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Statistical approximation in multivariate modular function spaces

Abstract. In this paper, using the concept of A-statistical convergence we prove a Korovkin type approximation theorem in multivariate modular function spaces. Furthermore, giving an example via bivariate operators of Kantorovich type, it is shown that our theorem is stronger than its classical case.

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1. Introduction and Background. The classical Korovkin theorem is one of the most interesting result in approximation theory [20]. Several modifications of this theorem has been considered in different functional spaces by many authors [12], [6], [25], [24], [1]. An extension of Korovkin theorem in abstract modular spaces was presented by Bardaro and Mantellini in [3]. They have given a modular version of this theorem in multivariate modular function spaces recently [4]. This generalization has been obtained for functions defined on open precompact sets in a Hausdorff locally compact space provided with a regular measure on Borel sets. The aim of this paper is to prove Korovkin theorem in multivariate modular function spaces by replacing strong or modular convergence with the more general $A-$statistical convergence. Finally, by giving an example via bivariate operators of Kantorovich type, we show that our theorem is stronger than its classical case. Our study is analogous that to of Karakuş, Demirci and Duman in [18]. They obtained various Korovkin type approximation theorems in modular spaces via the concept of A -statistical convergence. Moreover, by using $A-$statistical convergence of approximating operators defined on the space of all real valued continuous functions on a compact Hausdorff space, an abstract version of Korovkin theorem has been given by Duman and Orhan in [13].

Before further proceeding we recall some definitions and notations on statistical convergence and modular functions used in this paper.

Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$ the A transform of x, denoted by $Ax := \left((Ax)_j\right)$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series convergence for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$ [7].

Assume that A is a non-negative regular summability matrix and K is a subset of $\mathbb{N}$, the set of all positive integers. The $A-$density of K, is denoted by

$$\delta_A(K) := \lim_j \sum_{n \in K} a_{jn}$$

provided the limit exists. A sequence $x := (x_n)$ is said to be $A-$statistical convergent to the number L if for every $\varepsilon > 0$,

$$\delta_A \{n \in \mathbb{N} : |x_n - L| \geqslant \varepsilon\} = 0;$$

or equivalently

$$\lim_j \sum_{n:|x_n - L| \geqslant \varepsilon}^{\infty} a_{jn} = 0.$$

We denote this limit by $st_A - \lim x = L$ ([17], [8], [19]). For $A = C_1$, the Cesaro matrix, $A-$statistical convergence reduces to statistical convergence [14], [16], [15].

Also, if we take $A = I$, the identity matrix, $A-$statistical convergence coincides with the ordinary convergence. We note that if $A = (a_{nk})$ is a regular summability matrix for which $\lim_j \max_n \{a_{jn}\} = 0$, then $A-$statistical convergence is stronger than usual convergence (see, [19]).

The concepts of statistical limit superior and limit inferior have been introduced by Fridy and Orhan (see, [16]). $A-$statistical version of this definitions have been considered by Connor and Kline [9] and Demirci [10].

The $A-$statistical limit superior of a number sequence $x = (x_n)$ denoted by $st_A - \limsup x$, is defined by

$$st_A - \limsup x = \begin{cases} \sup B_x & , B_x \neq \emptyset \\ -\infty & , B_x = \emptyset \end{cases}$$

where $B_x = \{b \in \mathbb{R} : \delta_A \{n : x_n > b\} \neq 0\}$ and $\emptyset$ denotes the empty set. We denote that by $\delta_A \{K\} \neq 0$ either $\delta_A \{K\} > 0$ or K does not have $A-$density. Similarly, the $A-$statistical limit inferior of a number sequence (x_n), denoted by $st_A - \liminf x$, is defined by

$$st_A - \liminf x = \begin{cases} \sup A_x & , A_x \neq \emptyset \\ +\infty & , A_x = \emptyset \end{cases}$$

where $A_x = \{a \in \mathbb{R} : \delta_A \{n : x_n < a\} \neq 0\}$. If we take $A = C_1$ in above definitions we reach to the concepts of $st - \limsup x$ and $st - \liminf x$.

For any sequence $x=(x_n)$ satisfying $\delta_A\{n:|x_n|>M\}$ for some $M>0$ (equivalently for an $A-$statistically bounded sequence), $st_A-\lim x=L$ iff $st_A-\liminf x=st_A-\limsup x=L$ [10].

Now we turn our attention to the modular functionals.

Let G be a nonempty open set in a Hausdorff locally compact topological space H provided with a regular measure μ defined on the Borel sets of H. We will assume that $\overline{G}$ is compact. We will denote by $X(G)$ the space of all real-valued Borel measurable functions $f:G\to\mathbb{R}$ provided with equality μ a.e., by $C(G)$ the space of all continuous and bounded real functions defined on G and by $C_u(G)$ the subset of $C(G)$ whose elements have a continuous extension to $\overline{G}$. A functional $\varrho:X(G)\to\overline{\mathbb{R}_0^+}=[0,\infty]$ is called a *modular* on $X(G)$, if it satisfies following conditions for arbitrary $f,g\in X(G)$

i) $\varrho[f]=0$ if and only if $f=0$, a.e in G,

ii) $\varrho[-f]=\varrho[f]$, for every $f\in X(G)$,

iii) $\varrho[\alpha f+\beta g]\leqslant\varrho[f]+\varrho[g]$, for $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$.

The modular ϱ is called $N-$quasi convex if there is constant $N\geqslant 1$ such that

$$\varrho[\alpha f+\beta g]\leqslant N\alpha\varrho[Nf]+N\beta\varrho[Ng],$$

for every $f,g\in X(G)$, $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$. If $N=1$ we will say that ϱ is convex. By means of the functional ϱ, we introduce two vector subspaces of $X(G)$, denoted by $L^\varrho(G)$ and $E^\varrho(G)$, defined by

$$L^\varrho(G)=\left\{f\in X(G):\lim_{\lambda\to 0^+}\varrho[\lambda f]=0\right\}$$

and

$$E^\varrho(G)=\{f\in L^\varrho(G):\varrho[\lambda f]<\infty\text{ for all }\lambda>0\}.$$

The subspace $L^\varrho(G)$ is called the *modular space generated by* ϱ, and $E^\varrho(G)$ is called *the space of finite elements of* $L^\varrho(G)$. It is easy to see that when ϱ is $N-$quasi-convex then the modular space $L^\varrho(G)$ is characterized by

$$L^\varrho(G)=\{f\in X(G):\varrho[\lambda f]<\infty\text{ for some }\lambda>0\}.$$

In this paper following assumptions will be used on modular functionals:

a) ϱ is *monotone*, i.e. for $f,g\in X(G)$ and $|f|\leqslant|g|$ then $\varrho[f]\leqslant\varrho[g]$.

b) ϱ is *finite*, i.e. denoting by e_0 the function $e_0(t)=1$ for every $t\in G$, $e_0\in L^\varrho(G)$. Note that clearly $e_0\in C_u(G)$.

c) ϱ is *absolutely finite*, i.e. ϱ is finite and for every $\varepsilon>0$, $\lambda>0$ there is $\delta>0$ such that $\varrho[\lambda\chi_B]<\varepsilon$ for any measurable subset $B\subset G$ with $\mu(B)<\delta$. Here χ_B denotes the characteristic function of the set B.

d) ϱ is *strongly finite*, i.e. $e_0\in E^\varrho(G)$.

e) ϱ is *absolutely continuous*, i.e. there exists $\alpha>0$ such that for every $f\in X(G)$, with $\varrho[f]<\infty$, the following condition is satisfied: for every $\varepsilon>0$, there is $\delta>0$ such that $\varrho[\alpha f\chi_B]<\varepsilon$, for every measurable subset $B\subset G$ with $\mu(B)<\delta$.

For more details about the above definitions and notations see [21], [22] and [5]. Note that, since $\mu(G)<\infty$, if ϱ is strongly finite and absolutely continuous then it is also absolutely finite (see [2]).

A sequence of functions $\{f_n\} \subset L^{\varrho}(G)$ is called *modularly convergent* to a function $f \in L^{\varrho}(G)$, if

$$\lim_{n\to\infty} \varrho[\lambda(f_n - f)] = 0 \text{ for some } \lambda > 0.$$

Also $\{f_n\}$ is $F-$*norm-convergent* (or *strongly convergent*) to f iff

$$\lim_{n\to\infty} \varrho[\lambda(f_n - f)] = 0 \text{ for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\varrho[2f] < M\varrho[f]$, for every $f \in X(G)$ (see [21]). For example, this happens for every L^p-spaces and Orlicz spaces generated by $\varphi-$functions with the Δ_2-regularity condition (see [21], [5]). The modular convergence induces a topology on $L^{\varrho}(G)$, called modular topology. Given a subset $\mathcal{B} \subset L^{\varrho}(G)$, we will denote by $\overline{\mathcal{B}}$ the closure of $\mathcal{B}$ with respect to the modular topology. Then $f \in \overline{\mathcal{B}}$ if there is a sequence $\{f_n\} \subset \mathcal{B}$ such that f_n is modularly convergent to f.

Let us remark that $C(G) \subset L^{\varrho}(G)$ whenever ϱ is monotone and finite. Analogously, if ϱ is monotone and strongly finite, then $C(G) \subset E^{\varrho}(G)$ (see [4]).

2. Main Results. This section have been largely motivated by that of [5]. Assume that $f_1, \cdots, f_m \in C_u(G)$ have the following property:

There exist continuous functions $g_i \in C_u(G)$, $i = 1, \cdots, m$, such that for every $s, t \in \overline{G}$, the function

$$P_s(t) = \sum_{i=1}^{m} g_i(s) f_i(t), \tag{1}$$

is positive and equal to zero if and only if $s = t$.

Throughout this paper let $A = (a_{jn})$ be a non-negative regular summability matrix and let $T = \{T_n\}$ be a family of positive linear operators $T_n : D \to X(G)$, where $C_u(G) \subset D \subset X(G)$. Here D is the domain of the operators T_n.
Assume further that the family $\{T_n\}$ satisfies the following property
(*): There exists a subset $X_{\mathbf{T}} \subset D \cap L^{\varrho}(G)$ with $C_u(G) \subset X_{\mathbf{T}}$ and a constant $R > 0$ such that for every function $f \in X_{\mathbf{T}}$ we have $T_n f \in L^{\varrho}(G)$ and

$$st_A - \limsup_n \varrho[\lambda T_n f] \leqslant R\varrho[\lambda f]$$

for every $\lambda > 0$. Also, we will denote the value of $T_n f$ at a point s by $T_n(f; s)$. Furthermore, we will use the following assumption:

$$st_A - \lim_n \varrho[\lambda(T_n f_i - f_i)] = 0, \text{ for some } \lambda > 0 \text{ and } i = 1, \cdots, m. \tag{2}$$

LEMMA 2.1 *Let $A = (a_{jn})$ be a non-negative regular summability matrix and ϱ be a monotone modular and suppose that the condition (2) holds. Then, for every function defined by*

$$P(t) = \sum_{i=1}^{m} g_i f_i(t), \ g_1, g_2, \cdots, g_m \in \mathbb{R} \text{ and } t \in \overline{G},$$

we have

$$st_A - \lim_n \varrho[\lambda(T_n(P) - P)] = 0, \text{ for some } \lambda > 0.$$

PROOF Let $M := \max_{1 \leqslant i \leqslant m} |g_i|$ and let choose $\alpha > 0$ such that $\alpha m M \leqslant \lambda$. Then, using the property of the modular, we get

$$\begin{aligned} \varrho[\alpha(T_n P - P)] &\leqslant \sum_{i=1}^{m} \varrho[\alpha m M(T_n f_i - f_i)] \\ &\leqslant \sum_{i=1}^{m} \varrho[\lambda(T_n f_i - f_i)] = 0, \ i = 1, \cdots, m. \end{aligned} \tag{3}$$

For a given $\varepsilon > 0$ define the following sets:

$$\begin{aligned} D &= \{n : \varrho[\alpha(T_n P - P)] \geqslant \varepsilon\}, \\ D_i &= \left\{n : \varrho[\lambda(T_n f_i - f_i)] \geqslant \frac{\varepsilon}{m}\right\}, \ i = 1, \cdots, m. \end{aligned}$$

Then it is clear that $D \subset \bigcup_{i=1}^{m} D_i$. So, by (3) we may write

$$\sum_{n \in D} a_{jn} \leqslant \sum_{i=1}^{m} \left(\sum_{n \in D_i} a_{jn} \right).$$

Letting $j \to \infty$ on the both sides of the last inequality and using (2), we obtain the result. ∎

LEMMA 2.2 *Let $A = (a_{jn})$ be a non-negative regular summability matrix and ϱ be a monotone modular. Suppose that conditions* (1) *and* (2) *hold. Then for the function $P_s(t)$ in* (1) *we have*

$$st_A - \lim_n \varrho\left[\lambda(T_n P_{(.)})(.)\right] = 0 \text{ for some } \lambda > 0.$$

PROOF Choose a so large constant $M > 0$ such that $g_i(s) \leqslant M$ for every $i = 1, \cdots, m$ and for every $s \in G$. Let $\alpha > 0$ be such that $\alpha m M \leqslant \lambda$. Then

$$\begin{aligned} \varrho\left[\alpha(T_n P_{(.)})(.)\right] &= \varrho\left[\alpha\left((T_n P_{(.)})(.) - P_{(.)}\right)\right] \\ &\leqslant \sum_{i=1}^{m} \varrho[\alpha m M(T_n f_i - f_i)] \\ &\leqslant \sum_{i=1}^{m} \varrho[\lambda(T_n f_i - f_i)]. \end{aligned} \tag{4}$$

As Lemma 2.1, for a given $\varepsilon > 0$ define the following sets:

$$\begin{aligned} D &= \left\{n : \varrho\left[\alpha\left(\left(T_n P_{(.)}\right)(.) - P_{(.)}\right)\right] \geqslant \varepsilon\right\}, \\ D_i &= \left\{n : \varrho\left[\lambda\left(T_n f_i - f_i\right)\right] \geqslant \frac{\varepsilon}{m}\right\},\ i = 1, \cdots, m. \end{aligned}$$

Then it is clear that $D \subset \bigcup_{i=1}^{m} D_i$. So, by (4) we may write

$$\sum_{n \in D} a_{jn} \leqslant \sum_{i=1}^{m} \left(\sum_{n \in D_i} a_{jn} \right).$$

Letting $j \to \infty$ on the both sides of the last inequality and using (2), we obtain the result. ∎

Lemma 2.3 *Let $A = (a_{jn})$ be a non-negative regular summability matrix and ϱ be a finite, monotone and $N-$quasi-convex modular. Suppose that the conditions* (1) *and* (2) *hold. Let $f_s \in C_u(G)$, $s \in \overline{G}$, be a family of functions such that $f_s(t)$ is a continuous function of $(t, s) \in \overline{G} \times \overline{G}$ and $f_s(s) = 0$ for every $s \in \overline{G}$. Then we have*

$$st_A - \lim_n \varrho\left[\lambda (T_n f_{(.)})(.)\right] = 0 \text{ for some } \lambda > 0.$$

Proof We begin with the similar method used in Theorem 3.1 of [11]. There exists a P^* with $P^*(t) > 0$ for all $t \in G$; if $s_1 \neq s_2$ are two points of G, we can take $P^* = P^*(s_1) + P^*(s_2)$. So, we can write P^* of the form

$$P^*(t) = \sum_{i=1}^{m} g_i f_i(t).$$

Let us consider the diagonal $\Delta := \left\{(s, s) : s \in \overline{G}\right\}$. For a given $0 < \varepsilon < 1$, each point of Δ has an open neighbourhood U in $\overline{G} \times \overline{G}$ for which $|f_s(t)| < \varepsilon$ for every $(t, s) \in U$. Define the sets $V = \cup U$ and $F = \left(\overline{G} \times \overline{G}\right) \backslash V$. Then F is compact. Let $\omega = \min_{(t,s) \in F} P_s(t) > 0$ and $\Omega = \max_{(t,s) \in F} |f_s(t)|$. Then, for every $(s, t) \in \overline{G} \times \overline{G}$ we have

$$|f_s(t)| \leqslant \varepsilon + \frac{\omega}{\Omega} T_n(P_s; s).$$

Now using monotonicity and linearity of the operators T_n we have

$$|T_n(f_s; s)| \leqslant \varepsilon T_n(e_0; s) + \frac{\omega}{\Omega} T_n(P_s; s).$$

Then, for $\gamma > 0$ we have

$$\begin{aligned} \varrho\left[\gamma (T_n f_{(.)})(.)\right] &\leqslant \varrho\left[2\gamma\varepsilon\left(T_n e_0\right)(.)\right] + \varrho\left[2\gamma\frac{\omega}{\Omega}(T_n P_{(.)})(.)\right] \\ &\leqslant N\varepsilon\varrho\left[2\gamma N\left(T_n e_0\right)(.)\right] + \varrho\left[2\gamma\frac{\omega}{\Omega}(T_n P_{(.)})(.)\right] = I_1 + I_2. \end{aligned}$$

Firstly, let us consider I_1. We can choose a positive constant $a > 0$ such that $1 = e_0(t) \leqslant aP^*(t)$, $t \in \overline{G}$. So applying the modular we have

$$\begin{aligned} \varrho\left[2\gamma N\left(T_n e_0\right)(.)\right] &\leqslant \varrho\left[2\gamma Na\left(T_n P^*\right)(.)\right] \\ &\leqslant \varrho\left[4\gamma Na\left(\left(T_n P^*\right)(.) - P^*(.)\right)\right] + \varrho\left[4\gamma NaP^*(.)\right] \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

Now consider $I_{1,1}$. By Lemma 2.1, there exists a $K_1 \subseteq \mathbb{N}$ with $\delta_A(K_1) = 1$ and $\alpha > 0$ such that

$$\varrho\left[\alpha\left(\left(T_n P^*\right)(.) - P^*(.)\right)\right] < 1$$

for sufficiently large $n \in K_1$. For $I_{1,2}$, since the functions $f_1, \cdots, f_m \in L^{\varrho}(G)$, there exists $v > 0$ such that $\varrho[vf_i] < \infty$ for every $i = 1, \cdots, m$. Now, putting $M = \max_{i=1}^{m} |g_i|$ and taking γ such that $4\gamma mMNa < \nu$ and $4\gamma Na < \alpha$, by monotonicity of modular we have

$$\begin{aligned} I_{1,2} = \varrho\left[4\gamma NaP^*\right] &= \varrho\left[4\gamma Na\sum_{i=1}^{m} g_i f_i\right] \\ &\leqslant \sum_{i=1}^{m} \varrho\left[4\gamma mMNaf_i\right] \\ &\leqslant \sum_{i=1}^{m} \varrho\left[vf_i\right]. \end{aligned}$$

So, we have $I_1 < \varepsilon W$ for an absolute constant $W > 0$ and for sufficiently large $n \in K_1$.
Now consider I_2. By Lemma 2.2, there exists a $K_2 \subseteq \mathbb{N}$ such that $\delta_A(K_2) = 1$ and $\lim_{n \in K_2} I_2 = 0$ for $\gamma > 0$. Let $K = K_1 \cap K_2$. Observe that $\delta_A(K) = 1$. Hence, for every sufficiently large $n \in K$ and for sufficiently small $\gamma > 0$ we conlude that $\lim_{n \in K} T_n(f_s; s) = 0$, which completes the proof. ∎

LEMMA 2.4 *Let $A = (a_{jn})$ be a non-negative regular summability matrix and ϱ be a finite, monotone and $N-$quasi-convex modular. Suppose that the conditions* (1) *and* (2) *hold. Then for every $f \in C_u(G)$ we have*

$$st_A - \lim_n \varrho\left[\lambda\left(T_n f - f\right)\right] = 0, \text{ for some } \lambda > 0.$$

PROOF For a fixed $f \in C_u(G)$ define the function f_s by

$$f_s(t) = f(t) - \frac{f(s)}{P^*(s)} P^*(t)$$

where the function P^* is strictly positive in $\overline{G}$. By Lemma 2.3 there exists $\gamma > 0$ such that $st_A - \lim_n \varrho\left[\lambda(T_n f_{(.)})(.)\right] = 0$. This means that

$$st_A - \lim_{n \to \infty} \varrho\left[\gamma\left(\left(T_n f\right)(.) - \frac{f(.)}{P^*(.)}\left(T_n P^*\right)(.)\right)\right] = 0. \tag{5}$$

For a fixed $\lambda_0 > 0$ we have

$$\begin{aligned}\varrho\left[\lambda_0\left(T_nf-f\right)\right] &\leqslant \varrho\left[2\lambda_0\left(\left(T_nf\right)(.)-\tfrac{f(.)}{P^*(.)}\left(T_nP^*\right)(.)\right)\right]\\ &+\varrho\left[2\delta\left(\tfrac{f(.)}{P^*(.)}\left(T_nf\right)(.)-f(.)\right)\right]\\ &=\varrho\left[2\lambda_0\left(\left(T_nf\right)(.)-\tfrac{f(.)}{P^*(.)}\left(T_nP^*\right)(.)\right)\right]\\ &+\varrho\left[2\lambda_0\tfrac{f(.)}{P^*(.)}\left(\left(T_nP^*\right)(.)-P^*(.)\right)\right].\end{aligned}$$

If we choose λ_0 such that $2\lambda_0 < \gamma$, then from (5) we have

$$st_A-\lim_n \varrho\left[2\lambda_0\left(\left(T_nf\right)(.)-\frac{f(.)}{P^*(.)}\left(T_nP^*\right)(.)\right)\right]=0.$$

Let $\Lambda := \max_{s\in\overline{G}}\left|\frac{f(s)}{P^*(s)}\right|$, then by monotonicity of ϱ, we may write

$$\varrho\left[2\lambda_0\frac{f(.)}{P^*(.)}\left(\left(T_nP^*\right)(.)-P^*(.)\right)\right]\leqslant\varrho\left[2\lambda_0\Lambda\left(T_nP^*-P^*\right)\right].$$

Hence, from Lemma 2.1 we have $st_A-\lim_n\varrho\left[2\lambda_0\frac{f(.)}{P^*(.)}\left(\left(T_nP^*\right)(.)-P^*(.)\right)\right]=0$ for sufficiently small $\lambda_0>0$, which completes the proof. ■

REMARK 2.5 If we replace (2) with $st_A-\lim_n\varrho\left[\lambda\left(T_nf_i-f_i\right)\right]=0$, for all $\lambda>0$ and $i=1,\cdots,m$, then using exactly the same proof as before we can show that for every $f\in C_u(G)$, we have $st_A-\lim_n\varrho\left[\lambda\left(T_nf-f\right)\right]=0$, for some $\lambda>0$.

Now we can state the main result of this section.

THEOREM 2.6 *Let $A=(a_{jn})$ be a non-negative regular summability matrix and ϱ be a monotone, absolutely finite, absolutely continuous and $N-$quasi-convex modular on $X(G)$. Let $T=\{T_n\}$ be a sequence of positive linear operators satisfying property $(*)$. Let condition (1) holds and suppose that*

$$st_A-\lim_n\varrho\left[\lambda\left(T_nf_i-f_i\right)\right]=0,\ \textit{for every } \lambda>0 \textit{ and } i=1,\cdots,m.$$

Then for all f in $L^\varrho(G)\cap D$ such that $f-C_u(G)\subset X_{\mathbf{T}}$, we have

$$st_A-\lim_n\varrho\left[\lambda\left(T_nf-f\right)\right]=0,\ \textit{for some } \lambda>0.$$

PROOF Let $f\in L^\varrho(G)\cap D$ be a function such that $f-C_u(G)\subset X_{\mathbf{T}}$. By Proposition 1 of [4], we know that $\overline{C_u(G)}=L^\varrho(G)$. Hence, there exist a $\lambda>0$ and a sequence $\{f_k\}\subset C_u(G)$ such that $\varrho[3\lambda f]<\infty$ and $\lim_n\varrho[3\lambda(f_k-f)]=0$, i.e., for a given $\varepsilon>0$ there is positive number k_0 such that

$$\varrho[3\lambda(f_k-f)]<\varepsilon \text{ for every } k\geqslant k_0. \tag{6}$$

For fixed k_0, by the linearity and positivity of the T_n and monotonicity of ρ we can write

$$\begin{aligned}\varrho\left[\lambda\left(T_n f-f\right)\right] &\leqslant \varrho\left[3\lambda\left(T_n\left(f-f_{k_0}\right)\right)\right]+\varrho\left[3\lambda\left(T_n f_{k_0}-f_{k_0}\right)\right] \\ &\quad +\varrho\left[3\lambda\left(f_k-f_{k_0}\right)\right].\end{aligned} \tag{7}$$

From $(*)$ and (6), we can write

$$\begin{aligned}st_A-\limsup_n \varrho\left[3\lambda\left(T_n\left(f-f_{k_0}\right)\right)\right] &\leqslant \varrho\left[3\lambda\left(f_k-f_{k_0}\right)\right] \\ &< R\varepsilon.\end{aligned}$$

On the other hand, since f_{k_0} belongs to $C_u(G)$, it follows from Remark 2.5 that for every $\lambda>0$

$$st_A-\limsup_n \varrho\left[3\lambda\left(T_n f_{k_0}-f_{k_0}\right)\right]=0.$$

Now, taking $st_A-\limsup$ as $n\to\infty$ in (7), we obtain that

$$st_A-\limsup_n \varrho\left[\lambda(T_n f-f)\right]\leqslant \varepsilon\left(R+1\right),$$

which yields the proof from the arbitrariness of $\varepsilon>0$. ∎

Remark 2.7 Note that a similar result holds true in the case when G is compact replacing of course $C_u(G)$ with $C(G)$. The proof is exactly the same.

If we replace the matrix $A=(a_{jn})$ by the identity matrix, then the property $(*)$ reduces to following property.

$(*')$: There exists a subset $X_{\mathbf{T}}\subset D\cap L^{\varrho}(G)$ with $C_u(G)\subset X_{\mathbf{T}}$ and a constant $R>0$ such that for every function $f\in X_{\mathbf{T}}$ we have $T_n f\in L^{\varrho}(G)$ and

$$\limsup_n \varrho\left[\lambda T_n f\right]\leqslant R\varrho\left[\lambda f\right]$$

for every $\lambda>0$. Hence, our Theorem 2.6 coincides with Theorem 1 of [4].

Corollary 2.8 *Let ϱ be a monotone, absolutely finite, absolutely continuous and $N-$quasi-convex modular on $X(G)$ and $T=\{T_n\}$ be a sequence of positive linear operators satisfying property $(*')$. Let assumption* (1) *be satisfied. Then if*

$$\lim_{n\to\infty}\varrho\left[\lambda\left(T_n f_i-f_i\right)\right]=0,\ \textit{for every } \lambda>0 \textit{ and for } i=1,\cdots,m.$$

then $\lim_n T_n f=f$, modularly in $L^{\varrho}(G)$ for each $f\in L^{\varrho}(G)\cap D$ such that $f-C_u(G)\subset X_{\mathbf{T}}$.

3. Concluding Remarks. In this section we deal with an example of a positive linear operators for which Corollary 2.8 does not work but our Theorem 2.6 works.

Let $I = [0,1]^2 = [0,1] \times [0,1]$ and let Φ be class of all continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that φ is a convex function, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

For $\varphi \in \Phi$, we define the functional

$$\rho^{\varphi}[f] := \int_I \varphi(|f(s)|)\, ds = \int_0^1 \int_0^1 \varphi(|f(u,v)|)\, dudv$$

for every $f \in X(I)$. In this case, since φ is convex then ρ^{φ} is a convex modular on $X(I)$ which satisfies all assumptions listed in Section 1. Furthermore, the subspace

$$L_{\varphi}^{\rho}(I) := \{f \in X(I) : \rho^{\varphi}[\lambda f] < \infty, \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ (see [21]).

Now let us consider the bivariate operators of Kantorovich type on $L_{\varphi}^{\rho}(I)$ defined by

$$K_n(f;x,y) = (n+1)^2 \sum_{\substack{k,j=0 \\ k+j\leqslant n}}^{n} p_{n,k,j}(x,y) \int_{k/(n+1)}^{(k+1)/(n+1)} \int_{j/(n+1)}^{(j+1)/(n+1)} f(u,v)\, dudv$$

where $p_{n,k,j}$ is defined by

$$p_{n,k,j} = \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j},$$

for $k, j \geqslant 0$, $x, y \geqslant 0$ and $x + y \leqslant 1$. Also, $p_{n,k,j}$ satisfies the condition

$$\sum_{\substack{k,j=0 \\ k+j\leqslant n}} p_{n,k,j}(x,y) = 1 \tag{8}$$

(see, [23]).

Observe that K_n maps $L_{\varphi}^{\rho}(I)$ into itself. Because of the condition (8), as in the proof of [3] Lemma 5.1 ,we can use Jensen inequality and obtain that for every $f \in L_{\varphi}^{\rho}(I)$ and $n \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}[K_n f] \leqslant M \rho^{\varphi}[f]. \tag{9}$$

Let $A = C_1$ and define a real sequence (s_n) by

$$s_n = \begin{cases} 1 & \text{, if } n \text{ is square} \\ 0 & \text{, otherwise} \end{cases}$$

Although (s_n) is not convergent in ordinary sense, its statistical limit exists and equals to 1, i.e., $st - \lim_n s_n = 1$.

Now consider the sequence of positive linear operators $\mathbb{L} = \{L_n\}$ on $L_\varphi^\rho(I)$ defined by

$$L_n(f;x,y) = s_n K_n(f;x,y),\ x,y \geqslant 0,\ x+y \leqslant 1 \text{ and } n \in \mathbb{N}. \tag{10}$$

From the inequality (9), we have

$$\begin{aligned} \rho^\varphi[\lambda L_n f] &= \rho^\varphi[\lambda s_n K_n f] \\ &= s_n \rho^\varphi[\lambda K_n f] \\ &\leqslant s_n M \rho^\varphi[\lambda f] \end{aligned}$$

for every $f \in X_\mathbb{L} := L_\varphi^\rho(I)$ and $\lambda > 0$. Since (s_n) is statistically bounded sequence, we have $st - \limsup_n s_n = st - \lim_n s_n = 1$. So, taking $st - \limsup$ as $n \to \infty$ on both sides of the above inequality, we obtain

$$st - \limsup_n \rho^\varphi[\lambda L_n f] \leqslant M \rho^\varphi[\lambda f].$$

Hence the $(*)$ property is satisfied with the choice of $X_\mathbb{L} = L_\varphi^\rho(I)$. Since I is a compact subset of $\mathbb{R}^2$, we can replace $C_u(I)$ with $C(I)$. Hence, the condition $C(I) \subset X_\mathbb{L} = L_\varphi^\rho(I)$ is also satisfied.

Now choose the functions f_i and g_i in $C(I)$ such that the condition (1) holds. Let $f_1(u,v) = 1$, $f_2(u,v) = u$, $f_3(u,v) = v$, $f_4(u,v) = u^2 + v^2$ and $a_1(x,y) = x^2 + y^2$, $a_2(x,y) = -2x$, $a_3(x,y) = -2y$, $a_4(x,y) = 1$. Then the function P_s defined by

$$P(s,t) = \sum_{i=1}^{4} a_i(s) f_i(t) = \|s-t\|^2,\ s,t \in [0,1]^2$$

satisfies the condition (1), where $s = (x,y)$ and $t = (u,v)$.
We claim that

$$st - \lim_n \rho^\varphi[\lambda(L_n f_i - f_i)] = 0 \text{ for every } \lambda > 0 \text{ and } i = 0,1,2,3,4. \tag{11}$$

From [23], we see that

$$\begin{aligned} L_n(f_1;x,y) &= s_n \\ L_n(f_2;x,y) &= s_n\left(\frac{nx}{n+1} + \frac{1}{2(n+1)}\right) \\ L_n(f_3;x,y) &= s_n\left(\frac{ny}{n+1} + \frac{1}{2(n+1)}\right) \\ L_n(f_4;x,y) &= s_n\left[\frac{(n-1)(x(1-x)+y(1-y))}{(n+1)^2} + \frac{2}{3(n+1)^2}\right] \end{aligned}$$

For any $\lambda > 0$

$$\lambda |L_n(f_1;x,y) - f_1(x,y)| = \lambda(1-s_n)$$

which implies

$$\begin{aligned}\rho^{\varphi}\left[\lambda\left(L_{n}f_{1}-f_{1}\right)\right] &= \rho^{\varphi}\left[\lambda\left(1-s_{n}\right)\right]\\ &= \int_{I}\varphi\left(\lambda\left(1-s_{n}\right)\right)ds\\ &= \varphi\left(\lambda\left(1-s_{n}\right)\right)\mu\left(I\right)\\ &= \left(1-s_{n}\right)\varphi\left(\lambda\right)\end{aligned}$$

by the definition of (s_n). Here, $\mu(I)$ is Borel measure of unit square I and equals to 1. Since $st-\lim_{n}\left(1-s_{n}\right)=0$, we have

$$st-\lim_{n}\rho^{\varphi}\left[\lambda\left(L_{n}f_{1}-f_{1}\right)\right]=0 \text{ for every } \lambda>0.$$

Therefore (11) remains for $i=0$. On the other hand, for any $\lambda>0$

$$\begin{aligned}\lambda\left|\left(L_{n}\left(f_{2};x,y\right)-f_{2}\left(x,y\right)\right)\right| &= \lambda\left|s_{n}\frac{2nx+1}{2\left(n+1\right)}-x\right|\\ &\leqslant \lambda\left\{x\left(1-\frac{2n}{2\left(n+1\right)}s_{n}\right)+\frac{s_{n}}{2\left(n+1\right)}\right\}\\ &\leqslant \lambda\left\{\left(1-s_{n}\right)+\frac{3s_{n}}{2\left(n+1\right)}\right\}\end{aligned}$$

which implies

$$\begin{aligned}\rho^{\varphi}\left[\lambda\left(L_{n}f_{2}-f_{2}\right)\right] &\leqslant \rho^{\varphi}\left[\lambda\left(\left(1-s_{n}\right)+\frac{3s_{n}}{2\left(n+1\right)}\right)\right]\\ &\leqslant \rho^{\varphi}\left[2\lambda\left(1-s_{n}\right)\right]+\rho^{\varphi}\left[\frac{3\lambda s_{n}}{n+1}\right]\\ &\leqslant \left\{\rho^{\varphi}\left(2\lambda\left(1-s_{n}\right)\right)+\rho^{\varphi}\left[\frac{3\lambda s_{n}}{n+1}\right]\right\}\mu\left(I\right).\end{aligned}$$

Hence

$$\rho^{\varphi}\left[\lambda\left(L_{n}f_{2}-f_{2}\right)\right]\leqslant\left(1-s_{n}\right)\varphi\left(2\lambda\right)+s_{n}\varphi\left(\frac{3\lambda}{n+1}\right).$$

So, for all $\lambda>0$, we have $st-\lim_{n}\rho^{\varphi}\left[\lambda\left(L_{n}f_{2}-f_{2}\right)\right]=0$.

Similarly for all $\lambda>0$, we can obtain that

$$st-\lim_{n}\rho^{\varphi}\left[\lambda\left(L_{n}f_{3}-f_{3}\right)\right]=0.$$

Finally, let us consider the case $i = 4$ in (11).

$$\begin{aligned}
&\lambda \left| L_n \left(f_4; x, y \right) - f_4 \left(x, y \right) \right| \\
&= \lambda \left[s_n \frac{(n-1)\left(x\left(1-x\right) + y\left(1-y\right)\right)}{\left(n+1\right)^2} + \frac{2s_n}{3\left(n+1\right)^2} - \left(x^2+y^2\right) \right] \\
&\leqslant \lambda \left[\frac{(n-1)\left(x+y-\left(x^2+y^2\right)\right) s_n}{\left(n+1\right)^2} + \frac{2s_n}{3\left(n+1\right)^2} - \left(x^2+y^2\right) \right] \\
&\leqslant \lambda \left[\frac{(n-1)\, s_n}{\left(n+1\right)^2} + \frac{2s_n}{3\left(n+1\right)^2} + \left(x^2+y^2\right) \left(1 - \frac{(n-1)\, s_n}{\left(n+1\right)^2} \right) \right] \\
&\leqslant \lambda \left(\frac{s_n}{3\left(n+1\right)} + \left(1-s_n\right) \right)
\end{aligned}$$

which implies

$$\begin{aligned}
\rho^{\varphi} \left[\lambda \left(L_n f_4 - f_4 \right) \right] &\leqslant \rho^{\varphi} \left[\lambda \left(\left(1-s_n\right) + \frac{s_n}{3\left(n+1\right)} \right) \right] \\
&\leqslant \rho^{\varphi} \left[\frac{2\lambda s_n}{3\left(n+1\right)} \right] + \rho^{\varphi} \left[2\lambda \left(1-s_n\right) \right] \\
&= s_n \varphi \left(\frac{2\lambda}{3\left(n+1\right)} \right) + \left(1-s_n\right) \varphi \left(2\lambda\right).
\end{aligned}$$

Therefore $st - \lim_n \rho^{\varphi} \left[\lambda \left(L_n f_4 - f_4 \right) \right] = 0$, for all $\lambda > 0$.

So, condition (11) holds for $i = 1, 2, 3, 4$. From Theorem 2.6, for every $f \in X_{\mathbb{L}} = L_{\varphi}^{\rho}\left(I\right)$, there holds $st - \lim_n \rho^{\varphi} \left[\lambda \left(L_n f - f \right) \right] = 0$, for all $\lambda > 0$. However, for any $\lambda > 0$, since

$$\begin{aligned}
\rho^{\varphi} \left[\lambda \left(L_n f - f \right) \right] &= \rho^{\varphi} \left[\lambda \left(s_n K_n f - f \right) \right] \\
&= \begin{cases} \rho^{\varphi} \left[\lambda f \right] & \text{, if } n \text{ is square} \\ \rho^{\varphi} \left[\lambda \left(K_n f - f \right) \right] & \text{, otherwise} \end{cases}
\end{aligned}$$

the sequence $\{\rho^{\varphi} \left(\lambda \left(K_n f - f \right) \right)\}$ has two subsequences with different limit points. In other words, there is not any $\lambda > 0$ such that the sequence $\{\rho^{\varphi} \left(\lambda \left(K_n f - f \right) \right)\}$ tends to zero as $n \to \infty$. Hence, for the sequence of positive linear operators $\mathbb{L} = \{L_n\}$ defined by (10), it is shown that Theorem 2.6 is stronger than its classical case. Consequently, Theorem 2.6 is a generalization of Corollary 2.8.

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