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Continuity of real valued subquadratic functions

Abstract. The aim of this paper is to prove a regularity theorem for real valued subquadratic mappings that are solutions of the inequality

$$\varphi(x+y)+\varphi(x-y)\leqslant 2\varphi(x)+2\varphi(y), \qquad x,y\in X,$$

where $X=(X,+)$ is a topological group.

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1. Introduction. In the present paper we investigate real valued subquadratic functions that are solutions of the inequality

$$\varphi(x+y)+\varphi(x-y)\leqslant 2\varphi(x)+2\varphi(y), \qquad x,y\in X, \tag{1}$$

where X is a topological group. A crucial role in our proofs play an observation which is valid for arbitrary real functions defined on a topological group X having an additional property. It is done in section 2. Section 3 contains main regularity theorems for such type functions. By $\mathbb{R}$ and $\mathbb{N}$ we denote the sets of all real numbers and all positive integers, respectively.

2. An observation and some regular properties. At the beginning of this section we will remind some properties of subquadratic functions which are proved in [1] and [2].

Lemma 2.1 ([1],[2]) *Let $X=(X,+)$ be a group and let $\varphi: X\to\mathbb{R}$ be a subquadratic function. Then*

$$\varphi(0)\geqslant 0$$

and

$$\varphi(kx)\leqslant k^2\varphi(x), \qquad x\in X$$

for each positive integer k.

By a topological group we mean a group endowed with a topology such that the group operation as well as taking inverses are continuous functions. Theorem 2.2 below was proved in [1] where, with some reasons, subquadratic functions are called weakly subquadratic.

THEOREM 2.2 ([1]) *Let X be a 2-divisible topological abelian group which is generated by any neighbourhood of $0 \in X$ and let $\varphi : X \to \mathbb{R}$ be a subquadratic function. If φ is locally bounded from above at a point and locally bounded from below at a point then it is locally bounded (bilateraly) at every point of X.*

We adopt the following definition.

DEFINITION 2.3 We say that 2-divisible topological group X has the property $(\frac{1}{2})$ if and only if for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.

The following observation plays a crucial role in the proof of our Theorem 2.5 below.

OBSERVATION 1 Let X be a 2-divisible topological group having the property $(\frac{1}{2})$, $f : X \to \mathbb{R}$ be an arbitrary function. Then for every $u \in X$, each $\varepsilon > 0$ and every neighbourhood W_0 of $0 \in X$ there exists a neighbourhood W of zero in X such that $\frac{1}{2}W \subset W \subset W_0$ and

$$\inf\{f(v); v \in u + W\} + \varepsilon \geqslant \inf\{f(v'); v' \in u + \frac{1}{2}W\}.$$

PROOF Let us fixed $u \in X$. If f is not locally bounded from below at the point u, then for every $\varepsilon > 0$ and each neighbourhood W of zero in X the left and right hand side of the required inequality are equal to minus infinity. So, we may assume that f is locally bounded below at the point u. Then there exists a neighbourhood U of zero in X such that

$$\inf\{f(v); v \in u + U\} > -\infty.$$

For indirect proof assume that there exist an $\varepsilon > 0$ and a neighbourhood V_0 of zero in X such that for every neighbourhood V of zero, $V \subset V_0$, we have

$$\inf\{f(v); v \in u + V\} + \varepsilon < \inf\{f(v'); v' \in u + \frac{1}{2}V\}. \tag{2}$$

Due to the property $(\frac{1}{2})$ there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V_0 \cap U$. The set $\frac{1}{2}W$ is open, because it is a preimage of W under continuous function $s(x) = 2x,\ x \in X$. Define a sequence $(W_n)_{n\in\mathbb{N}}$ of the neighbourhoods of zero by the relation

$$W_n := \frac{1}{2^n}W, \qquad n \in \mathbb{N} \cup \{0\}$$

and we put

$$a_n := \inf\{f(v); v \in u + W_n\}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Hence and by virtue of (2) we get

$$a_n + \varepsilon < a_{n+1}, \qquad n \in \mathbb{N}.$$

Moreover, $(a_n)_{n\in\mathbb{N}}$ is nondecreasing, because $W_{n+1} \subset W_n$, $n \in \mathbb{N}$, and bounded above by $f(u)$. Therefore

$$\lim_{n\to\infty} a_n + \varepsilon \leqslant \lim_{n\to\infty} a_n,$$

which is impossible. This completes the proof of the Observation. ∎

THEOREM 2.4 *Let X be a 2-divisible topological abelian group having the property $(\frac{1}{2})$ and let $\varphi : X \to \mathbb{R}$ be a subquadratic function satisfying condition $\varphi(0) = 0$. If φ is locally bounded below at every point of X and upper semicontinuous at zero, then φ is continuous everywhere in X.*

PROOF We will proof that φ is upper and lower semicontinuous everywhere in X. Fix arbitrary $u \in X$ and an $\varepsilon > 0$. On account of the upper semicontinuity of φ at zero there exists a neighbourhood U_0 of zero in X such that

$$\varphi(t) \leqslant \frac{1}{4}\varepsilon, \qquad t \in U_0. \tag{3}$$

Without lost of the generality we may assume that φ is bounded below on the set $u + U_0$. According to our Observation there exists a neighbourhood W of zero in X such that $\frac{1}{2}W \subset W \subset U_0$ and

$$\inf\{\varphi(v); v \in u + W\} + \frac{1}{4}\varepsilon \geqslant \inf\{\varphi(v'); v' \in u + \frac{1}{2}W\}. \tag{4}$$

Replacing x by $\frac{v+u}{2}$ and y by $\frac{v-u}{2}$ in (1) we obtain

$$\varphi(v) + \varphi(u) \leqslant 2\varphi(\frac{v+u}{2}) + 2\varphi(\frac{v-u}{2}).$$

Take arbitrary $v \in u + W$. Then $\frac{v-u}{2} \in U_0$. It follows from the above inequality and (3) that

$$\varphi(v) + \varphi(u) \leqslant 2\varphi(\frac{v+u}{2}) + \frac{1}{2}\varepsilon.$$

Note that we also have $\frac{v+u}{2} \in u + \frac{1}{2}W$. Therefore

$$\varphi(u) + \inf\{\varphi(v); v \in u + W\} \leqslant 2 \inf\{\varphi(v') : v' \in u + \frac{1}{2}W\} + \frac{1}{2}\varepsilon.$$

Hence using also (4) we obtain

$$\varphi(u) \leqslant \inf\{\varphi(v); v \in u + W\} + \varepsilon.$$

Consequently,

$$\varphi(u) \leqslant \varphi(v) + \varepsilon, \qquad v \in u + W,$$

which means that φ is lower semicontinuous at the point u. It remains to show that φ is upper semicontinuous at each point of X.
Let us fix arbitrarily $y \in X$ and $\varepsilon > 0$. Let U_0 be a neighbourhood of zero such that

$$\varphi(t) \leqslant \frac{1}{3}\varepsilon, \qquad t \in U_0. \tag{5}$$

It follows from the lower semicontinuity of φ at y that there exists a neighbourhood $W \subset U_0$ of zero in X such that

$$\varphi(y) \leqslant \varphi(y+v) + \frac{1}{3}\varepsilon, \qquad v \in W. \tag{6}$$

Put $U_1 := W \cap (-W)$. Since φ is subquadratic function we have

$$\varphi(y+v) \leqslant 2\,\varphi(y) + 2\,\varphi(v) - \varphi(y-v), \qquad v \in U_1. \tag{7}$$

It follows from (7), (5) and (6) that

$$\varphi(y+v) \leqslant \varphi(y) + \varepsilon, \qquad v \in U_1,$$

which ends the proof of upper semicontinuity of φ at the point y. This completes the proof of Theorem 2.5. ∎

3. Main result. As a consequence of Lemma 2.1, Theorems 2.2 and 2.5 we obtain the main result of the paper.

THEOREM 3.1 *Let X be a 2-divisible topological abelian group having the property $(\frac{1}{2})$, which is generated by any neighbourhood of zero in X. Assume that a subquadratic function $\varphi : X \to \mathbb{R}$ satisfies the following conditions:*
(i) $\varphi(0) \leqslant 0$;
(ii) φ is locally bounded from below at a point of X;
(iii) φ is upper semicontinuous at zero.
Then φ is continuous everywhere in X.

The following example shows that the assumption of the upper semicontinuity at zero cannot be replaced by the assumption of the upper semicontinuity at other points.

EXAMPLE 3.2 The function $\varphi : \mathbb{R} \to \mathbb{R}$ given by the formula

$$\varphi(x) = \begin{cases} 1 & if\, x \neq 0 \\ 0 & if\, x = 0 \end{cases},$$

is subquadratic and discontinuous at the point $x = 0$.

We end our paper with a following remark.

REMARK 3.3 Recently A. Gilányi and K. Troczka-Pawelec proved [1] that if a real valued subquadratic function φ defined on a 2-divisible topological abelian group X, which is generated by any neighbourhood of zero in X, is continuous at the point $x = 0$ and $\varphi(0) = 0$, then φ is continuous everywhere in X. Notice that, in the case where X has the property $(\frac{1}{2})$, it is a consequence of our Theorem 3.1.

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