COMMENTATIONES MATHEMATICAE Vol. 51, No. 1 (2011), 55-69

Lidia Skóra

Impulsive differential equations with initial data difference

Abstract. In this paper, we present some results on impulsive differential inequalities and equations with initial and impulsive data difference.

2000 Mathematics Subject Classification: 34A37, 34K45, 34K05.

Key words and phrases: impulsive differential equation, impulsive differential inequalities, existence, monotone iterative method, extremal solutions.

Ministry of Science and Higher Education Republic of Poland

1. Introduction. Impulsive differential equations describe processes which have a short-term rapid change of their state at fixed times or variable times. During recent years, the impulsive differential equations have been an object of intensive investigation because of the wide possibilities for their applications in various fields of science and technology as physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. When studying the initial value problems the authors have been keeping the initial time unchanged. However, the fact of changing initial time needs to be taken into consideration since it is difficult to keep this time fixed in experimentations. The investigations of initial value problems of differential equations with initial time difference have been initiated in the papers [9],[10]. Enlightened by these works the study of functional differential equations and impulsive differential equations with initial time difference was initiated in [17],[16]. We refer the reader to the papers [1]-[6], [11], [14]-[18] and the references therein for more complete information on the subject.

The aim of this paper is to prove some new impulsive differential inequalities with initial data difference. We also establish some new existence results and develop monotone method. We use techniques given in [7],[8],[12],[13].

2. Preliminaries. Let $I = [t_0, b], t_0 < b < \infty$. Suppose that

$$t_0 < t_1 < ... < t_p < t_{p+1} = b$$

are given numbers. We define

$$PC\left(I,\mathbb{R}\right)=\{u:I\to\mathbb{R},u \text{ is continuous in } I\setminus\{t_k\}_{k=1}^{p},\text{the limits}\\ u(t_k^+),u(t_k^-),k=1,...,p \text{ exist and } u(t_k^+)=u(t_k),k=1,...,p\}$$

and

$$PC^1\left(I,\mathbb{R}\right)=\{u\in PC\left(I,\mathbb{R}\right),u \text{ is continuously differentiable for any}\\ t\in I\setminus\{t_k\}_{k=1}^{p},u'(t_k^+),u'(t_k^-) \text{ exist },k=1,...,p\ \}.$$

For $u\in PC\left(I,\mathbb{R}\right)$, $\triangle u(t_k)$ denotes the jump of $u(t)$ at $t=t_k$, i.e.

$$\triangle u(t_k)=u(t_k)-u(t_k^-).$$

Let τ_0 be such that $t_0<\tau_0<t_1$. Denote

$$\tau_0-t_0=\eta,\\ \tau_k=t_k+\eta,k=1,...,p+1,\\ J=[\tau_0,b+\eta].$$

Analogously

$$PC\left(J,\mathbb{R}\right)=\{u:J\to\mathbb{R},u \text{ is continuous in } J\setminus\{\tau_k\}_{k=1}^{p},\text{the limits}\\ u(\tau_k^+),u(\tau_k^-),k=1,...,p \text{ exist and } u(\tau_k^+)=u(\tau_k),k=1,...,p\}$$

and

$$PC^1\left(J,\mathbb{R}\right)=\{u\in PC\left(J,\mathbb{R}\right),u \text{ is continuously differentiable for any}\\ t\in J\setminus\{\tau_k\}_{k=1}^{p},u'(\tau_k^+),u'(\tau_k^-) \text{ exist },k=1,...,p\ \}.$$

3. Comparison theorem. We shall now prove the following comparison result.

THEOREM 3.1 *Suppose that:*

(i) *Functions* $f_i,i=1,2$, *where* $f_1\in C\left(I\times\mathbb{R},\mathbb{R}\right),f_2\in C\left(J\times\mathbb{R},\ \mathbb{R}\right)$ *satisfy inequality*

$$f_1(t,x)\leqslant f_2(t+\eta,x),$$

for $t\in I,x\in\mathbb{R}$.

(ii) *Functions* $x\in PC\left(I,\mathbb{R}\right),y\in PC\left(J,\mathbb{R}\right)$ *satisfy the differential inequality*

$$D_-x(t)-f_1(t,x(t))<D_-y^*(t)-f_2(t+\eta,y^*(t)),\tag{3.1}$$

$t\in I,t\neq t_k,k=1,..,p,y^*(t)=y(t+\eta)$ *and* $D_-x(t)=\liminf\limits_{h\to0^-}\dfrac{x(t+h)-x(t)}{h}$ *is the Dini derivative.*

(iii) *Functions* $g_1 : \{t_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}, g_2 : \{\tau_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}$ *satisfy the inequality*

$$g_1(t_k, x) \leqslant g_2(\tau_k, x),\ x \in \mathbb{R}, k = 1, ..., p$$

and for each $t = t_k, k = 1, ..., p$ *the function* $\sigma_1 : \mathbb{R} \ni s \to s + g_1(t, s)$ *or for each* $t = \tau_k, k = 1, ..., p$ *the function* $\sigma_2 : \mathbb{R} \ni s \to s + g_2(t, s)$ *is non-decreasing on* $\mathbb{R}$.

(iv) *Impulsive inequalities*

$$\Delta x(t_k) - g_1\left(t_k, x(t_k^-)\right) < \Delta y^*(t_k) - g_2\left(t_k + \eta, y^*(t_k^-)\right),\ k = 1, ..., p \tag{3.2}$$

are satisfied.

(v) $x(t_0) < y(\tau_0)$.

Then

$$x(t) < y(t + \eta),\ t \in [t_0, b]. \tag{3.3}$$

Proof If (3.3) is false than the set

$$Z = \{t \in [t_0, b] :\ x(t) \geqslant y(t + \eta)\}$$

is non empty. Let

$$\bar{t} = \inf Z.$$

It follows from (v) that $\bar{t} > t_0$ and

$$x(t) < y(t + \eta),\ t \in [t_0, \bar{t}), \tag{3.4}$$

$$x(\bar{t}) = y(\bar{t} + \eta) = y^*(\bar{t}). \tag{3.5}$$

We then have the following two cases:
Case 1. Suppose that $\bar{t} \neq t_k, k = 1, ..., p$. Then for small $h < 0$ we have

$$\frac{x(\bar{t} + h) - x(\bar{t})}{h} > \frac{y^*(\bar{t} + h) - y^*(\bar{t})}{h}.$$

Hence

$$D_- x(\bar{t}) \geqslant D_- y^*(\bar{t}).$$

Using above and Assumption (i) we obtain

$$\begin{aligned} D_- x(\bar{t}) - f_1(\bar{t}, x(\bar{t})) &\geqslant D_- y^*(\bar{t}) - f_1(\bar{t}, x(\bar{t})) \\ &\geqslant D_- y^*(\bar{t}) - f_2(\bar{t} + \eta, x(\bar{t})) \geqslant D_- y^*(\bar{t}) - f_2(\bar{t} + \eta, y^*(\bar{t})), \end{aligned}$$

which contradicts Assumption (ii).
Case 2. There exists $k, 1 \leqslant k \leqslant p$, such that $\bar{t} = t_k$. From (3.4)-(3.5), we have

$$\begin{aligned} x(t_k^-) &< y^*(t_k^-), \\ x(t_k) &= y^*(t_k). \end{aligned} \tag{3.6}$$

On the other hand from (iv) we have

$$x(t_k) - y^*(t_k) < x(t_k^-) + g_1(t_k, x(t_k^-)) - \left(y^*(t_k^-) + g_2(t_k + \eta, y^*(t_k^-)\right). \tag{3.7}$$

By (iii) we have

$$g_1(t_k, x(t_k^-)) \leqslant g_2(t_k + \eta, x(t_k^-)) \tag{3.8}$$

and

$$g_1(t_k, y^*(t_k^-)) \leqslant g_2(t_k + \eta, y^*(t_k^-)). \tag{3.9}$$

From (3,6),(3.7) and (3.8) we obtain

$$0 = x(t_k) - y^*(t_k) < x(t_k^-) + g_2(t_k + \eta, x(t_k^-)) - \left(y^*(t_k^-) + g_2(t_k + \eta, y^*(t_k^-)\right),$$

which contradicts (iii). From (3.6),(3.7) and (3.9)we obtain

$$0 = x(t_k) - y^*(t_k) < x(t_k^-) + g_1(t_k, x(t_k^-)) - \left(y^*(t_k^-) + g_1(t_k, y^*(t_k^-)\right),$$

which still contradicts (iii). ∎

Corollary 3.2 *The conclusion (3.3) remains valid when the inequality (3.1) is replaced by*

$$\begin{aligned} &D_- x(t) < f_1(t, x(t)), \\ &D_- y^*(t) \geqslant f_2(t + \eta, y^*(t)), \qquad t \in I \setminus \{t_k\}_{k=1}^p \end{aligned}$$

or

$$\begin{aligned} &D_- x(t) \leqslant f_1(t, x(t)), \\ &D_- y^*(t) > f_2(t + \eta, y^*(t)), \quad t \in I \setminus \{t_k\}_{k=1}^p \end{aligned}$$

and the inequalities (3.2) are replaced by

$$\begin{aligned} &\triangle x(t_k) < g_1(t_k, x(t_k^-)), \\ &\triangle y^*(t_k) \geqslant g_2(t_k + \eta, y^*(t_k^-)), \quad k = 1, ..., p \end{aligned}$$

or

$$\begin{aligned} &\triangle x(t_k) \leqslant g_1(t_k, x(t_k^-)), \\ &\triangle y^*(t_k) > g_2(t_k + \eta, y^*(t_k^-)), \quad k = 1, ..., p. \end{aligned}$$

4. Weak inequalities. Now we consider weak impulsive differential inequalities.

THEOREM 4.1 *Suppose that:*

(i) *One of the functions* $f_i, i = 1, 2$, *where* $f_1 \in C(I \times \mathbb{R}, \mathbb{R}), f_2 \in C(J \times \mathbb{R}, \mathbb{R})$ *is strictly decreasing with respect to the argument* $x \in \mathbb{R}$ *for each* $t \in I$ $(t \in J)$ *and*

$$f_1(t, x) \leqslant f_2(t + \eta, x),$$

for $t \in I, x \in \mathbb{R}$.

(ii) *Functions* $x \in PC^1(I, \mathbb{R}), y \in PC^1(J, \mathbb{R})$ *satisfy the differential inequality*

$$x'(t) - f_1(t, x(t)) \leqslant (y^*)'(t) - f_2(t + \eta, y^*(t)), \tag{4.1}$$

where $t \in I, t \neq t_k, k = 1, .., p, y^*(t) = y(t + \eta)$.

(iii) *Functions* $g_1 : \{t_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}, g_2 : \{\tau_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}$ *satisfy the inequality*

$$g_1(t_k, x) \leqslant g_2(\tau_k, x), \ x \in \mathbb{R}, k = 1, ..., p$$

and one of the function $g_1(t_k, \cdot), g_2(\tau_k, \cdot)$ *is strictly decreasing on* $\mathbb{R}, k = 1, ..., p$.

(iv) *Impulsive inequalities*

$$\Delta x(t_k) - g_1(t_k, x(t_k)) \leqslant \Delta y^*(t_k) - g_2(t_k + \eta, y^*(t_k)), \ k = 1, ..., p \tag{4.2}$$

are satisfied.

(v) $x(t_0) \leqslant y(\tau_0)$.

Then

$$x(t) \leqslant y(t + \eta), \ t \in [t_0, b]. \tag{4.3}$$

PROOF Without loss generality, we only prove the case when the functions $f_2(t, \cdot)$, $g_2(\tau_k, \cdot)$, $k = 1, ..., p$ are strictly decreasing on $\mathbb{R}$.

Let $\varepsilon = \sup\{x(t) - y(t + \eta), t \in I\}$. Suppose (4.3) is not true, then $\varepsilon > 0$ and there exists $\bar{t} \in I$ such that

$$\begin{aligned} &x(\bar{t}) - y^*(\bar{t}) = \varepsilon, \\ &x(t) - y^*(t) \leqslant \varepsilon, \ t \in I. \end{aligned} \tag{4.4}$$

We consider the following cases:
Case 1: Suppose that $\bar{t} \in I, \bar{t} \neq t_k, k = 1, ..., p$. Since $x - y^*$ attains its maximum at $t = \bar{t}$ we have $(x - y^*)'(\bar{t}) = 0$. On the other hand it follows from the inequality (4.1) and Assumption (i) that

$$0 = (x - y^*)'(\bar{t}) \leqslant f_1(\bar{t}, x(\bar{t})) - f_2(\bar{t} + \eta, y^*(\bar{t})) \leqslant f_2(\bar{t} + \eta, x(\bar{t})) - f_2(\bar{t} + \eta, y^*(\bar{t})).$$

Then

$$f_2(\bar{t}+\eta, x(\bar{t})) \geqslant f_2(\bar{t}+\eta, y^*(\bar{t}))$$

which is a contradiction because for each $t \in J$, the function $f_2(t,\cdot)$ is strictly decreasing.
Case 2: Suppose that $\bar{t} = t_k$ for some $k, 1 \leqslant k \leqslant p$. Then we have

$$\begin{aligned} x(t_k) - y^*(t_k) &= \varepsilon, \\ x(t_k^-) - y^*(t_k^-) &\leqslant \varepsilon. \end{aligned} \tag{4.5}$$

Making use of (4.2) and Assumption (iii) we obtain

$$\begin{aligned} x(t_k) - y^*(t_k) &\leqslant x(t_k^-) + g_1(t_k, x(t_k)) - y^*(t_k^-) - g_2(t_k+\eta, y^*(t_k)) \\ &\leqslant x(t_k^-) + g_2(t_k+\eta, x(t_k)) - y^*(t_k^-) - g_2(t_k+\eta, y^*(t_k)) < \varepsilon, \end{aligned}$$

which contradicts to (4.5).
Case 3: If $x(t) - y^*(t) < \varepsilon$ for all $t \in I$, then

$$x(t_k^-) - y^*(t_k^-) = \varepsilon$$

for some $1 \leqslant k \leqslant p+1$.

Let $m : [t_{k-1}, t_k] \to \mathbb{R}$ be defined by

$$m(t) = \begin{cases} x(t) - y^*(t), & t \in [t_{k-1}, t_k), \\ \varepsilon, & t = t_k. \end{cases}$$

Thus m is continuous function. Since $m(t) < \varepsilon$ for $t \in [t_{k-1}, t_k)$, then there exists a sequence $\{\nu_n\}$ such that $\nu_n \in [t_{k-1}, t_k), \nu_n < \nu_{n+1}$, $\lim\limits_{n\to\infty} \nu_n = t_k$ and $D_- m(\nu_n) \geqslant 0$ for $n = 1, 2, ...$

Let N_0 be such a positive integer that for $n \geqslant N_0$ we have $x(\nu_n) - y^*(\nu_n) > 0$. Then it follows that

$$\begin{aligned} 0 \leqslant D_- m(\nu_n) \leqslant (x - y^*)'(\nu_n) &\leqslant f_1(\nu_n, x(\nu_n)) - f_2(\nu_n+\eta, y^*(\nu_n)) \\ &\leqslant f_2(\nu_n+\eta, x(\nu_n)) - f_2(\nu_n+\eta, y^*(\nu_n)). \end{aligned}$$

Then

$$f_2(\nu_n+\eta, x(\nu_n)) \geqslant f_2(\nu_n+\eta, y^*(\nu_n)),$$

which is a contradiction because for each $t \in J$, the function $f_2(t,\cdot)$ is strictly decreasing. ∎

5. Existence result. Consider the initial value problems for impulsive differential equations

$$\begin{cases} x'(t) = f(t, x(t)), & t \in I \setminus \{t_k\}_{k=1}^p, \\ \triangle x(t_k) = I_k(x(t_k^-)), & k = 1, ..., p, \\ x(t_0) = x_0 \end{cases} \tag{5.1}$$

and

$$\left\{\begin{array}{l} y'(t) = f(t, y(t)), \quad t \in J \setminus \{\tau_k\}_{k=1}^{p}, \\ \triangle y(\tau_k) = I_k(y(\tau_k^-)), \quad k = 1, ..., p, \\ y(\tau_0) = y_0 \end{array}\right. \tag{5.2}$$

We introduce the following assumptions:

(H1) There exist $v \in PC^1(I, \mathbb{R}), w \in PC^1(J, \mathbb{R})$ such that $v(t) \leqslant w(t+\eta),\ t \in I$, where

$$\left\{\begin{array}{l} v'(t) \leqslant f(t, v(t)), \quad t \in I \setminus \{t_k\}_{k=1}^{p}, \\ \triangle v(t_k) \leqslant I_k(v(t_k^-)), \quad k = 1, ..., p, \\ v(t_0) \leqslant v_0 \end{array}\right. \tag{5.3}$$

and

$$\left\{\begin{array}{l} w'(t) \geqslant f(t, w(t)), \quad t \in J \setminus \{\tau_k\}_{k=1}^{p}, \\ \triangle w(\tau_k) \geqslant I_k(w(\tau_k^-)), \quad k = 1, ..., p, \\ w(\tau_0) \geqslant w_0. \end{array}\right. \tag{5.4}$$

(H2) $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}), f(t, u)$ is nondecreasing in t for each $u \in \mathbb{R}$ and

$$\sup_{v(t) \leqslant u \leqslant w(t+\eta)} |f(t, u)| \leqslant \lambda(t) \quad \text{a.e. on } I,$$

where $\lambda \in L^1(I)$.

(H3) $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing for each $k = 1, ..., p$.

We are now in a position to state and prove existence result for the problem (5.1).

THEOREM 5.1 *Let assumptions (H1)-(H3) hold. Then problem (5.1), where $v_0 \leqslant x_0 \leqslant w_0$ has a solution x such that*

$$v(t) \leqslant x(t) \leqslant w(t+\eta), \quad t \in I.$$

PROOF Let

$$w^*(t) = w(t+\eta), \quad t \in [t_0, t_1].$$

We have

$$w^*(t_0) = w(t_0+\eta) = w(\tau_0) \geqslant v(t_0)$$

and

$$(w^*)'(t) = w'(t+\eta) \geqslant f(t+\eta, w(t+\eta)) = f(t+\eta, w^*(t)), t \in [t_0, t_1].$$

Consider the initial problem

$$\left\{\begin{array}{l} x'(t) = F(t, x(t)), \quad t \in [t_0, t_1], \\ x(t_0) = x_0, \end{array}\right. \tag{5.5}$$

where $F : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t,x) = \begin{cases} f(t, w^*(t)) + \dfrac{w^*(t) - x}{1 + |x|}, & x > w^*(t), \\ f(t,x), \quad v(t) \leqslant x \leqslant w^*(t), \\ f(t, v(t)) + \dfrac{v(t) - x}{1 + |x|}, & x < v(t). \end{cases} \tag{5.6}$$

Since

$$\sup_{x \in \mathbb{R}} |F(t,x)| \leqslant \lambda(t), \text{a.e. on } [t_0, t_1]$$

then (5.5) has a solution x_1 on $[t_0, t_1]$ (see Theorem 2.7.1,[7]).
We prove that

$$v(t) \leqslant x_1(t) \leqslant w^*(t), \ t \in [t_0, t_1], \tag{5.7}$$

hence x_1 is a solution of (5.1), $t \in [t_0, t_1]$.
We shall now prove that

$$x_1(t) \leqslant w^*(t), \quad t \in [t_0, t_1].$$

If it is not true, than the function

$$m(t) = x_1(t) - w^*(t), \quad t \in [t_0, t_1]$$

attains a positive maximum at some $\bar{t} \in (t_0, t_1]$ and

$$\begin{aligned} m(\bar{t}) &> 0, \\ m'(\bar{t}) &\geqslant 0. \end{aligned}$$

In consequence

$$\begin{aligned} x_1(\bar{t}) &> w^*(\bar{t}), \\ x_1'(\bar{t}) &\geqslant (w^*)'(\bar{t}). \end{aligned}$$

On the other hand

$$0 \leqslant x_1'(\bar{t}) - (w^*)'(\bar{t}) \leqslant f(\bar{t}, w^*(\bar{t})) + \frac{w^*(\bar{t}) - x_1(\bar{t})}{1 + |x_1(\bar{t})|} - f(\bar{t} + \eta, w^*(\bar{t})). \tag{5.8}$$

Moreover, in view of H2

$$f(\bar{t}, w^*(\bar{t})) - f(\bar{t} + \eta, w^*(\bar{t})) \leqslant 0.$$

From this and (5.8) we obtain the contradiction

$$0 \leqslant x_1'(\bar{t}) - (w^*)'(\bar{t}) \leqslant \frac{w^*(\bar{t}) - x_1(\bar{t})}{1 + |x_1(\bar{t})|} < 0.$$

Analogously we can prove that

$$v(t) \leqslant x_1(t), \quad t \in [t_0, t_1].$$

Since

$$v(t_1^-) \leqslant x_1(t_1^-)) \leqslant w^*(t_1^-)$$

and I_1 is nondecreasing, we get

$$I_1(v(t_1^-)) \leqslant I_1(x_1(t_1^-)) \leqslant I_1(w^*(t_1^-)).$$

From this, by (5.3) and (5.4) we have

$$\begin{aligned} v(t_1) &\leqslant v(t_1^-) + I_1(v(t_1^-)) \leqslant x_1(t_1^-) + I_1(x_1(t_1^-)) \\ &\leqslant w^*(t_1^-) + I_1(w^*(t_1^-)) = w(\tau_1^-) + I_1(w(\tau_1^-)) \leqslant w(\tau_1) = w(t_1+\eta) = w^*(t_1). \end{aligned}$$

Repeating the same arguments we can show that the problem

$$\begin{cases} x'(t) = F(t, x(t)), \quad t \in [t_1, t_2], \\ x(t_1) = x_1(t_1^-) + I_1(x_1(t_1^-)), \end{cases}$$

where F is defined analogously as in (5.6), has a solution x_2 on $[t_1, t_2]$ such that

$$v(t) \leqslant x_2(t) \leqslant w(t+\eta), \quad t \in [t_1, t_2].$$

So forth and so on for $t \in [t_p, t_{p+1}]$, we consider the initial problem

$$\begin{cases} x'(t) = F(t, x(t)), \quad t \in [t_p, t_{p+1}], \\ x(t_p) = x_p(t_p^-) + I_p(x_p(t_p^-)). \end{cases}$$

Analogously, we can prove that this problem has a solution x_{p+1} such that

$$v(t) \leqslant x_{p+1}(t) \leqslant w(t+\eta), \quad t \in [t_p, t_{p+1}].$$

Continuing the proof, let

$$x(t) = \begin{cases} x_1(t), & t \in [0, t_1); \\ x_2(t), & t \in [t_1, t_2); \\ \ldots & \ldots \\ x_{p+1}(t), & t \in [t_p, t_{p+1}]. \end{cases}$$

Then x is a solution of problem (5.1) and $v(t) \leqslant x(t) \leqslant w(t+\eta),\ t \in I$. ∎

6. Monotone method. Next result develops monotone iterative technique in the present frame work.

LEMMA 6.1 *([13]). Let $m \in PC^1(I, \mathbb{R}), a \in \mathbb{R}$ such that*

$$\begin{aligned} &u'(t) \leqslant au(t), \quad t \in I \setminus \{t_k\}_{k=1}^p, \\ &\triangle u(t_k) \leqslant 0, \quad k = 1, ..., p, \\ &u(t_0) \leqslant 0. \end{aligned}$$

Then $u(t) \leqslant 0, t \in I$.

THEOREM 6.2 *Suppose that assumptions of Theorem 5.1 hold and there exists* $M \geqslant 0$ *such that*

$$f(t,x) - f(t,y) \geqslant -M(x-y) \tag{6.1}$$

whenever $t \in I, v(t) \leqslant y \leqslant x \leqslant w(t+\eta)$.
Then there exist monotone sequences $\{v_n\}, \{\bar{w}_n\}$ *such that* $\lim\limits_{n\to\infty} v_n(t) = \rho(t)$, $\lim\limits_{n\to\infty} \bar{w}_n(t) = \bar{r}(t), \bar{r} = r(t+\eta)$ *monotonically and piecewise uniformly on* I, *where* ρ *is the minimal solution of (5,1) with* $x_0 = v_0$ *and* r *is maximal solution of (5.2) with* $y_0 = w_0$ *in the sector* $[v, \bar{w}], \bar{w}(t) = w(t+\eta), t \in I$.

PROOF We define

$$\begin{aligned} \bar{w}_1(t) &= w(t+\eta), \\ v_1(t) &= v(t),\ t \in I \end{aligned}$$

as the first iterations. Next define the sequences $\{v_n\}, n > 1$ as the solution of

$$\begin{aligned} & v'_{n+1}(t) = f(t, v_n(t)) - M\left(v_{n+1}(t) - v_n(t)\right),\ t \in I \setminus \{t_k\}_{k=1}^p, \\ & \triangle v_{n+1}(t_k) = I_k\left(v_n(t_k^-)\right),\ k = 1, ..., p, \\ & v_{n+1}(t_0) = v_0 \end{aligned} \tag{6.2}$$

and the sequence $\{\bar{w}_n\}, n > 1$ as the solution of

$$\begin{aligned} & \bar{w}'_{n+1}(t) = f(t+\eta, \bar{w}_n(t)) - M\left(\bar{w}_{n+1}(t) - \bar{w}_n(t)\right),\ t \in I \setminus \{t_k\}_{k=1}^p, \\ & \triangle \bar{w}_{n+1}(t_k) = I_k\left(\bar{w}_n(t_k^-)\right),\ k = 1, ..., p, \\ & \bar{w}_{n+1}(t_0) = w_0, \end{aligned} \tag{6.3}$$

where $\bar{w}_n(t) = w_n(t+\eta), t \in I$.
The existence and uniqueness of solutions for the above systems are guarantee by general results on the initial value problem of impulsive differential equations. It can be shown by induction that

$$v_n(t) \leqslant v_{n+1}(t) \leqslant \bar{w}_{n+1}(t) \leqslant \bar{w}_n(t), \quad t \in I, n \geqslant 1. \tag{6.4}$$

If we consider $m(t) = v_1(t) - v_2(t), t \in I$ then

$$\begin{aligned} m'(t) &= v'_1(t) - v'_2(t) \\ &\leqslant f(t, v_1(t)) - f(t, v_1(t)) + M\left(v_2(t) - v_1(t)\right) = -Mm(t),\ t \in I \setminus \{t_k\}_{k=1}^p, \\ \triangle m(t_k) &= \triangle v_1(t_k) - \triangle v_2(t_k) \leqslant I_k\left(v_1(t_k^-)\right) - I_k\left(v_1(t_k^-)\right) = 0, \quad k = 1, ..., p, \\ m(t_0) &= v_1(t_0) - v_2(t_0) = v_0 - v_0 = 0. \end{aligned}$$

By Lemma 6.1 we have $m(t) \leqslant 0,\ t \in I$. In consequence $v_1(t) \leqslant v_2(t), t \in I$.
Analogously let

$$m(t) = \bar{w}_2(t) - \bar{w}_1(t), t \in I.$$

We have

$$\begin{aligned} m'(t) &= \bar{w}_2'(t) - \bar{w}_1'(t) \\ &\leqslant f(t+\eta, \bar{w}_1(t)) - M\left(\bar{w}_2(t) - \bar{w}_1(t)\right) - f(t+\eta, \bar{w}_1(t)) \\ &= -Mm(t), \quad t \in I \setminus \{t_k\}_{k=1}^p, \\ \triangle m(t_k) &= \triangle \bar{w}_2(t_k) - \triangle \bar{w}_1(t_k) \leqslant I_k\left(\bar{w}_1(t_k^-)\right) - I_k\left(\bar{w}_1(t_k^-)\right) = 0, \quad k = 1, ..., p, \\ m(t_0) &= \bar{w}_2(t_0) - \bar{w}_1(t_0) = w_0 - w_0 = 0. \end{aligned}$$

By Lemma 6.1 we have $m(t) \leqslant 0,\ t \in I$. In consequence $\bar{w}_2(t) \leqslant \bar{w}_1(t), t \in I$.

Now let $m(t) = v_2(t) - \bar{w}_2(t)$. We have

$$\begin{aligned} m'(t) = v_2'(t) - \bar{w}_2'(t) &= f(t, v_1(t)) - M\left(v_2(t) - v_1(t)\right) \\ &\quad - f(t+\eta, \bar{w}_1(t) + M\left(\bar{w}_2(t) - \bar{w}_1(t)\right), \quad t \in I \setminus \{t_k\}_{k=1}^p. \end{aligned}$$

Using H2, we have

$$f(t, \bar{w}_1(t)) \leqslant f(t+\eta, \bar{w}_1(t)), \quad t \in I.$$

This together with the condition (6.1) give

$$\begin{aligned} m'(t) &\leqslant f(t, v_1(t)) - M\left(v_2(t) - v_1(t)\right) - f(t, \bar{w}_1(t)) + M\left(\bar{w}_2(t) - \bar{w}_1(t)\right) \\ &\leqslant M\left(\bar{w}_1(t) - v_1(t)\right) - M\left(v_2(t) - v_1(t)\right) + M\left(\bar{w}_2(t) - \bar{w}_1(t)\right) \\ &= -M\left(v_2(t) - \bar{w}_2(t)\right) = -Mm(t), \quad t \in I \setminus \{t_k\}_{k=1}^p, \\ \triangle m(t_k) &= \triangle v_2(t_k) - \triangle \bar{w}_2(t_k) = I_k\left(v_1(t_k^-)\right) - I_k\left(\bar{w}_1(t_k^-)\right) \leqslant 0, \quad k = 1, ..., p, \\ m(t_0) &= v_2(t_0) - \bar{w}_2(t_0) = v_0 - w_0 \leqslant 0. \end{aligned}$$

By Lemma 6.1 we have $m(t) \leqslant 0,\ t \in I$. In consequence $v_2(t) \leqslant \bar{w}_2(t), t \in I$. We see that inequalities (6.4) are valid for $n = 1$. Assuming that inequalities (6.4) are valid for $n \geqslant 1$ we show that they are valid for $n+1$.

Let

$$m(t) = \bar{w}_{n+1}(t) - \bar{w}_n(t), \quad t \in I.$$

We have

$$\begin{aligned} m'(t) &= \bar{w}_{n+1}'(t) - \bar{w}_n'(t) = f(t+\eta, \bar{w}_n(t)) - M\left(\bar{w}_{n+1}(t) - \bar{w}_n(t)\right) \\ &\quad - f(t+\eta, \bar{w}_{n-1}(t)) + M\left(\bar{w}_n(t) - \bar{w}_{n-1}(t)\right) \\ &\leqslant M\left(\bar{w}_{n-1}(t) - \bar{w}_n(t)\right) - M\left(\bar{w}_{n+1}(t) - \bar{w}_n(t)\right) + M\left(\bar{w}_n(t) - \bar{w}_{n-1}(t)\right) \\ &= -M\left(\bar{w}_{n+1}(t) - \bar{w}_n(t)\right) = -Mm(t), \quad t \in I \setminus \{t_k\}_{k=1}^p, \\ \triangle m(t_k) &= \triangle \bar{w}_{n+1}(t_k) - \triangle \bar{w}_n(t_k) = I_k\left(\bar{w}_n(t_k^-)\right) - I_k\left(\bar{w}_{n-1}(t_k^-)\right) \leqslant 0,\ k = 1, ..., p, \\ m(t_0) &= \bar{w}_{n+1}(t_0) - \bar{w}_n(t_0) = w_0 - w_0 \leqslant 0. \end{aligned}$$

By Lemma 6.1 we have $m(t) \leqslant 0, t \in I$. In consequence $\bar{w}_{n+1}(t) \leqslant \bar{w}_n(t), t \in I$.

Let

$$m(t) = v_{n+1}(t) - \bar{w}_n(t), \quad t \in I.$$

We have

$$\begin{aligned} m'(t) = v_{n+1}'(t) - \bar{w}_n'(t) &= f(t, v_n(t)) - M\left(v_{n+1}(t) - v_n(t)\right) \\ &\quad - f(t+\eta, \bar{w}_n(t)) + M\left(\bar{w}_{n+1}(t) - \bar{w}_n(t)\right). \end{aligned}$$

By H2 we have

$$f(t,\bar{w}_n(t)) \leqslant f(t+\eta,\bar{w}_n(t))$$

and in consequence using (6.1) we obtain

$$\begin{aligned}
m'(t) &\leqslant f(t,v_n(t)) - M\left(v_{n+1}(t)-v_n(t)\right) - f(t,\bar{w}_n(t)) + M\left(\bar{w}_{n+1}(t)-\bar{w}_n(t)\right)\\
&\leqslant M\left(\bar{w}_n(t)-v_n(t)\right) - M\left(v_{n+1}(t)-v_n(t)\right) + M\left(\bar{w}_{n+1}(t)-\bar{w}_n(t)\right)\\
&= -M\left(v_{n+1}(t)-\bar{w}_{n+1}(t)\right) = -Mm(t), \quad t\in I\setminus\{t_k\}_{k=1}^{p},\\
\triangle m(t_k) &= \triangle v_{n+1}(t_k) - \triangle\bar{w}_{n+1}(t_k) = I_k\left(v_n(t_k^-)\right) - I_k\left(\bar{w}_n(t_k^-)\right) \leqslant 0, \quad k=1,...,p,\\
m(t_0) &= v_{n+1}(t_0) - \bar{w}_{n+1}(t_0) = v_0 - w_0 \leqslant 0.
\end{aligned}$$

By Lemma 6.1 we have $m(t)\leqslant 0,\ t\in I$. In consequence $v_{n+1}(t)\leqslant\bar{w}_{n+1}(t), t\in I$. Analogously we can prove that $v_n(t)\leqslant v_{n+1}(t), t\in I$. By induction we have proved (6.4).

It than follows from standard arguments that $\lim\limits_{n\to\infty} v_n(t)=\rho(t)$, $\lim\limits_{n\to\infty}\bar{w}_n(t)=\bar{r}(t), \bar{r}(t)=r(t+\eta)$ piecewise uniformly on I, and that ρ, r are solutions of (5.1),(5.2) respectively in view of the fact that v_n satisfy (6.2) and $\bar{w}_n$ satisfy (6.3).

To prove that r is maximal solution of (5.2) on $[v,\bar{w}]$, let y be any solution of (5.2) with $y_0=w_0$ on $[v,\bar{w}]$. It is obvious that

$$y(t+\eta)\leqslant w(t+\eta)=\bar{w}_1(t),\ t\in I.$$

Suppose that for some n we have

$$y(t+\eta)\leqslant w_n(t+\eta)=\bar{w}_n(t),\ t\in I.$$

Then setting $m(t)=y(t+\eta)-\bar{w}_{n+1}(t)$ we get

$$\begin{aligned}
m'(t) &= y'(t+\eta) - \bar{w}'_{n+1}(t)\\
&= f(t+\eta,y(t+\eta)) - f(t+\eta,\bar{w}_n(t)) + M\left(\bar{w}_{n+1}(t)-\bar{w}_n(t)\right)\\
&\leqslant M\left(\bar{w}_n(t)-y(t+\eta)\right) + M\left(\bar{w}_{n+1}(t)-\bar{w}_n(t)\right)\\
&= -M\left(y(t+\eta)-\bar{w}_{n+1}(t)\right) = -Mm(t), \quad t\in I\setminus\{t_k\}_{k=1}^{p},\\
\triangle m(t_k) &= \triangle y(t_k+\eta) - \triangle\bar{w}_{n+1}(t_k) = I_k\left(y(t_k^-+\eta)\right) - I_k\left(\bar{w}_n(t_k^-)\right) \leqslant 0,\ k=1,...,p,\\
m(t_0) &= y(t_0+\eta) - \bar{w}_{n+1}(t_0) = w_0 - w_0 \leqslant 0.
\end{aligned}$$

By Lemma 6.1 we have $m(t)\leqslant 0$. In consequence

$$y(t+\eta)\leqslant\bar{w}_{n+1}(t), \quad t\in I.$$

Now it follows by induction that

$$y(t+\eta)\leqslant\bar{w}_n(t),\ t\in I, n\in N.$$

Thus passing the limit we may conclude that

$$y(t+\eta)\leqslant r(t+\eta), \quad t\in I.$$

Let x be any solution of (5.1)with $x_0 = v_0$ on $[v, \bar{w}]$. The same arguments prove that

$$v_n(t) \leqslant x(t), \quad t \in I, n \in N.$$

By taking limits as $n \to \infty$ and using Theorem 5.1 we may conclude that

$$\rho(t) \leqslant x(t) \leqslant y(t+\eta) \leqslant r(t+\eta), \quad t \in I.$$

The proof is complete. ∎

7. Differential-functional inequalities. Let $I = [t_0, b], t_0 < b \leqslant \infty$. Let $\tau > 0$ be a fixed real number. We denote

$$I_0 = [t_0 - \tau, t_0], \quad I_{t_0} = [t_0 - \tau, b],$$

$$PC([-\tau, 0], \mathbb{R}) = \{\phi : [-\tau, 0] \to \mathbb{R},\ \phi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \phi(\bar{t}^-) \text{ and } \phi(\bar{t}^+) \text{ exist and } \phi(\bar{t}^+) = \phi(\bar{t})\}.$$

Suppose that $t_0 < t_1 < ... < t_p < t_{p+1} = b$ are given numbers. We define

$$PC(I_{t_0}, \mathbb{R}) = \{u : I_{t_0} \to \mathbb{R}, u \text{ is continuous in } I_{t_0} \setminus \{t_k\}_{k=1}^{p}, \text{the limits } u(t_k^+), u(t_k^-), k = 1, ..., p \text{ exist and } u(t_k^+) = u(t_k), i = 1, ..., p\}.$$

If $x \in PC(I_{t_0}, \mathbb{R})$, then for any $t \in I$ define the function $x_t \in PC([-\tau, 0], \mathbb{R})$ by

$$x_t(\Theta) = x(t + \Theta), \quad -\tau \leqslant \Theta \leqslant 0.$$

As we see x_t is the graph of x on $[t - \tau, t]$ shifted to the interval $[-\tau, 0]$.
Let τ_0 be such that $t_0 < \tau_0 < t_1$. Denote

$$\begin{aligned} &\tau_0 - t_0 = \eta, \\ &\tau_k = t_k + \eta, k = 1, ..., p+1, \\ &J = [\tau_0, b + \eta], \\ &J_{\tau_0} = [\tau_0 - \tau, b + \eta]. \end{aligned}$$

Analogously

$$PC(J_{\tau_0}, \mathbb{R}) = \{u : J_{\tau_0} \to \mathbb{R}, u \text{ is continuous in } J_{\tau_0} \setminus \{\tau_k\}_{k=1}^{p}, \text{the limits } u(\tau_k^+), u(\tau_k^-), k = 1, ..., p \text{ exist and } u(\tau_k^+) = u(\tau_k), k = 1, ..., p\}.$$

If $y \in PC(J_{\tau_0}, \mathbb{R})$, then for any $t \in J$ define the function $y_t \in PC([-\tau, 0], \mathbb{R})$ by

$$y_t(\Theta) = y(t + \Theta), \quad -\tau \leqslant \Theta \leqslant 0.$$

The following theorem generalizes results obtained in [17].

THEOREM 7.1 *Suppose that:*

(i) *One of the functions* $f_i, i = 1, 2$, *where*

$$f_1 \in C\left(I \times \mathbb{R} \times PC([-\tau, 0], \mathbb{R}), \mathbb{R}\right),$$

$$f_2 \in C\left(J \times \mathbb{R} \times PC([-\tau, 0], \mathbb{R}), \mathbb{R}\right)$$

is nondecreasing with respect to the functional argument $\varphi \in PC([-\tau, 0], \mathbb{R})$ *for each* $t \in I (t \in J), x \in \mathbb{R}$ *and*

$$f_1(t, x, \varphi) \leqslant f_2(t + \eta, x, \varphi),$$

for $t \in I, x \in \mathbb{R}, \varphi \in PC([-\tau, 0], \mathbb{R})$.

(ii) *Functions* $x \in PC\left(I_{t_0}, \mathbb{R}\right), y \in PC\left(J_{\tau_0}, \mathbb{R}\right)$ *satisfy the functional-differential inequality*

$$D_- x(t) - f_1(t, x(t), x_t) < D_- y^*(t) - f_2(t + \eta, y^*(t), y_t^*),$$

where $t \in I, t \neq t_k, k = 1, .., p, y^*(t) = y(t + \eta)$.

(iii) *Functions* $g_1 : \{t_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}, g_2 : \{\tau_k\}_{k=1}^p \times \mathbb{R} \to \mathbb{R}$ *satisfy the inequality*

$$g_1(t_k, x) \leqslant g_2(\tau_k, x), \ x \in \mathbb{R}, k = 1, ..., p$$

and for each $t = t_k, k = 1, ..., p$ *the function* $\sigma_1 : \mathbb{R} \ni s \to s + g_1(t, s)$ *or for each* $t = \tau_k, k = 1, ..., p$ *the function* $\sigma_2 : \mathbb{R} \ni s \to s + g_2(t, s)$ *is non-decreasing on* $\mathbb{R}$.

(iv) *Impulsive inequalities*

$$\Delta x(t_k) - g_1\left(t_k, x(t_k^-)\right) < \Delta y^*(t_k) - g_2\left(t_k + \eta, y^*(t_k^-)\right), \ k = 1, ..., p$$

are satisfied.

(v) $x_{t_0} < y_{\tau_0}$ *in* $[-\tau, 0]$.

Then

$$x(t) < y(t + \eta), \ t \in [t_0, b].$$

The proof is similar to the proof of Theorem 1.1 and we omit it.

REFERENCES

[1] B. Ahmed, *Integro-differential equations with initial time difference,* Dynam. Systems Appl. **16**(2007), no.3, 497-506.

[2] Jun Yan Bao, Chun Xia Gao, Xian Wang, *Boundedness of differential equations with initial time difference,* Math. Practice Theory **38**(2008), 215-218.

[3] T. Jankowski, *Delay integro-differential inequalities with initial time difference and applications,* J. Math. Anal. Appl. **291**(2004), 605-624.

[4] T. Jankowski, *Systems of differential inequalities with initial time difference,* Ukraïn. Math. Zh. **56**(2004), no.I, 139-145.

[5] T. Jankowski, *Quadratic convergence of monotone iterations for differential equations with initial time difference,* Dynam. Systems Appl. **14**(2005), no. 2, 245-251.

[6] T. Jankowski, *Delay differential inequalities with initial time difference,* Proceedings of Neural, Parallel, and Scientific Computations Vol.3, 82-88, Dynamic, Atlanta, GA, 2006.

[7] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of impulsive differential equations,* World Scientific Publishing, (1989).

[8] V. Lakshmikantham, S. Leela, *Differential and integral inequalities,* vol.I, Academic Press (1969).

[9] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Another approach to the theory of differential inequalities relative to changes in the initial times,* J. Inequal. Appl. **4** (1999), 163-174.

[10] V. Lakshmikantham, A.S. Vatsala, *Differential inequalities with initial time difference and applications,* J. Inequal. Appl. **3** (1999), 233-244.

[11] M.D. Shaw, C. Yakar, *Generalized variation of parameters with initial time difference and a comparison result in terms Lyapunov like functions,* International Journal of Non-Linear Differential Equations **5**(1999), 86-108.

[12] L. Skóra, *Remarks on first order impulsive ordinary differential equations with anti-periodic boundary conditions,* Fasciculi Mathematici **36** (2005), 103-108.

[13] L. Skóra, *Monotone iterative method for differential systems with impulses and anti-periodic boundary condition,* Ann. Soc. Math. Polon., Series I: Commentationes Mathematicae **XLII (2)** (2002), 237-249.

[14] Xinyu Song, Senlin Li, An Li, *Practical stability of nonlinear differential equation with initial time difference,* Appl. Math. Comput. **203**(2008), 157-162.

[15] Xinyu Song, An Li, Zhixiang Wang, *Study on the stability of nonlinear differential equations with initial time difference,* Nonlinear Anal. Real World Appl. **11**(2010), 1304-1311.

[16] Y. Zhang, B. Zhang, *Impulsive differential equations with initial time difference and applications,* Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal. **9** (2002), 439-447.

[17] Y. Zhang, Yi Zhang, *Theory of functional differential equation with initial data difference,* Dynamic Systems and Applications **10** (2001), 523-532.

[18] C. Yakar, *Variation-of-parameters formulae and Lipschitz stability criteria for nonlinear matrix differential equations with initial time difference,* Differential & difference equations and applications, 1201-1216, Hindawi Publ. Corp., New York, 2006.

Lidia Skóra
Cracow Technical University, Institute of Mathematics
Warszawska 24, 31-155 Kraków
E-mail: lskora@usk.pk.edu.pl

(Received: 9.11.2010)