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Approximate homomorphisms and derivation in multi-Banach algebras

Abstract. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in multi-Banach algebras and derivations on multi-Banach algebras for the additive functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right)$$

for each $m \in \mathbb{N}$ with $m \geqslant 2$.

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1. Introduction and preliminaries. The stability problem of functional equations originated from a question of Ulam [34] concerning the stability of group homomorphisms:

Let $(G_1, *)$ be a group and $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. For any $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that, if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta(\epsilon)$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

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for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is *stable*. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the functional equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let X and Y be Banach spaces. Assume that $f : X \to Y$ is a mapping satisfying the condition that there exists $\varepsilon \geqslant 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leqslant \varepsilon$$

for all $x \in X$.

Rassias [27] provided a generalization of Hyers' Theorem and also the unbounded Cauchy differences have been considered in [1, 2].

THEOREM 1.1 ([27]) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if, for any $x \in E$, the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms or the stability of the equation of homomorphism was studied by a number of mathematicians. Găvruta [9] generalized the Rassias' result and a number of authors have extensively investigated the stability problems of several functional equations (see [11], [14]–[25], [28]–[30]).

THEOREM 1.2 ([31, 32, 33]) *Let X be a real normed linear space and Y be a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geqslant 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leqslant \frac{\theta}{|2^p - 2|}\|x\|^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

We recall two fundamental results in fixed point theory for our main results in this paper.

THEOREM 1.3 ([3, 4, 26]) *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, that is, there exists a Lipschitz constant $L < 1$ such that*

$$d(Jx, Jy) \leqslant Lf(x, y)$$

for all $x, y \in X$. Then we have the following:

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, that is, $\lim_{n\to\infty} J^n x = x^*$ for any starting point $x \in X$;*

(3) *one has the following estimation inequalities:*

$$d(J^n x, x^*) \leqslant L^n d(x, x^*), \quad d(J^n x, x^*) \leqslant \frac{1}{1-L} d(J^n x, J^{n+1} x),$$

$$d(x, x^*) \leqslant \frac{1}{1-L} d(x, Jx)$$

for all $n \geqslant 0$ and $x \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

THEOREM 1.4 ([8]) *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for any $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all $n \geqslant 0$ or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geqslant n_0$*;*

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

2. Multi-normed spaces. The notion of multi-normed space was introduced by Dales and Polyakov in [5]. This concept is somewhat similar to the operator sequence space and has some connections with the operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [5, 6, 12].

Let $(\mathcal{E}, \|\cdot\|)$ be a complex normed space and let $k \in \mathbb{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ consisting of k-tuples $(x_1, \cdots, x_k)$, where $x_1, \cdots, x_k \in \mathcal{E}$. The linear operations on $\mathcal{E}^k$ are defined coordinate-wise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \cdots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1 A *multi-norm* on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ is a sequence $(\|cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on $\mathcal{E}^k$ for each $k \in \mathbb{N}$ with $k \geqslant 2$ satisfying the following conditions:

(A1) $\|(x_{\sigma(1)}, \cdots, x_{\sigma(k)})\|_k = \|(x_1, \cdots, x_k)\|_k \quad (\sigma \in \Sigma_k,\ \ x_1, \cdots, x_k \in \mathcal{E})$;

(A2) $\|(\alpha_1 x_1, \cdots, \alpha_k x_k)\|_k \leqslant (\max_{i \in \mathbb{N}_k} |\alpha_i|)\, \|(x_1, \cdots, x_k)\|_k$
$(\alpha_1, \cdots, \alpha_k \in \mathbb{C},\ \ x_1, \cdots, x_k \in \mathcal{E})$;

(A3) $\|(x_1, \cdots, x_{k-1}, 0)\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in \mathcal{E})$;

(A4) $\|(x_1, \cdots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \cdots, x_{k-1})\|_{k-1} \quad (x_1, \cdots, x_{k-1} \in \mathcal{E})$.

Then $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a *multi-normed space.*

Lemma 2.2 ([12]) *Suppose that* $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ *is a multi-normed space and take* $k \in \mathbb{N}$*. Then*

(1) $\|(x, \cdots, x)\|_k = \|x\| \quad (x \in \mathcal{E})$*;*

(2) $\max_{i \in \mathbb{N}_k} \|x_i\| \leqslant \|x_1, \cdots, x_k\|_k \leqslant \sum_{i=1}^k \|x_i\| \leqslant k \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \cdots, x_k \in \mathcal{E})$.

It follows from (2) of Lemma 2.2 that, if $(\mathcal{E}, \|\cdot\|)$ is a Banach space, then $(\mathcal{E}^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case, $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ is called a *multi-Banach space.*

Now, we state two important examples of multi-norms for an arbitrary normed space $\mathcal{E}$ (see [5]).

EXAMPLE 2.3 The sequence $(\|\cdot\|_k : k \in \mathbb{N})$ on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ defined by

$$\|x_1, \cdots, x_k\|_k := \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \cdots, x_k \in \mathcal{E})$$

for each $k \in \mathbb{N}$ is a multi-norm, which is called the *minimum multi-norm*. The terminology 'minimum' is justified by the property (2) of Lemma 2.2.

EXAMPLE 2.4 Let $\{(\|\cdot\|_k^\alpha : k \in \mathbb{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{\mathcal{E}^k : k \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, set

$$\|(x_1, \cdots, x_k)\|_k := \sup_{\alpha \in A} \|(x_1, \cdots, x_k)\|_k^\alpha \quad (x_1, \cdots, x_k \in \mathcal{E}).$$

Then $(\|\cdot\|_k : k \in \mathbb{N})$ is a multi-norm on $\{\mathcal{E}^k : k \in \mathbb{N}\}$, which is called the *maximum multi-norm*.

We need the following observation which deduces from (2) of Lemma 2.2.

LEMMA 2.5 *Let $(x_1, \cdots, x_k) \in \mathcal{E}^k$ for each $k \in \mathbb{N}$ and, for each $j \in \{1, \cdots, k\}$, let $(x_n^j)_{n \geqslant 1}$ be a sequence in $\mathcal{E}$ such that $\lim_{n\to\infty} x_n^j = x_j$. Then, for each $(y_1, \cdots, y_k) \in \mathcal{E}^k$, we have*

$$\lim_{n\to\infty} (x_n^1 - y_1, \cdots, x_n^k - y_k) = (x_1 - y_1, \cdots, x_k - y_k).$$

DEFINITION 2.6 Let $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space.

(1) A sequence (x_n) in $\mathcal{E}$ is a *multi-null sequence* if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \|(x_n, \cdots, x_{n+k-1})\|_k < \varepsilon$$

for all $n \geqslant n_0$.

(2) A sequence (x_n) is said to be *multi-convergent* to apoint $x \in \mathcal{E}$ (write $\lim_{n\to\infty} x_n = x$) if $(x_n - x)$ is a multi-null sequence.

DEFINITION 2.7 [5, 13] Let $(A, \|\cdot\|)$ be a normed algebra such that $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. Then $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed algebra if

$$\|(a_1 b_1, \cdots, a_k b_k)\|_k \leqslant \|(a_1, \cdots, a_k)\|_k \cdot \|(b_1, \cdots, b_k)\|_k$$

for all $k \in \mathbb{N}$ and $a_1, \cdots, a_k, b_1, \cdots, b_k \in A$. Further, the multi-normed algebra $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra if $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

EXAMPLE 2.8 [5, 13] Let p, q with $1 \leqslant p \leqslant q < \infty$ and let $A = \ell^p$. The algebra A is a Banach sequence algebra with respect to the coordinatewise multiplication of sequences (see [7]). Let $(\|\cdot\|_k : k \in \mathbb{N})$ be the standard (p, q)-multi-norm on $\{A^k : k \in \mathbb{N}\}$. Then $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach algebra.

3. Stability of homomorphisms in multi-Banach algebras. Let A and B be complex Banach algebras. For any mapping $f : A \to B$, we define

$$D_\mu f(x_1, \cdots, x_m) := \sum_{i=1}^{m} \mu f\left(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\right) + \mu f\left(\sum_{i=1}^{m} x_i\right) - 2f\left(\mu \sum_{i=1}^{m} mx_i\right)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and $x_1, \cdots, x_m \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in multi-Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

Now, we prove the generalized Hyers-Ulam stability of homomorphisms in multi-Banach algebras for the functional equation $D_\mu f(x_1, \cdots, x_m) = 0$.

THEOREM 3.1 *Let A, B be complex Banach algebras and $((B^k, \|.\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^{mk} \to [0, \infty)$ and $\psi : A^{2k} \to [0, \infty)$ such that*

$$\lim_{j\to\infty} m^{-j}\varphi(m^j x_{11}, \cdots, m^j x_{1m}, \cdots, m^j x_{k1}, \cdots, m^j x_{km}) = 0, \tag{2}$$

$$\begin{aligned} &\|(D_\mu f(x_{11}, \cdots, x_{1m}), \cdots, D_\mu f(x_{k1}, \cdots, x_{km}))\|_k \\ \leqslant\ &\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}), \end{aligned} \tag{3}$$

$$\begin{aligned} &\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \\ \leqslant\ &\psi(x_1, y_1, \cdots, x_k, y_k), \end{aligned} \tag{4}$$

$$\lim_{j\to\infty} m^{-2j}\psi(m^j x_1, m^j y_1, \cdots, m^j x_k, m^j y_k) = 0 \tag{5}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, , y_1, \cdots, y_k \in A$. If there exists a constant $L < 1$ such that

$$\begin{aligned} &\varphi(\overbrace{mx_{11}, 0, \cdots, 0}^{m}, \overbrace{mx_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{mx_{k1}, 0, \cdots, 0}^{m}) \\ &\qquad \leqslant mL\varphi(\overbrace{x_{11}, 0, \cdots, 0}^{m}, \overbrace{x_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_{k1}, 0, \cdots, 0}^{m}) \end{aligned}$$

for all $x_{11}, x_{21}, \cdots, x_{k1} \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\begin{aligned} &\|(f(x_1) - H(x_1), \cdots, f(x_k) - H(x_k))\|_k \\ &\qquad \leqslant \frac{1}{m - mL}\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}) \end{aligned} \tag{6}$$

for all $x_1, \cdots, x_k \in A$.

PROOF Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|(g(x_1) - h(x_1), \cdots, g(x_k) - h(x_k))\|_k \leqslant C\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}),\ \forall x_1, \cdots, x_k \in A\}.$$

It is easy to show that (X, d) is complete (see [6]).

Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{m} g(mx)$$

for all $x \in A$. By [3, Theorem 3.1] (also see [6, Lemma 3.2]),

$$d(Jg, Jh) \leqslant Ld(g,h)$$

for all $g, h \in X$. Letting $\mu = 1$, $x_{i1} = x_i$ and $x_{i2} = \cdots = x_{im} = 0$, $1 \leqslant i \leqslant k$, in (3.2), we get

$$\|(f(mx_1) - mf(x_1), \cdots, f(mx_k) - mf(x_k))\|_k \leqslant \varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}) \tag{7}$$

for all $x_1, \cdots, x_k \in A$ and so

$$\|(f(x_1) - \frac{1}{m} f(mx_1), \cdots, f(x_k) - \frac{1}{m} f(mx_k))\|_k \leqslant \frac{1}{m}\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m})$$

for all $x_1, \cdots, x_k \in A$. Hence $d(f, Jf) \leqslant \frac{1}{m}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(mx) = mH(x) \tag{8}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (3.7) such that there exists $C \in (0, \infty)$ satisfying

$$\|(H(x_1) - f(x_1), \cdots, H(x_k) - f(x_k))\|_k \leqslant C\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m})$$

for all $x_1, \cdots, x_k \in A$. (2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^n} = H(x) \tag{9}$$

for all $x \in A$.

(3) $d(f, H) \leqslant \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leqslant \frac{1}{m - mL}.$$

This implies that the inequality (3.5) holds.

It follows from (3.1), (3.2) and (3.8) that

$$\begin{aligned}
&\left\| \left(\sum_{i=1}^{m} H\left(mx_{1i} + \sum_{j=1,\, j\neq i}^{m} x_{1j} \right) + H\left(\sum_{i=1}^{m} x_{1i} \right) - 2H\left(\sum_{i=1}^{m} mx_{1i} \right), \right.\right. \\
&\left.\left. \cdots, \sum_{i=1}^{m} H\left(mx_{ki} + \sum_{j=1,\, j\neq i}^{m} x_{kj} \right) + H\left(\sum_{i=1}^{m} x_{ki} \right) - 2H\left(\sum_{i=1}^{m} mx_{ki} \right) \right) \right\|_k \\
= \lim_{n\to\infty} \frac{1}{m^n} &\left\| \left(\sum_{i=1}^{m} f\left(m^{n+1} x_{1i} + \sum_{j=1,\, j\neq i}^{m} m^n x_{1j} \right) + f\left(\sum_{i=1}^{m} m^n x_{1i} \right) - 2f\left(\sum_{i=1}^{m} m^{n+1} x_{1i} \right), \right.\right. \\
&\left.\left. \cdots, \sum_{i=1}^{m} f\left(m^{n+1} x_{ki} + \sum_{j=1,\, j\neq i}^{m} m^n x_{kj} \right) + f\left(\sum_{i=1}^{m} m^n x_{ki} \right) - 2f\left(\sum_{i=1}^{m} m^{n+1} x_{ki} \right) \right) \right\|_k \\
\leqslant \lim_{n\to\infty} \frac{1}{m^n} &\varphi(m^n x_{11}, \cdots, m^n x_{1m}, \cdots, m^n x_{k1}, \cdots, m^n x_{km}) = 0
\end{aligned}$$

for all $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km} \in A$ and so

$$\sum_{i=1}^{m} H\left(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j \right) + H\left(\sum_{i=1}^{m} x_i \right) = 2H\left(\sum_{i=1}^{m} mx_i \right) \tag{10}$$

for all $x_1, \cdots, x_m \in A$ and hence H is additive.

By a similar method as above, we get

$$\mu H(mx) = H(m\mu x)$$

for all $\mu \in \mathbb{T}^1$ and $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear. Further, it follows from (3.3), (3.4) and (3.8) that

$$\begin{aligned}
&\|(H(x_1y_1) - H(x_1)H(y_1), \cdots, H(x_ky_k) - H(x_k)H(y_k))\|_k \\
&= \lim_{n\to\infty} \frac{1}{m^{2n}} \left\| \left(f(m^{2n}x_1y_1) - f(m^n x_1) f(m^n y_1), \cdots, f(m^{2n}x_ky_k) - f(m^n x_k) f(m^n y_k) \right) \right\|_k \\
&\leqslant \lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x_1, m^n y_1, \cdots, m^n x_k, m^n y_k) = 0
\end{aligned}$$

for all $x_1, y_1, \cdots, x_k, y_k \in A$ and so

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$. Thus $H : A \to B$ is a homomorphism satisfying (3.5). This completes the proof. ∎

COROLLARY 3.2 *Let A, B be complex Banach algebras and $((B^k, \|.\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $r < 1$ and θ be nonnegative real numbers and $f : A \to B$ be a mapping such that*

$$\|(D_\mu f(x_{11}, \cdots, x_{1m}), \cdots, D_\mu f(x_{k1}, \cdots, x_{km}))\|_k \leqslant \theta \left(\sum_{j=1}^{m} \|x_{1j}\|_A^r + \cdots + \sum_{j=1}^{m} \|x_{kj}\|_A^r \right), \tag{11}$$

$$\|(f(x_1y_1) - f(x_1)f(y_1), \cdots, f(x_ky_k) - f(x_k)f(y_k))\|_k \leqslant \theta \left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r \right) \tag{12}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|(f(x_1) - H(x_1), \cdots, f(x_k) - H(x_k))\|_k \leqslant \frac{\theta}{m - m^r} \left(\|x_1\|_A^r + \cdots + \|x_k\|_A^r \right)$$

for all $x_1, \cdots, x_k \in A$.

PROOF The proof follows from Theorem 3.1 by taking

$$\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}) := \theta \left(\sum_{j=1}^{m} \|x_{1j}\|_A^r + \cdots + \sum_{j=1}^{m} \|x_{km}\|_A^r \right)$$

and

$$\psi(x_1, y_1, \cdots, x_k, y_k) := \theta \left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r \right)$$

for all $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$ and $L = m^{r-1}$. ■

THEOREM 3.3 *Let A, B be complex Banach algebras and $((B^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^{mk} \to [0, \infty)$ and $\psi : A^{2k} \to [0, \infty)$ such that the inequalities (3.2), (3.3) and*

$$\lim_{j \to \infty} m^j \varphi(m^{-j}x_{11}, \cdots, m^{-j}x_{1m}, \cdots, m^{-j}x_{k1}, \cdots, m^{-j}x_{km}) = 0, \tag{13}$$

$$\lim_{j \to \infty} m^{2j} \psi(m^{-j}x_1, m^{-j}y_1, \cdots, m^{-j}x_k, m^{-j}y_k) = 0 \tag{14}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. If there exists a constant $L < 1$ such that

$$\varphi\left(\overbrace{x_{11}, 0, \cdots, 0}^{m}, \overbrace{x_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_{k1}, 0, \cdots, 0}^{m} \right) \leqslant \frac{L}{m} \varphi\left(\overbrace{mx_{11}, 0, \cdots, 0}^{m}, \overbrace{mx_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{mx_{k1}, 0, \cdots, 0}^{m} \right)$$

for all $x_{11}, x_{21}, \cdots, x_{k1} \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|(f(x_1) - H(x_1), \cdots, f(x_k) - H(x_k))\|_k \leqslant \frac{L}{m - mL}\varphi\left(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}\right) \tag{15}$$

for all $x_1, \cdots, x_k \in A$.

Proof We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in A$. It follows from (3.6) that

$$\begin{aligned}\left\|\left(f(x_1) - mf\left(\frac{x_1}{m}\right), \cdots, f(x_k) - mf\left(\frac{x_k}{m}\right)\right)\right\|_k \\ \leqslant \varphi\left(\overbrace{\frac{x_1}{m}, 0, \cdots, 0}^{m}, \overbrace{\frac{x_2}{m}, 0, \cdots, 0}^{m}, \cdots, \overbrace{\frac{x_k}{m}, 0, \cdots, 0}^{m}\right) \\ \leqslant \frac{L}{m}\varphi\left(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}\right)\end{aligned} \tag{16}$$

for all $x_1, \cdots, x_k \in A$. Hence

$$d(f, Jf) \leqslant \frac{L}{m}.$$

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(mx) = mH(x) \tag{17}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (3.7) such that there exists $C \in (0, \infty)$ satisfying

$$\|(H(x_1) - f(x_1), \cdots, H(x_k) - f(x_k))\|_k \leqslant C\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m})$$

for all $x_1, \cdots, x_k \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) = H(x)$$

for all $x \in A$.

(3) $d(f,H) \leqslant \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,H) \leqslant \frac{L}{m-mL}.$$

Hence the inequality (3.16) holds.

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. ∎

COROLLARY 3.4 *Let A, B be complex Banach algebras and $((B^k, \|.\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $r > 1$ and θ be nonnegative real numbers and $f : A \to B$ be a mapping satisfying the inequalities (3.10) and (3.11) for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. Then there exists a unique homomorphism $H : A \to B$ such that*

$$\begin{aligned} &\|(f(x_1) - H(x_1), \cdots, f(x_k) - H(x_k))\|_k \\ &\qquad \leqslant \frac{\theta}{m^r - m} \left(\|x_1\|_A^r + \cdots + \|x_k\|_A^r\right) \end{aligned}$$

for all $x_1, \cdots, x_k \in A$.

PROOF The proof follows from Theorem 3.3 by taking

$$\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}) := \theta \left(\sum_{j=1}^{m} \|x_{1j}\|_A^r + \cdots + \sum_{j=1}^{m} \|x_{kj}\|_A^r \right)$$

and

$$\psi(x_1, y_1, \cdots, x_k, y_k) := \theta \left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r\right)$$

for all $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$ and $L = m^{1-r}$. ∎

4. Stability of derivations on multi-Banach algebras. Let A be a complex Banach algebra. Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

Now, we prove the generalized Hyers-Ulam stability of derivations on multi-Banach algebras for the functional equation

$$D_\mu f(x_1, \cdots, x_m) = 0$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and $x_1, \cdots, x_m \in A$, where $f : A \to A$ is a mapping.

THEOREM 4.1 *Let A be a complex Banach algebra and $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^{mk} \to [0,\infty)$ and $\psi : A^{2k} \to [0,\infty)$ such that (3.1), (3.2), (3.4) hold and*

$$\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \leqslant \psi(x_1, y_1, ..., x_k, y_k) \tag{18}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. If there exists a constant $L < 1$ such that

$$\varphi(\overbrace{mx_{11}, 0, \cdots, 0}^{m}, \overbrace{mx_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{mx_{k1}, 0, \cdots, 0}^{m}) \leqslant mL\varphi(\overbrace{x_{11}, 0, \cdots, 0}^{m}, \overbrace{x_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_{k1}, 0, \cdots, 0}^{m})$$

for all $x_{11}, x_{21}, \cdots, x_{k1} \in A$, then there exists a unique derivation $\delta : A \to A$ such that

$$\|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leqslant \frac{1}{m - mL}\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}) \tag{19}$$

for all $x_1, \cdots, x_k \in A$.

PROOF By the same reasoning as the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (4.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n} \tag{20}$$

for all $x \in A$. It follows from (4.1), (3.4) and (4.3) that

$$\begin{aligned}
&\|(\delta(x_1y_1) - \delta(x_1)y_1 - x_1\delta(y_1), \cdots, \delta(x_ky_k) - \delta(x_k)y_k - x_k\delta(y_k))\|_k \\
&= \lim_{n\to\infty} \frac{1}{m^{2n}}\|(f(m^{2n}x_1y_1) - f(m^n x_1)\cdot m^n y_1 - m^n x_1 f(m^n y_1), \\
&\quad \cdots, f(m^{2n}x_ky_k) - f(m^n x_k)\cdot m^n y_k - m^n x_k f(m^n y_k))\|_k \\
&\quad \leqslant \lim_{n\to\infty} m^{-2n}\psi(m^n x_1, m^n y_1, \cdots, m^n x_k, m^n y_k) \\
&= 0
\end{aligned}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in A$ and so

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (4.3). This completes the proof. ∎

COROLLARY 4.2 *Let A be a complex Banach algebra and $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.10) and such that*

$$\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \leqslant \theta\left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r\right) \tag{21}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leqslant \frac{\theta}{m - m^r}(\|x_1\|_A^r + \cdots + \|x_k\|_A^r)$$

for all $x_1, \cdots, x_k \in A$.

PROOF The proof follows from Theorem 4.1 by taking

$$\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}) := \theta\left(\sum_{j=1}^{m}\|x_{1j}\|_A^r + \cdots + \sum_{j=1}^{m}\|x_{kj}\|_A^r\right)$$

and

$$\psi(x_1, y_1, \cdots, x_k, y_k) := \theta\left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r\right)$$

for all $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$ and $L = m^r$. ∎

THEOREM 4.3 *Let A be a complex Banach algebra and $((A^k, \|.\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^{mk} \to [0, \infty)$ and $\psi : A^{2k} \to [0, \infty)$ satisfying (3.2), (3.12) (3.13) and (4.1) for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. If there exists a constant $L < 1$ such that*

$$\varphi(\overbrace{mx_{11}, 0, \cdots, 0}^{m}, \overbrace{mx_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{mx_{k1}, 0, \cdots, 0}^{m}) \leqslant \frac{L}{m}\varphi(\overbrace{x_{11}, 0, \cdots, 0}^{m}, \overbrace{x_{21}, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_{k1}, 0, \cdots, 0}^{m})$$

for all $x_{11}, x_{21}, \cdots, x_{k1} \in A$, then there exists a unique derivation $\delta : A \to A$ such that

$$\|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leqslant \frac{L}{m - mL}\varphi(\overbrace{x_1, 0, \cdots, 0}^{m}, \overbrace{x_2, 0, \cdots, 0}^{m}, \cdots, \overbrace{x_k, 0, \cdots, 0}^{m}) \tag{22}$$

for all $x_1, \cdots, x_k \in A$.

COROLLARY 4.4 *Let A be a complex Banach algebra and $((A^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach algebra. Let $r > 1$ and θ be nonnegative real numbers and $f : A \to A$ be a mapping such that (3.10) holds and*

$$\|(f(x_1y_1) - f(x_1)y_1 - x_1f(y_1), \cdots, f(x_ky_k) - f(x_k)y_k - x_kf(y_k))\|_k \leqslant \theta\left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r\right) \tag{23}$$

for all $\mu \in \mathbb{T}^1$ and $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|(f(x_1) - \delta(x_1), \cdots, f(x_k) - \delta(x_k))\|_k \leqslant \frac{\theta}{m^r - m}(\|x_1\|_A^r + \cdots + \|x_k\|_A^r)$$

for all $x_1, \cdots, x_k \in A$.

PROOF The proof follows from Theorem 4.3 by taking

$$\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}) := \theta\left(\sum_{j=1}^{m}\|x_{1j}\|_A^r + \cdots + \sum_{j=1}^{m}\|x_{kj}\|_A^r\right)$$

and

$$\psi(x_1, y_1, \cdots, x_k, y_k) := \theta\left(\|x_1\|_A^r \cdot \|y_1\|_A^r + \cdots + \|x_k\|_A^r \cdot \|y_k\|_A^r\right)$$

for all $x_{11}, \cdots, x_{1m}, \cdots, x_{k1}, \cdots, x_{km}, x_1, \cdots, x_k, y_1, \cdots, y_k \in A$ and $L = m^{1-r}$. ∎

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