now using (40) and the Tchebyshev inequality in (39) we have

$$|\{\Gamma f > t\}|^* \leqslant \frac{C}{\varphi'(0)} \int_{\{|f-s|>t/2\}} \varphi'(|f - s|)\,dy + \frac{C}{\varphi'(t/2)} \int_{\{|f-s|>t/2\}} \varphi'(|f - s|)\,dy.$$

That is we need to estimate $\int_{\{|f-s|>t/2\}} \varphi'(|f-s|)\,dy$. Thus we split the above integral as follows.

$$\int_{\{|f-s|>t/2\}} \varphi'(|f-g|)\,dy + \int_{\{|f-s|>t/2\}} \varphi'(|g-s|)\,dy = I + J.$$

Where g is a simple function such that $|g| \leqslant |f|$ and

$$\int_{\{|f-g|>t\}\cap\mathcal{O}} \varphi'(|f|)\,dy < \varepsilon,$$

and $\mathcal{O}$ is an open set of finite measure such that $\{f \neq 0\} \subset \mathcal{O}$.

Taking into account that $\varphi'(|f-g|) \leqslant \varphi'(2|f|) \leqslant C\varphi'(|f|)$, for a fixed g we estimate I by

$$C\int_{\{|f-g|>t/4\}\cap\mathcal{O}} \varphi'(|f|)\,dy + C\int_{\{|g-s|>t/4\}\cap\mathcal{O}} \varphi'(|f|)\,dy. \tag{41}$$

Now, for the fixed simple function $g = \sum_{i=1}^{l} \lambda_i \chi_{E_i}$ with $E_i \subset \mathcal{O}$, we choose $s = \sum_{i=1}^{l} \lambda_i \chi_{I_i}$, where I_i is a finite union of intervals contained in $\mathcal{O}$, in such a way that the measure of $\cup_{i=1}^{l}(E_i \triangle I_i)$ is small enough. Thus the second integral in (41) will be less than ε.

In order to estimate J we observe that

$$\varphi'(|g(x)-s(x)|) \leqslant C_l\Big(\sum_{i=1}^{l} \varphi'(|\lambda_i|) + \varphi'(0)\Big),$$

where the constant C_l depends on the Δ_2 condition of the function φ'. Then we also must choose $\cup_{i=1}^{l}(E_i \triangle I_i)$ such that

$$|\cup_{i=1}^{l}(E_i \triangle I_i)|\Big(\sum_{i=1}^{l} \varphi'(|\lambda_i|) + \varphi'(0)\Big) < \varepsilon. \tag{42}$$

In fact we split the integral J on the two following regions $|s| \leqslant |g|$ and $|s| > |g|$. The first one is treated as the integral I and the second one is bounded by (42).

3. About the ∇_2 condition.. For a function $\eta \in \Phi$ we will prove some equivalences to the inequality (6), that is $\eta \in \nabla_2$, and observe that in the literature the symbol ∇_2 is only used for convex functions. The next lemma is quoted as **3.4.2** on page 19 of [19]. Besides it is known that for a N-function η the next inequality (43) is equivalent to (7) which is used as a definition of the ∇_2 condition, see [20].

LEMMA 3.1 *For an absolutely continuous function $\eta : (0,\infty) \to (0,\infty)$, the following statements are equivalents:*

a) There exists $\beta > 1$ such that

$$t\frac{\eta'(t)}{\eta(t)} \geqslant \beta, \tag{43}$$

for almost every $t > 0$.

b) There exists $\beta > 1$ *such that*

$$\frac{\eta(t_1)}{t_1^{\beta}} \leqslant \frac{\eta(t_2)}{t_2^{\beta}}, \tag{44}$$

for every $0 < t_1 < t_2$.

Proof Given $l > 1$ and assuming a) we have for $t > 0$,

$$\log \frac{\eta(lt)}{\eta(t)} = \int_t^{lt} \frac{\eta'(s)}{\eta(s)}\, ds \geqslant \int_t^{lt} \frac{\beta}{s}\, ds = \beta \log l.$$

Thus we have

$$\frac{\eta(lt)}{\eta(t)} \geqslant l^{\beta} = \frac{(lt)^{\beta}}{t^{\beta}},$$

for every $l > 1$ and $t > 0$, and so we have b).

On the other hand, given $0 < t < s$ the statement b) implies

$$\int_t^s \frac{\eta'(r)}{\eta(r)}\, dr = \log \frac{\eta(s)}{\eta(t)} \geqslant \log(\frac{s}{t})^{\beta} = \int_t^s \frac{\beta}{r}\, dr.$$

Now using the Lebesgue differentiation theorem we obtain $\frac{\eta'(t)}{\eta(t)} \geqslant \frac{\beta}{t}$ a.e. t. ∎

The following result can be found in [10], page 7.

Lemma 3.2 *Let* η *be a function in* Φ *and suppose that there exists* $\alpha > 1$ *such that* $2\alpha\eta(t) < \eta(\alpha t)$ *for* $t > 0$. *Then there exists* $\alpha_1 \in (0, 1)$ *such that*

$$2\alpha^2 \eta^{\alpha_1}(t) < \eta^{\alpha_1}(\alpha^2 t),$$

for $t > 0$.

Now we can proof.

Proposition 3.3 *For a function* $\eta \in \Phi$ *the following statements are equivalent*

a)There exists $\alpha > 1$ *such that*

$$\eta(t) < \frac{1}{2\alpha}\eta(\alpha t),$$

for all $t > 0$.

ã)There exists $\tilde{\alpha} > 1$ *such that*

$$\eta(t) \leqslant \frac{1}{2\tilde{\alpha}}\eta(\tilde{\alpha} t),$$

for all $t > 0$.

b)There exists $\beta > 1$ and $K \geqslant 1$ such that

$$\frac{\eta(t_1)}{t_1^\beta} \leqslant K^\beta \frac{\eta(Kt_2)}{t_2^\beta},$$

for $0 < t_1 < t_2$.

c)There exists a positive constant C such that

$$\int_0^t \frac{\eta(s)}{s^2} ds \leqslant C\frac{\eta(Ct)}{t}, \qquad 0 < t < \infty.$$

Proof First we prove a) implies b). By Lemma 3.2 there exists a constant $\alpha_2 \in (0,1)$ such that

$$\eta^{\alpha_2}(t) < \frac{1}{2a_1}\eta^{\alpha_2}(a_1 t),$$

where $a_1 = \alpha^2$ and $t > 0$. Then applying Lemma 1.2.3 (p.7) of [10] η^{α_2} is a quasi convex function and we get

$$\frac{\eta^{\alpha_2}(t_1)}{t_1} \leqslant K\frac{\eta^{\alpha_2}(Kt_2)}{t_2}, \tag{45}$$

if $0 < t_1 < t_2$. See Lemma 1.1.1 in [10] where it is proved that condition (45) is equivalent to the concept of quasi convex function given by these authors. Then we have

$$\frac{\eta(t_1)}{t_1^{\frac{1}{\alpha_2}}} \leqslant K^{\frac{1}{\alpha_2}} \frac{\eta(Kt_2)}{t_2^{\frac{1}{\alpha_2}}},$$

and b) holds with $\beta = \frac{1}{\alpha_2}$.

Now we assume b) and we have for $\alpha > 1$

$$\frac{\eta(K\alpha t)}{\eta(t)} \geqslant \frac{(\alpha t)^\beta}{(Kt)^\beta} = \Big(\frac{\alpha}{K}\Big)^\beta \geqslant 2K\alpha,$$

the last inequality holds if we select α such that $\alpha \geqslant 2^{\frac{1}{\beta-1}} K^{\frac{\beta+1}{\beta-1}}$. Thus we have $\tilde{a}$) with the constant $\tilde{\alpha} = K\alpha$.

We prove that $\tilde{a}$) implies a). In fact, we have $2\tilde{\alpha}\eta(\tilde{\alpha}t) \leqslant \eta(\tilde{\alpha}^2 t)$ for $t > 0$. Then $\tilde{\alpha}\eta(\tilde{\alpha}t) < 2\tilde{\alpha}\eta(\tilde{\alpha}t) \leqslant \eta(\tilde{\alpha}^2 t)$, and $t > 0$. Therefore $\eta(t) \leqslant \frac{1}{2\tilde{\alpha}}\eta(\tilde{\alpha}t) < \frac{1}{2\tilde{\alpha}^2}\eta(\tilde{\alpha}^2 t)$, for any $t > 0$, that is we have a) with constant $\alpha = \tilde{\alpha}^2$. We have used that $\eta(s) > 0$ for $\eta \in \Phi$ and $s > 0$. For a proof of $a) \Leftrightarrow c)$, see [10]. ∎

Remark 3.4 If we assume that η is also a convex function the constant K in Proposition 3.3 is equal to 1.

In fact when η is a convex function property $\tilde{a}$) in Proposition 3.3 is equivalent to property a) in Lemma 3.1, see page 23 in [20], which is equivalent to $t_2^\beta \eta(t_1) \leqslant t_1^\beta \eta(t_2),\ t_1 < t_2$.

REMARK 3.5 If η is a differentiable function in Φ, then condition c) of Proposition 3.3 is equivalent to the following inequality

$$\int_0^t \frac{\eta'(s)}{s}\, ds \leqslant C\, \frac{\eta(Ct)}{t},$$

for some constant $C > 0$. See [10], page 6.

We analyze in more details the condition $\theta(t^{\frac{1}{p}}) \in \nabla_2$, which appears in Corollary 2.8. For this purpose it will be useful to introduce the following definition.

DEFINITION 3.6 For $p > 0$ we will say that a function $\eta : (0,\infty) \to (0,\infty)$ satisfies a ∇_{p+1} condition ($\eta \in \nabla_{p+1}$) if there exists $\beta > 1$ such that

$$\eta(t) < \frac{1}{2\beta^p}\eta(\beta t), \tag{46}$$

for all $t > 0$.

In the literature the ∇_{p+1} is only considered for $p = 1$, that is the classical ∇_2 condition. Also note that $\eta(t^{\frac{1}{p}}) \in \nabla_2$ if and only if $\eta \in \nabla_{p+1}$. The definition 3.6 is meaningful for all $p \in \mathbb{R}$, though if $\eta \in \Phi$ always we have $\eta \in \nabla_{p+1}$, for $p < 0$. Also it follows directly from the definitions that $\frac{\eta(t)}{t^r} \in \nabla_2 \Leftrightarrow \eta \in \nabla_{r+2},\ r > -1$ and $\frac{\eta(t)}{t^{p-1}} \in \nabla_2 \Leftrightarrow \eta \in \nabla_{p+1},\ p > 0$.

LEMMA 3.7 *Given $\eta \in \Phi$ and $p > 0$ the following statements are equivalent*

a) The function η satisfies a ∇_{p+1} condition.

b) There exists $C > 0$ such that

$$\int_0^t \frac{\eta(s)}{s^{p+1}}\, ds \leqslant \frac{C^p}{p}\frac{\eta(Ct)}{t^p}, \tag{47}$$

for $t > 0$.

c) There exists $\beta > 1$ and $K > 0$ such that

$$\frac{\eta(s)}{s^{\beta p}} \leqslant K^p \frac{\eta(Kt)}{t^{\beta p}}, \tag{48}$$

for $s < t$.

If η is a differentiable function and the constant K in the above statement is one, then c) is equivalent to

d) There exists $\beta > 1$ such that

$$\frac{\eta'(t)t}{\eta(t)} > \beta p. \tag{49}$$

PROOF Since a) is equivalent to $\eta(t^{\frac{1}{p}}) \in \nabla_2$, then use Proposition 3.3 to see that a) is equivalent to b) and c). Then use Lemma 3.1 for the equivalence with d). ∎

Note that if we assume that $\eta(t^{\frac{1}{p}})$ is a ∇_2 convex function the statement c) holds with $K = 1$, see Remark 3.4. Besides in this case the equivalence c) and d) appears in Theorem 3.2 of [14].

For a pair of functions η and ψ in Φ we make some considerations about the ∇_2 condition on $\eta \circ \psi$.

REMARK 3.8 If a function $\eta \in \Phi$ satisfies the ∇_2 condition, then the function $\eta \circ \psi$ satisfies the ∇_2 condition, for any convex function ψ in Φ.

PROOF Using the ∇_2 condition on the function η applied on $\psi(t)$ we get

$$\eta(\psi(t)) < \frac{1}{2\alpha}\eta(\alpha\psi(t))$$

and now by the convexity on ψ we obtain

$$\eta(\psi(t)) < \frac{1}{2\alpha}\eta(\psi(\alpha t)).$$

The next Remark follows using that $\psi(at) \leqslant a\psi(t)$, for a concave function $\psi \in \Phi$ and $a > 1$.

REMARK 3.9 Consider a function $\eta \in \Phi$ and a concave function $\psi \in \Phi$. If $\eta \circ \psi$ satisfies the ∇_2 condition, then η also satisfies the ∇_2 condition.

Given a function $\eta \in \Phi$ such that $\frac{\eta(t)}{t} \to \infty$ for $t \to \infty$ we consider the complementary function $\eta^* \in \Phi$ defined by

$$\eta^*(t) = \sup_{s \geqslant 0}(ts - \eta(s)).$$

In some specific cases the function η^* coincides with the complementary function used in (25). The basic properties of η^* can be found in [13].

REMARK 3.10 Note that for functions $\eta_1, \eta_2 \in \Phi$ such that $\eta_1(t) \leqslant \eta_2(t)$ for every $t \geqslant 0$, it follows straightforward that $\eta_2^*(t) \leqslant \eta_1^*(t)$ for every $t \geqslant 0$.

Now we get some properties on η^* assuming the ∇_{p+1} condition on η. First we need the following auxiliary known result.

LEMMA 3.11 *Given a function $\eta \in \Phi$ such that $\frac{\eta(t)}{t} \to \infty$ for $t \to \infty$, set $\eta_1(t) = a\eta(bt)$, for any $a, b > 0$. Then*

$$\eta_1^*(t) = a\eta^*(\frac{t}{ab}).$$

PROOF From the definition of η^* we have

$$\eta_1^*(t) = \sup_{s \geqslant 0}(ts - \eta_1(s)) = \frac{1}{b}\sup_{s \geqslant 0}(tsb - ab\eta(bs)) = a\eta^*(\frac{t}{ab}).$$

PROPOSITION 3.12 *Let η be in Φ such that $\frac{\eta(t)}{t} \to \infty$ for $t \to \infty$ and $\eta \in \nabla_{p+1}$ with $p > 0$. Then*

$$\eta^*(2l^{p-1}t) \leqslant 2l^p\eta^*(t), \tag{50}$$

for any $t \geqslant 0$ and for some $l > 1$.

PROOF Since $\theta(t) = \frac{1}{2l^p}\eta(lt)$ is a function in Φ we obtain, by Lemma 3.11

$$\theta^*(t) = \frac{1}{2l^p}\eta^*(2l^{p-1}t).$$

Now, using Remark 3.10 we get

$$\frac{1}{2l^p}\eta^*(2l^{p-1}t) \leqslant \eta^*(t),$$

for $t \geqslant 0$, which concludes the proof. ∎

Similar arguments of Proposition 3.12 and the fact that $\eta^{**}(t) = \eta(t)$, according to Theorem 8.5, page 54 of [13], are used to prove the following.

REMARK 3.13 Given a convex function $\eta \in \Phi$ such that (50) holds for its conjugate function η^*, then $\eta \in \nabla_{p+1}$.

Observe that the inequality $\eta(2l^{p-1}t) \leqslant 2l^p\eta(t)$ appearing in Proposition 3.12 is a sort of Δ_{p+1} condition for the function η, which is exactly the Δ_2 condition for $p = 1$. For $p > 1$ the condition is stronger than the Δ_2 condition while for $0 < p < 1$ the condition holds for any function $\eta \in \Phi$.

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