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Local properties of certain solutions of differential equations of elliptic type

In this paper we shall prove some theorems dealing with the local behaviour of the following equations:

$$\Delta u(X)+k(r)u(X)=0, \tag{1.1}$$

$$\Delta u(X)+\sum_{i=1}^{p} b_i(X)u_{x_i}+c(X)u(X)=0, \tag{2.1}$$

$$\Delta^2 u(X)-k^2(r)u(X)=0, \tag{3.1}$$

$$\Delta^{(p)}u(X)+C_p\Delta^{(p-1)}u(X)+\ldots+C_0u(X)=0, \tag{4.1}$$

$$\sum_{i,k=1}^{p} a_{ik}u''_{x_ix_k}(X)+\sum_{j=1}^{p} b_ju_{x_j}+p(X)u(X)=0. \tag{5.1}$$

Here $X=(x_1,\ldots,x_p)$, p is a positive integer, $C_0,\ldots,C_p$ in (4.1) are constants, as well as $a_{ik}=a_{ki}$ and b_j in (5.1). Further $\Delta^2 u=\Delta\Delta u$, $\Delta u=\sum_{i=1}^{p} u''_{x_ix_i}$ and $r^2=x_1^2+\ldots+x_p^2$.

Some solutions of those equations will be obtained by separation of variables. Lemmas 6 and 7 and Theorems 1, 2 and 3 of section 1 are generalizations of the results of Niemeyer [4] dealing with the Helmholtz equation $\Delta u+ku=0$ with k constant. In sections 2-5 we transfer the results of H. Niemeyer into the equations more general than (1.1). In section 1 we deal with equation (1.1) and in the following ones with equations (2.1)-(5.1).

1. We deal first with equation (1.1).

Let $H_n(X)=H_n(x_1,\ldots,x_p)$ be a harmonic polynomial of degree n and let $X=rX_0$, $\|X_0\|=1$, where $\|\ \|$ is the Euclidean norm; then

$$H_n(X)=r^nK_n(X_0).$$

$K_n(X_0)$ is called the *spherical function of order* n, ω will denote the area of the surface of the p-dimensional unit sphere.

It is well known [2], [4] that there exist exactly

$$N = N(p, n) = \frac{2n+p-2}{n}\binom{n+p-3}{n-1} = \binom{n+p-1}{p-1} - \binom{n+p-3}{p-1} \tag{1.0}$$

homogeneous, linearly independent, harmonic polynomials of degree n.

Let us introduce the spherical coordinates $r, \varphi_1, \ldots, \varphi_{p-1}$; then

$$x_i = r\Phi_i(\varphi_1, \ldots, \varphi_{p-1}), \quad i = 1, \ldots, p, \tag{1.2}$$

where

$$\Phi_1 = \prod_{i=1}^{p-1} \cos\varphi_i, \quad \Phi_2 = \prod_{i=1}^{p-2} \cos\varphi_i, \quad \Phi_3 = \prod_{i=1}^{p-3} \cos\varphi_i, \ldots, \quad \Phi_p = \sin\varphi_1,$$

$$0 < r \leqslant R, \quad D_1 = \{0 \leqslant \varphi_1 \leqslant \pi, \ldots, 0 \leqslant \varphi_{p-2} \leqslant \pi, \ 0 \leqslant \varphi_{p-1} \leqslant 2\pi\}.$$

Let us write

$$u(X) = U(r, \Phi);$$

then equation (1.1) takes on the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{p-1}{r}\frac{\partial U}{\partial r} + r^{-2}\Delta_1 U + k(r)\, U = 0, \tag{1.3}$$

where Δ_1, called the *Beltrami operator*, satisfies the condition (see [4])

$$\Delta_1 K_n(X_0) + n(n+p-2)\, K_n(X_0) = 0.$$

Let us deal with the solutions of (1.3) of form

$$U(r, \Phi) = w(r)\, Y(\Phi),$$

where $U(r, \Phi)$ is of class C^2 in $D_1 \times (0, R]$. Inserting this into (1.3) and separating the variables we obtain

$$\frac{w''(r)}{w(r)} + \frac{p-1}{r}\frac{w'(r)}{w(r)} + r^{-2}\frac{\Delta_1 Y(\Phi)}{Y(\Phi)} + k(r) = 0.$$

Thus we are lead to the equations

$$\Delta_1 Y(\Phi) + \lambda Y(\Phi) = 0, \tag{1.4}$$

$$w''(r) + (p-1)r^{-1}w'(r) + [k(r) + \lambda r^{-2}]w(r) = 0, \tag{1.5}$$

where λ is a constant. Supposing that the solution of (1.4) is of class C^2 in D, we obtain $\lambda_n = n(n+p-2)$, $n = 0, 1, 2, \ldots$, for the possible values of the constant λ. To those values correspond the spherical functions

$$Y_n(\Phi) = K_n(X_0),$$

as the solutions of (1.4). Then the solution of (1.5) may be written as

$$r^2 w_n''(r)+(p-1)rw_n'(r)+[r^2k(r)-n(n+p-2)]w_n(r)=0, \tag{1.6}$$

where $n=0,1,2,\ldots$

Let

$$r^2k(r)=\sum_{i=1}^{\infty}b_ir^i, \qquad k_n(r)=-n(n+p-2)+\sum_{i=1}^{\infty}b_ir^i,$$

where b_i are constants, $i=1,2,\ldots$, and let $b_0(n)=-n(n+p-2)$, $p\geqslant 2$, $n=0,1,2,\ldots$

Applying the results of monograph [5], vol. 3, part II, pp. 339-345, we can state the following lemmas.

LEMMA 1. *Let $r^2k(r)$ be an entire function of the variable r, and let*

$$\varrho_1(n)=\tfrac{1}{2}\{-(p-2)+[(p-2)(p-2+4n)+4n^2]^{1/2}\}=n$$

be a positive solution of the equation

$$f_0(\varrho)=\varrho^2+(p-2)\varrho-n(n+p-2)=0;$$

then there exists a sequence of entire functions

$$w_n^{(1)}=r^n\sum_{s=0}^{\infty}C_s^{(n)}r^s, \tag{0}$$

where $C_0^{(n)}=1$, $n=0,1,2,\ldots$, which are solutions of equations (1.6) *or $r\geqslant 0$.*

The coefficients $C_m^{(n)}$ are determined by the equations

$$C_m^{(n)}f_0(n+m)+C_{m-1}^{(n)}f_1(m+n-1)+\ldots+C_0^{(n)}f_m(n)=0,$$

where $m=0,1,2,\ldots$, $n=0,1,2,\ldots$, and

$$f_0(n+m)=m^2+2nm+(p-2)m,\ f_1=b_1,\ f_2=b_2,\ldots,f_m=b_m,\ C_0^{(n)}=1.$$

Thus

$$C_m^{(n)}[m^2+2nm+(p-2)m]+C_{m-1}^{(n)}b_1+\ldots+C_0^{(n)}b_m=0,$$

whence

$$|C_m^{(n)}|\leqslant m^{-2}[|b_1||C_{m-1}^{(n)}|+\ldots+|C_1^{(n)}||b_{m-1}|+|b_m|].$$

By well-known estimations (see [5]) we obtain

$$|C_m^{(n)}|\leqslant R^{-m}P^m,$$

where P is a constant greater than 1, and R is an arbitrary positive constant, independent of n. Therefore the series

$$\sum_{m=0}^{\infty}(R^{-1}P)^m r^m$$

converging for $|r| \leqslant P^{-1}R$ is a majorant for series (0), whence series (0) converges for arbitrary R.

By Lemma 1

$$w_n^{(1)}(r) = r^n + r^{n+1}[C_1^{(n)} + C_2^{(n)} r + \ldots] = r^n + O(r^{n+1}),$$

where $C_1^{(n)}$, $C_2^{(n)}$ are polynomials of degree at most 1 of the variables n and $|b_k|$ $(k = 1, \ldots, n)$ and $|b_k| \leqslant R^{-k}M$, where R is an arbitrary positive number and M is a constant. Therefore there exists a constant $C > 0$ such that $|w_n^{(1)}(r)| = r^n + O(r^{n+1})$, as $n \to \infty$ uniformly for $r \in [0, C]$.

LEMMA 2. *Let $r^2 k(r)$ be an entire function of r and let $\varrho_2(n) = 2 - p - n$ be a negative solution of the equation*

$$f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n+p-2) = 0;$$

then there exists a sequence

$$w_n^{(2)}(r) = r^{2-p-n} Q(r) + a w_n^{(1)}(r) \log r$$

of entire functions, where a is a constant and $Q(r)$ is polynomial solution of equation (1.6) *for $r > 0$.*

By Lemma 1 there exists a sequence of solutions of (1.1) of class C^2 in a neighbourhood of the origin

$$U_n^{(1)}(r, \Phi) = [A_n^{(1)} Y_1^{(n)} + \ldots + A_n^N Y_N^{(n)}] w_n^{(1)}(r), \tag{1.7}$$

where $A_n^{(1)}, \ldots, A_n^N$ are arbitrary constants.

By Lemma 2 there exist solutions of (1.1) of form

$$U_n^{(1)}(r, \Phi) = [B_n^{(1)} Y_1^{(n)} + \ldots + B_n^N Y_N^{(n)}] w_n^{(2)}(r) \tag{1.8}$$

having at the origin a pole of order $n + p - 2$; here $B_n^{(1)}, \ldots, B_n^N$ are arbitrary constants. The functions $U_n^{(1)}(r, \Phi)$ and $U_n^{(2)}(r, \Phi)$ are analytic in the entire space, except at the origin and

$$Y_1^{(n)}(\Phi) = K_n^{(1)}(X_0), \ldots, Y_N^{(n)}(\Phi) = K_n^N(X_0).$$

Thus the functions

$$V_n(X) = U_n^{(1)}(r, \Phi) + U_n^{(2)}(r, \Phi) \tag{1.9}$$

are solutions of equation (1.1) for $r > r_0 > 0$.

Let us now quote certain lemmas, the first of which is due to Walter [7].

LEMMA 3. *Let $u(X)$ be a solution of class C^2 in the ball $\overline{K(R, X)}$ of radius R and with the centre $X = (x_1, \ldots, x_p)$ and let*

$$M(R, X, u) = \frac{1}{\omega R^{p-1}} \operatorname*{SSS}_{K(R,X)} u(Y)\, dS \tag{1.10}$$

denote the mean value of the function $u(X)$ over the surface of the ball $K(R, X)$; then

$$1^\circ\ M(R, X, u) = u(X) + a_{p,1} R^2 \Delta u(X) + o(R^2),$$

$$2^\circ\ \lim_{R\to 0} (R^2 a_{p,1})^{-1} [M(R, X, u) - u(X)] = \Delta u(X),$$

where

$$a_{p,i} = [2^i \cdot i!\, p(p+2)\dots(p+2i-2)]^{-1} \quad \text{for } i \geqslant 1, \tag{1.11}$$
$$a_{p,0} = 1,$$

in the two-dimensional case $(p = 2)$ $\omega R^{p-1} = 2\pi R$, $(R^2 a_{p,1})^{-1} = 4R^{-2}$, $a_{2,1} = \frac{1}{4}$.

LEMMA 4. *Let $q(X)$ be continuous and let $u(X)$ be function of class C^2 solving the equation*

$$\Delta u(X) + q(X) u(X) = 0 \tag{1.12}$$

in the ball $\overline{K(R, X)}$; then

$$M(R, X, u) = u(X) + Q_2, \tag{1.13}$$

$$Q_2 = -(\omega)^{-1} \operatorname*{SSS}_{K(R,X)} q(Y) u(Y) H_1(|Y-X|, R)\, dY \tag{1.14}$$

and

$$H_1(r, R) = \int_r^R s^{1-p} ds = (p-2)^{-1}(r^{2-p} - R^{2-p}), \quad r = \|X-Y\|,\ p \geqslant 3.$$

In the two-dimensional case $(p = 2)$ the function Q_2 is of form

$$Q_2 = -(2\pi)^{-1} \iint_{K(R,X)} q(Y) u(Y) \left[\int_r^R \log \frac{R}{r}\, dr \right] dY,$$

where $K(R, X)$ is a circle of radius R with the centre R.

Let us prove now converse to Lemma 4.

LEMMA 5. *Let the function $u(X)$ of class C^2 satisfy condition* (1.13) *in each ball $\overline{K(R, X)} \subset D$; then it satisfies equation* (1.12) *in the region D.*

Proof. Let X be an arbitrary point of the region D and let $K(R, X) \subset D$. Since $r^{2-n} - R^{2-n} \geqslant 0$, we obtain, applying the mean value theorem to the integral defined by (1.14),

$$Q_2 = -(\omega)^{-1}(p-2)^{-1} q(P) u(P) \operatorname*{SSS}_{K(R,X)} (r^{2-p} - R^{2-p})\, dY, \quad P \in \overline{K(R, X)}.$$

By (1.2)

$$Q_2 = -[(p-2)\omega]^{-1} q(P) u(P) \int_0^R \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} (\varrho^{2-p} - R^{2-p}) |J(\varrho, \Phi)|\, d\varrho\, d\Phi,$$

where

$$J(\varrho, \Phi) = \varrho^{p-1}\sin\varphi_1(\sin\varphi_2)^2\ldots(\sin\varphi_{p-1})^{p-1} = \varrho^{p-1}J(1, \Phi)$$

is the Jacobi determinant of transformation (1.2) and $\int_0^{\pi}\ldots\int_0^{\pi}\int_0^{2\pi}|J(1, \Phi)|\,d\Phi = \omega$. Therefore

$$Q_2 = -\big(\omega(p-2)\big)^{-1}q(P)u(P)(2p)^{-1}(p-2)R^2\omega = -(2p)^{-1}R^2q(P)u(P).$$

Since by (1.13)

$$2pR^{-2}[M(R, X, u)-u] = -u(P)q(P)$$

we obtain by Lemma 3, equation (1.12) passing to the limit $R \to 0$.

The next lemma is an analogue to the well-known theorem of Harnack.

LEMMA 6. *Let $\{u_n(X)\}$ be a sequence of solutions of class C^2 of equation* (1.1) *converging almost uniformly in the ball $|X| < C$ to the function $u(X)$ of class C^2; then the function $u(X)$ satisfies equation* (1.1) *for $|X| < C$.*

Proof. By Lemma 4 we have for each ball $K(C, 0)$

$$M(C, X, u_n) = u_n(X)-\big(\omega(p-2)\big)^{-1}\operatorname*{SSS}_{K(C, X)} q(Y)u_n(Y)(r^{2-p}-R^{2-p})\,dY;$$

passing to the limit as $n \to \infty$ we obtain

$$M(C, X, u) = u(X)-\big(\omega(p-2)\big)^{-1}\operatorname*{SSS}_{K(C,X)} q(Y)u(Y)(r^{2-p}-R^{2-p})\,dY$$

and then apply Lemma 6.

An analogous results hold for the exterior of the ball $|X| < C$.

Lemma 6 implies immediately two theorems.

THEOREM 1. *Let*

$$u(X) = \sum_{n=0}^{\infty} u_n(X),$$

where the functions $u_n(X)$ of class C^2 are determined by (1.7) *and the series $\sum u_n(X)$ is almost uniformly covergent in the ball $|X| < C$ to the function $u(X)$ of class C^2; then the function $u(X)$ satisfies equation* (1.1) *in the ball $|X| < C$.*

THEOREM 2. *Let*

$$u(X) = \sum_{n=0}^{\infty} V_n(X),$$

where the functions $V_n(X)$ of class C^2 are determined by (1.9) *and the series $\sum V_n(X)$ is almost uniformly convergent in the exterior of the ball $|X| < C$ to the function $u(X)$ of class C^2; then the function $u(X)$ satisfies equation* (1.1) *in the exterior of the ball $|X| < C$.*

Let $K_{n,j}(X_0)$ be the orthonormal system of $N(p,n)$ spherical functions of order n and let

$$K_n(X_0) = \sum_{j=1}^{N} C_{n,j} K_{n,j}(X_0) \quad (n = 0, 1, 2, \ldots), \tag{1.15}$$

$C_{n,j}$ being arbitrary constants, be a sequence of spherical functions. By the orthonormality of the system $K_{n,j}(X_0)$

$$\SS_{\partial K(1,0)} K_n^2(X_0)\,dS = \sum_{j=1}^{N} C_{n,j}^2 = C_n^2 \quad (C_n \geqslant 0). \tag{1.16}$$

It is known [8] that

$$\sum_{j=1}^{N} K_{n,j}^2(X_0) \leqslant (\omega)^{-1} N,$$

whence by (1.15) and (1.16) and by the Schwarz inequality, we have

$$K_n^2(X_0) \leqslant \sum_{j=1}^{N} C_{n,j}^2 \sum_{j=1}^{N} K_{n,j}^2(X_0) \leqslant (\omega)^{-1} N \SS_{\partial K(1,0)} K_n^2(X_0)\,dS. \tag{1.17}$$

LEMMA 7. *Let the sequence* $\{K_n(X_0)\}$ *of spherical functions satisfy* (1.16) *and let the series*

$$\sum_{n=0}^{\infty} [W_n^{(1)}(r)]^2 C_n^2 \tag{1.18}$$

be convergent for $0 \leqslant r \leqslant r_0$ (*where* $r_0 > 0$); *then for each* $\alpha > 0$ *the series*

$$\sum_{n=0}^{\infty} n^{\alpha} w_n^{(1)}(r) K_n(X_0)$$

is almost uniformly and absolutely convergent in the ball $|X| < r_0$.

Proof. The convergence of series (1.18) implies

$$\lim_{n\to\infty} |w_n^{(1)}(r_0)|\, C_n = 0.$$

Thus there exists a constant A_1 such that for n sufficiently large and $w_n^{(1)}(r_0) \neq 0$ we have

$$C_n \leqslant \frac{A_1}{|w_n^{(1)}(r_0)|}.$$

By (1.16) and (1.17)

$$K_n^2(X_0) \leqslant (\omega)^{-1} N C_n^2.$$

Thu

$$|K_n(X_0)| \leqslant \left(\frac{N}{\omega}\right)^{1/2} C_n \leqslant A_0 C_n n^{(p-1)/2}.$$

By Lemma 1, $w_n^{(1)}(r_0) \sim r_0^n$ for a certain $r_0 > 0$. Then $C_n \leqslant A_2 r_0^{-n}$, whence $|w_n^{(1)}(r)| \leqslant A_3 r^n$, where A_2 and A_3 are constants. So we obtain

$$|n^a K_n(X_0) w_n^{(1)}(r)| \leqslant n^a w_n^{(1)}(r) \left(\frac{N}{\omega}\right)^{1/2} C_n \leqslant A_4 n^{a+(p-1)/2} r_0^{-n} r^n$$

$$= A_5 n^{a+(p-1)/2} \left(\frac{r}{r_0}\right)^n,$$

where A_4 and A_5 are constants. Thus the radius of convergence of the series

$$\sum_{n=0}^{\infty} n^{a+(p-1)/2} \left(\frac{r}{r_0}\right)^n$$

is equal to r_0.

LEMMA 8 [4]. *Let $g(X_0)$ be a continuous function on $\partial K(1,0)$ and let*

$$\SS_{\partial K(1,0)} g(X_0) K_{n,j}(X_0)\, dS_{X_0} = 0$$

for each integrer $n \geqslant 0$, $j = 1, \ldots, N$; then $g(X_0) \equiv 0$.

This lemma states that the system $K_{n,j}(X_0)$ is complete.

LEMMA 9 [4]. *Let $g(X_0)$ be a continuous function on $\partial K(1,0)$ and let*

$$C_{n,j} = \SS_{\partial K(1,0)} K_{n,j}(X_0) g(X_0)\, dS_{X_0};$$

then

$$\SS_{\partial K(1,0)} |g(X_0)|^2 dS_{X_0} = \sum_{n=0}^{\infty} \sum_{j=1}^{N} C_{n,j}^2.$$

This Lemma states that the system $K_{n,j}(X_0)$ is closed.

Applying Lemmas 7, 8 and 9 we shall prove a theorem converse to Theorem 1.

THEOREM 3. *Let the function $u(X)$ of class C^2 satisfy equation* (1.1) *in the ball $|X| < C$; then there exist a sequence $\{K_n(X_0)\}$ such that*

$$u(X) = \sum_{n=0}^{\infty} U_n(X),$$

where $U_n(X)$ are determined by formula (1.7) *and the series $\sum U_n(X)$ is almost uniformly convergent in the ball $|X| < C$.*

Proof. The functions

$$F_{n,j}(r) = \SS_{|X|=1} u(rX_0) K_{n,j}(X_0)\, dS, \tag{1.19}$$

where $j = 1, 2, \ldots, N$, are determined by $0 \leqslant r < C$ and are of class C^2.

By (1.19)

$$r^{p-1}F'_{n,j}(r) = \SS_{|X|=1} \frac{\partial u}{\partial n} K_{n,j}(X_0)\, dS_{X_0}. \tag{1.20}$$

On the other hand, by Green's formula

$$\SSS_{|X|\leqslant R} [K_{n,j}(X_0)\, \Delta u - u \Delta K_{n,j}(X_0)]\, dV = \SS_{|X|=R} \left[K_{n,j}(X_0) \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} K_{n,j}(X_0)\right] dV. \tag{1.21}$$

Since

$$\frac{\partial}{\partial n} K_{n,j}(X_0) = \frac{\partial}{\partial r} K_{n,j}(X_0) = \frac{\partial}{\partial r} K_{n,j}\left(\frac{X}{r}\right),$$

we have

$$\frac{\partial}{\partial r} K_{n,j}(X_0) = 0. \tag{1.22}$$

Moreover,

$$\Delta K_{n,j}(X_0) = \frac{\partial^2}{\partial r^2} K_{n,j}(X_0) + r^{-1}(p-1) \frac{\partial}{\partial r} K_{n,j}(X_0) + r^{-2} \Delta_1 K_{n,j}(X_0).$$

Thus, by (1.22),

$$\Delta K_{n,j}(X_0) = r^{-2} \Delta_1 K_{n,j}(X_0) = r^{-2}[-n(n+p-2) K_{n,j}(X_0)].$$

So by (1.1), (1.21) and (1.22) we obtain

$$\SS_{|X|=r} K_{n,j}(X_0) \frac{\partial u}{\partial n}\, dS_X = -\SSS_{|X|\leqslant r} [K_{n,j}(X_0)\, k(\varrho)\, u - \varrho^{-2} n(n+p-2) K_{n,j}(X_0)\, u]\, dV,$$

where $0 \leqslant \varrho \leqslant R$. Taking into account (1.20) we get

$$r^{p-1}F'_{n,j}(r) = \SSS_{|X|\leqslant r} K_{n,j}(X_0)\, k(\varrho)\, u\, dV + n(n+p-2) \SSS_{|X|\leqslant r} \varrho^{-2} K_{n,j}(X_0)\, u(X)\, dV,$$

whence

$$r^{p-1}F'_{n,j}(r) = -\int_0^r k(\varrho)\, \varrho^{p-1} \SS_D K_{n,j}(X_0)\, u(\varrho X_0)\, |J|\, d\Phi d\varrho + n(n+p-2) \int_0^r \varrho^{-3+p} \SS_D K_{n,j}(X_0)\, u(\varrho X_0)\, |J|\, d\Phi d\varrho,$$

since

$$\SS_D K_{n,j}(X_0)\, u(\varrho X_0)\, |J|\, d\Phi = \SS_{|X|=\varrho} K_{n,j}(X_0)\, u(\varrho X_0)\, dS$$

and by (1.19) it follows that

$$r^{p-1}F'_{n,j}(r) = -\int_0^r \varrho^{p-1} k(\varrho) F_{n,j}(\varrho)\, d\varrho + n(n+p-2) \int_0^r \varrho^{p-3} F_{n,j}(\varrho)\, d\varrho.$$

Differentiation of both sides of this equality with respect to r gives (1.6). By Lemma 1 a regular integral for $r \geqslant 0$ of this equation is the function

$$F_{n,j}(r) = C_{n,j} w_n^{(1)}(r) \tag{1.23}$$

for $0 \leqslant r < C$. By (1.15), (1.16), (1.23) and by Lemma 9

$$u(rX_0) = \sum_{n=0}^{\infty} a_{n,j} K_{n,j}(X_0),$$

$$a_{n,j}(r) = \operatorname*{SS}_{\partial K(1,0)} u(rX_0) K_{n,j}(X_0) = F_{n,j}(r),$$

$$w^2(rX_0) = \Big[\sum_{n=0}^{\infty} F_{n,j}(r) K_{n,j}(X_0)\Big]^2 = \Big[\sum_{n=0}^{\infty} C_{n,j} w_n^{(1)}(r) K_{n,j}(X_0)\Big]^2,$$

$$\operatorname*{SS}_{\partial K(1,0)} u^2(rX_0)\, dS = \sum_{n=0}^{\infty} (w_n^{(1)})^2 \sum_{j=1}^{N} C_{n,j}^2 = \sum_{n=0}^{\infty} (w_n^{(1)})^2 C_n^2$$

and finally

$$\operatorname*{SS}_{\partial K(1,0)} u^2(rX_0)\, dS = \sum_{n=0}^{\infty} \big(w_n^{(1)}(r)\big)^2 C_n^2,$$

whence

$$\operatorname{SS} |u(rX_0)|^2 dS = \sum_{n=0}^{\infty} |w_n^{(1)}(r)|^2 C_n^2.$$

By Lemma 7 the series

$$\psi(rX_0) = \sum_{n=0}^{\infty} w_n^{(1)}(r) K_n(X_0) = \sum_{n=0}^{\infty} w_n^{(1)}(r) \sum_{j=1}^{\infty} C_{n,j} K_{n,j}(X_0)$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^{N} C_{n,j} w_n^{(1)}(r) K_{n,j}(X_0) = \sum_{n=0}^{\infty} \sum_{j=1}^{N} F_{n,j}(r) K_{n,j}(X_0) = \sum_{n=0}^{\infty} U_n(X),$$

is uniformly and absolutely convergent for $a = 0$, $0 \leqslant r < C$. Completeness of the system $K_{n,j}(X_0)$ implies $u(rX_0) = \psi(rX_0)$, whence $u(rX_0) = \sum U_n(X)$ for $0 \leqslant r < C$, i.e., $u(rX_0) \equiv \psi(rX_0)$.

2. In this section we shall deal with the properties of the solutions of equation (2.1). To be able to apply Lemmas 1 and 2 and method of separation of variables we shall first give.

LEMMA 10. *Let*

$$\sum_{i=1}^{p} b_i(X)\, dX_i \tag{2.2}$$

be a complete differential and let $b_i(X) = B_i(r)$, $c(X) = C(r)$ *and* $\sum_{i=1}^{p} \frac{\partial b_i(X)}{\partial x_i} = f(r)$; *then the transformation*

$$u(X) = v(X) \exp\big(-\tfrac{1}{2} Q(X)\big), \tag{2.3}$$

where

$$Q(X) = \sum_{i=1}^{p} \int_{x_i^0}^{x_i} b_i(S_i)\,ds_i \tag{2.4}$$

and $S_i = (x_1, \ldots, x_{i-1}, s_i, x_{i+1}, \ldots, x_p)$, *transforms equation* (2.1) *into*

$$Lv(X) = \Delta v(X) + K(r)v(X) = 0, \tag{2.5}$$

where

$$K(r) = -\tfrac{1}{2}, \quad b_i^2 - \tfrac{1}{2}\Big[\sum_{i=1}^{p} \frac{\partial b_i(X)}{\partial x_i}\Big] + c(X). \tag{2.6}$$

We omit the simple proof.

Applying to (2.5), analogously as in section 1, the method of separation of variables we obtain an equation of form (1.6) in which the function $k(r)$ is to be replaced by the function $K(r)$ defined by formula (2.6). Next, assuming that $r^2K(r)$ is an entire function of the variable $r \geqslant 0$ and that $\varrho_1(n) = n$ is a positive solution of the equation $f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n-p-2) = 0$, we apply Lemma 1 and formula (2.3) and so we obtain a sequence of solutions of class C^2 for $r \geqslant 0$ of equation (2.1) of form

$$U_n^{(1)}(r, \Phi) = \exp(-\tfrac{1}{2}Q)[A_1^{(1)}Y_1 + \ldots + A_n^N Y_N]w_n^{(1)}(r), \tag{2.7}$$

where $A_1^{(1)}, \ldots, A_n^N$ are constants and the function $Q(X)$ is defined by (2.4); it corresponds to sequence (1.6). Then applying Lemma 2 under the assumption that $r^2K(r)$ is an entire function for $r > 0$ and that $\varrho_2(n) = 2-p-n$ is a negative solution of the equation $f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n+p-2)$, we obtain by Lemma 2 and formula (2.3) another sequence of solutions of (2.1)

$$U_n^{(2)}(r, \Phi) = \exp(-\tfrac{1}{2}Q)[B_1^{(1)}Y_1 + \ldots + B_n^N Y_N]w_n^{(2)}(r), \tag{2.8}$$

where $B_1^{(1)}, \ldots, B_n^N$ are constants, and the function Q is defined by (2.4). This sequence corresponds to (1.7). The functions $U_n^{(1)}$ and $U_n^{(2)}$ are analytic in the entire space except the origin. Thus the functions

$$V_n(X) = \exp(-\tfrac{1}{2}Q)[U_n^{(1)}(r, \Phi) + U_n^{(2)}(r, \Phi)] \tag{2.9}$$

are integrals of (2.1) for $r > r_0 > 0$. Sequence (2.9) corresponds to sequence (1.9). The next lemma play a similar role as Lemma 3.

LEMMA 11. *Let the function* $u(X)$ *of class* C^2 *satisfy in the ball* $K(R, X)$ *equation* (1.2); *then*

$$M(R, X, u) = u(X) + \frac{1}{\omega} \SSS_{K(R,X)} \Big[\sum_{i=1}^{p} b_i(Y)\frac{\partial u}{\partial y_i} + c(Y)u(Y)\frac{(r^{2-p} - R^{2-p})}{p-2}\Big] dY, \tag{2.10}$$

where $r = |X-Y|$. *In the two-dimensional case* ($p = 2$) *formula* (2.10) *takes on the form*

$$M(R, X, u) = u(X) - \frac{1}{2\pi} \iint_{K(R,X)} \left[\sum_{i=1}^{p} b_i(Y) \frac{\partial u}{\partial y_i} + c(Y) u(Y) \log \frac{R}{r} \right] dY.$$

The analogue of Lemma 5 is

LEMMA 12. *Let the function* $u(X)$ *of class* C^2 *satisfy in the ball* $\overline{K(R, X)}$ *condition* (2.10). *Then it satisfies equation* (2.1).

The proof, based on Lemma 3, is similar to this of Lemma 5.

Making use of Lemmas 10, 11 and 12 we can prove, analogously as Lemma 5, the

LEMMA 13. *Let* $\{u_n(X)\}$ *be a sequence of integrals of class* C^2 *of equation* (2.1), *converging almost uniformly in the ball* $|X| < C$ *to a function* $u(X)$ *of class* C^2. *If the functions* $\partial u_n/\partial x_i$ ($i = 1, 2, \ldots, p$) *converge almost uniformly to* $\partial u/\partial x_i$ *in the ball* $|X| < C$, *then the function* $u(X)$ *satisfies equation* (2.1).

The last lemma implies two theorems.

THEOREM 4. *If*

1° $U_n(X)$ *are functions defined by formula* (2.7),

2° $u(X) = \sum U_n(X)$,

3° *the series* $\sum U_n(X)$ *converges to* $u(X)$ *almost uniformly in the ball* $|X| < C$ *to the function* $u(X)$, *as well as the series* $\sum \partial U_n/\partial x_i$ *converges almost uniformly to* $U'_{x_i}(X)$,

then the function $u(X)$ *satisfies in the ball* $|X| < C$ *equation* (2.1).

A similar theorem may be obtained for the exterior of the ball $|X| < C$. It coresponds to Theorem 2, and the functions $V_n(r, \Phi)$ are in this case defined by formula (2.9).

Using Lemmas 7 and 10 we can prove, similarly as Theorem 3, the following converse to Theorem 4:

THEOREM 5. *Let the function* $u(X)$ *of class* C^2 *satisfy equation* (2.1); *then there exists a sequence* $\{K_n(X_0)\}$ *of harmonic polynomials of degree* n *respectively such that* $u(X) = \sum U_n(X)$, *the functions* $U_n(X)$ *being defined by* (2.7), *and the series* $\sum U_n(X)$ *being almost uniformly convergent in the ball* $|X| < C$.

All the preceding lemmas apply also to the self-adjoint equation

$$\sum_{i,j=1}^{p} \frac{\partial}{\partial x_i} \left[p(r) \frac{\partial u}{\partial x_j} \right] + k(r) u(X) = 0, \tag{2.11}$$

since it is a particular case of equation (2.1). Indeed, it may be rewritten as

$$\Delta u + \sum_{i=1}^{p} A_i \frac{\partial u}{\partial x_i} + \frac{k(r)}{p(r)} u(X) = 0, \tag{2.11a}$$

where $A_i = (rp(r))^{-1} p'(r) x_i$. The function $p(r)$ is supposed here to be analytic and positive for $r \geqslant 0$. It is easy to check that $\sum A_i(X)\,dx_i$ is a exact differential; hence, by Lemma 10, the transformation

$$u(X) = \exp(-\tfrac{1}{2}Q_1)\,v(X), \tag{2.12}$$

where

$$Q_1(X) = \sum_{i=1}^{p} \int_{x_i^0}^{x_i} A_i(S_i)\,ds_i \tag{2.13}$$

and $S_i = (x_1, \ldots, x_{i-1}, s_i, x_{i+1}, \ldots, x_p)$, converts equation (2.11) into

$$Lv(X) = \Delta v + K_1(r)\,v = 0, \tag{2.14}$$

where

$$K_1(r) = -\tfrac{1}{4}(p(r))^{-2}(p'(r))^2 - \tfrac{1}{2}\left[p\frac{p'(r)}{rp(r)} + B'(r)\right] + \frac{k(r)}{p(r)}, \tag{2.15}$$

or

$$K_1(r) = -\tfrac{1}{4}\sum_{i=1}^{p} A_i^2 - \tfrac{1}{2}\sum_{i=1}^{p} \frac{\partial A_i}{\partial x_i} + \frac{k(r)}{p(r)}.$$

Applying the method of separation of variables we obtain an equation of form (1.6) in which the function $k(r)$ is to be replaced by $K_1(r)$ defined by (2.15). Supposing that the function $r^2K_1(r)$ is entire for $r \geqslant 0$ and that $\varrho_1(n) = n$ is a positive solution of the equation $f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n+p-2) = 0$ we apply Lemma 1 and formula (2.12) to obtain a sequence of solutions of class C^2 of the equation

$$U_n^{(1)}(r, \Phi) = \exp(-\tfrac{1}{2}Q_1)(A_1^{(1)}Y_1 + \ldots + A_1^N Y_N) w_n^{(1)}(r), \tag{2.16}$$

where $A_1^{(1)}, \ldots, A_1^N$ are constants and the function $Q_1(X)$ is defined by (2.13) and N by (1.0).

Similarly, applying Lemma 2, supposing that $r^2K_1(r)$ is an entire function for $r > 0$ and that $\varrho_2(n) = 2-p-n$ is a negative solution of the equation $f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n+p-2)$ we obtain in virtue of Lemma 2 and formula (2.12) another sequence of solutions of equation (2.14)

$$U_n^{(2)}(r, \Phi) = \exp(-\tfrac{1}{2}Q_1)(B_1^{(1)}Y_1 + \ldots + B_n^N Y_N) w_n^{(2)}(r), \tag{2.17}$$

where $B_1^{(1)}, \ldots, B_n^N$ are constants and the function $Q_1(X)$ is defined by formula (2.13). Sequence (2.16) corresponds to sequences (1.7) and (2.7),

and (2.17) corresponds to (1.8) and (2.8). The functions $U_n^{(1)}$ and $U_n^{(2)}$ are analytic in the whole space except at the origin. For $r > r_0 > 0$ the function

$$V_n(X) = \exp(-\tfrac{1}{2}Q_1)\big(U_n^{(1)}(r, \Phi) + U_n^{(2)}(r, \Phi)\big), \tag{2.18}$$

form a sequence of solutions of equation (2.1). This sequence corresponds to sequences (1.9) and (2.9). The analogous of Lemma 3 is Lemma 14. Let the function $u(X)$ of class C^2 satisfy in the ball $\overline{K(R, X)}$ equation (2.11); then

$$M(R, X, u) = u(X) + \big(\omega(p-2)\big)^{-1} \operatorname*{SSS}_{K(R,X)} \Big[\frac{p'(r)}{rp(r)} \sum_{i=1}^{p} y_i \frac{\partial u}{\partial y_i} + \frac{k(r)}{p(r)} u(Y)\Big] (r^{2-p} - R^{2-p})\, dY. \tag{2.19}$$

In the two-dimensional case, formula (2.19) may written in form

$$M(R, X, u) = u(X) - \frac{1}{2\pi} \operatorname*{SS}_{K(R,X)} \Big[\frac{p'(r)}{rp(r)} \sum_{i=1}^{2} \frac{\partial u}{\partial y_i} + \frac{k(r)}{p(r)} u(Y)\Big] \log \frac{R}{r}\, dY.$$

The analogue of Lemma 5 is

LEMMA 15. *Let the function $u(X)$ of class C^2 satisfy condition* (2.19) *in the ball $K(R, X)$; then it satisfies equation* (2.11).

The proof is similar to this of Lemma 5.

By an argument based upon Lemmas 6 and 13 we can prove an analogue of Lemma 13. Theorems 4 and 6 may be transferred to case we deal now under the assumption that the functions $U_n(X)$ and $V_n(X)$ are defined by formulas (2.16) and (2.18).

3. In this section we deal with certain solutions of equation (3.1). Using, similarly as in section 1, the method of separation of variables we obtain equations (1.4) and (1.6) as well as the equation

$$w_n''(r) + r^{-1}(p-1)w_n'(r) - [k(r) + n(n+p-2)r^{-2}]w_n(r) = 0. \tag{3.2}$$

Let $r^2 k(r)$ be an entire function of the variable $r \geqslant 0$ and let $\varrho_1(n) = n$ be the positive solution of the equation $f_0(\varrho) = \varrho^2 + (p-2)\varrho - n(n+p-2) = 0$; then, according to Lemma 1, $\{w_{n,1}^{(1)}(r)\}$ is a sequence of solution of equation (3.2) of class C^2 for $r \geqslant 0$ and $\{w_{n,2}^{(1)}(r)\}$ is a sequence of solutions of (1.6) of class C^2. Thus there exists at the origin a sequence of solutions of (3.1) of class C^2,

$$U_{n,1}^{(1)}(r, \Phi) = [D_n^{(1)} Y_1 + \ldots + D_n^N Y_N] w_{n,1}^{(1)}(r) + [E_n^{(1)} Y_1 + \ldots + E_n^N Y_N] w_{n,2}^{(1)}(r), \tag{3.3}$$

where $D_n^{(1)}, \ldots, D_n^N, E_n^{(1)}, \ldots, E_n^N$ are arbitrary constants. Let now $r^2 k(r)$ be an entire function for $r > 0$ and let $\varrho_2(n) = 2-p-n$ be the negative solution of the equation $f_0(\varrho) = \varrho^2+(p-2)\varrho-n(n+p-2) = 0$; then, by Lemma 2, $\{w_n^3(r)\}$ and $\{w_n^{(4)}(r)\}$ are a sequences of solutions of equations (3.2) and (1.6), respectively. These solutions have a pole of order $n+p-2$ at the origin. Thus the functions

$$U_{n,2}^{(2)}(r, \Phi) = (G_n^{(1)} Y_1+\ldots+G_n^N Y_N) w_n^{(3)}(r)+ \\ +[H_n^{(1)} Y_1+\ldots+H_n^N Y_N] w_n^{(4)}(r) \tag{3.4}$$

form a sequence of solutions of (3.1) having at the origin the only pole of order $n+p-2$. Here $G_n^{(1)}, \ldots, G_n^N; H_n^{(1)}, \ldots, H_n^N$ are arbitrary constants. The functions $U_{n,1}^{(1)}$ and $U_{n,2}^{(2)}$ are analytic in the entire space except at the origin. Thus the functions

$$V_n(r, \Phi) = U_{n,1}^{(1)}(r, \Phi)+U_{n,2}^{(2)}(r, \Phi) \tag{3.5}$$

form a sequence of integrals of equation (3.1) for $r > r_0 > 0$.

Following Courant and Hilbert (see [1], vol. 2, p. 245) and Walter [7] we shall give two lemmas.

LEMMA 16. *Let the function $u(X)$ be of class C^4 in the ball $K(R, X)$; then it satisfies formula* (1.10) *and*

$$\lim_{R\to 0} (R^4 a_{p,2})^{-1}[M(R, X, u)-u(X)-a_{p,1} R^2 \Delta u] = \Delta^2 u(X),$$

where

$$M(R, X, u) = u(X)+a_{p,1} R^2 \Delta u+a_{p,2} R^4 \Delta^2 u+o(R^4),$$

the constants $a_{p,i}$ being defined by (1.11). *In the two-dimensional case we have $(R^4 a_{p,2})^{-1} = R^{-4} 64$.*

LEMMA 17. *Let the function $u(X)$ of class C^4 satisfy in the ball $\overline{K(R, X)}$ the equation*

$$\Delta^2 u(X)-q(X)u(X) = 0; \tag{3.6}$$

then

$$M(R, X, u) = u(X)+a_{p,1} R^2 \Delta u+Q_3, \tag{3.7}$$

where

$$Q_3 = (\omega)^{-1} \operatorname*{SSS}_{K(R,X)} q(Y) u(Y) H_2(|Y-X|, R)\, dY \tag{3.8}$$

and

$$H_2 = \frac{1}{4} \int_r^R \frac{(R^2-\varrho^2)(\varrho^2-r^2)}{\varrho^{p+1}}\, d\varrho, \quad r = |X-Y|,\ p \geqslant 3.$$

In the two-dimensional case ($p = 2$)

$$Q_3 = (\omega)^{-1} \iint\limits_{K(R,X)} \left[\int\limits_r^R \varrho \log \frac{R}{\varrho} \log \frac{\varrho}{r} \, d\varrho \right] q(Y) u(Y) \, dY.$$

LEMMA 18. *Let the function* $u(X)$ *of class* C^4 *satisfy in the ball* $K(R, X)$ *condition* (3.7); *then it satisfies equation* (3.6).

Proof. Since $(R^2 - \varrho^2)(\varrho^2 - r^2) > 0$, we obtain applying to integral (3.8) the mean value theorem

$$Q_3 = (\omega)^{-1} q(P) u(P) \operatorname*{SSS}_{K(R,X)} \left[\int\limits_r^R \frac{(R^2 - \varrho^2)(\varrho^2 - r^2)}{\varrho^{p+1}} \, d\varrho \right] dY,$$

where $P \in \overline{K(R, X)}$. Further, applying transformation (1.2) we obtain

$$Q_3 = u(P) q(P) \int\limits_0^R r^{p-2} \int\limits_r^R (R^2 - \varrho^2)(\varrho^2 - r^2) \varrho^{-2-p} \, d\varrho r dr.$$

Now introducing now variables $T = R^2$, $z = \varrho^2$, $r^2 = t$, and changing the order of differentiation we get

$$Q_3 = B \int\limits_0^T \left[\int\limits_0^z t^{\frac{1}{2}(p-2)} (z - t) \, dt \right] (T - z) z^{-\frac{1}{2}(p+2)} \, dz,$$

where $B = \frac{1}{4} u(P) q(P)$. Hence $Q_3 = [2p(p+2)]^{-1} R^4 u(P) q(P)$, and by (3.7) we get

$$R^{-4} 2p(p+2)[M(R, X, u) - u(X) - a_{p,1} R^2 \Delta u] = u(P) q(P). \tag{3.9}$$

Passing to the limit as $R \to 0$ by Lemma 14 we obtain (3.6).

In the two-dimensional case ($p = 2$) formula (3.9) takes an the form

$$32 R^{-4}[M(R, X, u) - u(X) - 4^{-1} R^2 \Delta u] = u(P) q(P).$$

LEMMA 19. *Let* $\{u_n(X)\}$ *be a sequence of solutions of equation* (3.1) *of class* C^4 *converging almost uniformly in the ball* $|X| < C$ *to a function* $u(X) \in C^4$ *and let the functions* $\Delta u_n(X)$ *be convergent almost uniformly to* $\Delta u(X)$ *for* $|X| < C$; *then the function* $u(X)$ *satisfies equation* (3.6).

The proof of Lemma 19, based on Lemmas 17 and 18, is similar to this of Lemma 6.

Lemma 19 implies directly the following theorems analogous to Theorems 1 and 2.

THEOREM 6. *Let the functions* $U_n(X)$ *be defined by formula* (3.3) *and let the series* $\sum U_n(X)$ *be almost uniformly convergent in the ball* $|X| < C$ *to a function* $u(X)$ *of class* C^4. *If the functions* $\sum \Delta U_n(X)$ *are almost uniformly converging to* $\Delta u(X)$ *for* $|X| < C$, *then the function* $u(X)$ *satisfies equation* (3.1) *in the ball* $|X| < C$.

An analogous result we obtain for the outside of the ball $|X| < C$ and the functions $V_n(X)$ defined by (3.5).

4. We shall deal now with certain properties of the solutions of equations (4.1). Let us set

$$E(u) = \prod_{i=1}^{s} (\Delta - k_i) u(X) = 0, \tag{4.1a}$$

where k_i are real constants. Applying the separation of variables to each factor of (4.1a) we obtain system (1.4) of equations and

$$w''_{n,i}(r) + (p-1) r^{-1} w'_{n,i}(r) - [k_i + n(n+p-2) r^{-2}] w_{n,i}(r) = 0, \tag{4.2}$$

where $i = 1, \ldots, s$. Let the functions $\{w_n^{1,i}(r)\}$ and $\{w_n^{2,i}(r)\}$ form for $r > 0$ a sequence of integrals of class C^2 of equations (4.2). Then the functions

$$U_n^{(1)}(r, \Phi) = \sum_{i=1}^{s} [D_{n,i}^{(1)} Y_1 + \ldots + D_{n,i}^{N} Y_N] w_n^{(1,i)}(r) \tag{4.3}$$

are for $r \geqslant 0$ the solutions of (4.1) and the functions

$$U_n^{(2)}(r, \Phi) = \sum_{i=1}^{s} [F_{n,i}^{(1)} Y_1 + \ldots + F_{n,i}^{N} Y_N] w_n^{(2,i)}(r) \tag{4.4}$$

are solutions of (4.1) for $r > 0$. Here $D_{n,i}^{(1)}, \ldots, D_{n,i}^{N}, F_{n,i}^{(1)}, \ldots, F_{n,i}^{N}$ are arbitrary constants.

For the outside of the ball $|X| < C$ solutions of (4.1) are

$$V_n(r, \Phi) = U_n^{(1)}(r, \Phi) + U_n^{(2)}(r, \Phi). \tag{4.5}$$

LEMMA 20 [7]. *Let $u(X)$ be of class C^{2s} in the ball $\overline{K(R, X)}$; then*

$$M(R, X, u) = \sum_{i=0}^{s-1} a_{p,i} R^{2i} \Delta^{(i)} u(X) + a_{p,s} R^{2s} \Delta^{(s)} u(\bar{X}), \tag{4.6}$$

where $\bar{X} \in \overline{K(R, X)}$ and $a_{p,i}$ are defined by (1.11) *and*

$$\lim_{R \to 0} (a_{p,s} R^{2s})^{-1} \Big(M(R, X, u) - \sum_{i=0}^{s-1} a_{p,i} R^{2i} \Delta^{(s)} u(\bar{X}) \Big) = \Delta^{(s)} u(X).$$

It follows from Lemma 20

LEMMA 21. *Let the function $u(X)$ of class C^{2s} in the ball $\overline{K(R, X)}$ satisfy equation* (4.1); *then*

$$M(R, X, u) = \sum_{i=1}^{s-1} a_{p,i} \Delta^{(i)} u(X) + Q_s, \tag{4.7}$$

where

$$(4.8)\quad Q_s = (\omega)^{-1} \operatorname*{SSS}_{K(R,X)} [C_{p-1}\Delta^{(s-1)}u(X)+\ldots+C_0 u(X)]H_{s-1}(|Y-X|, R)\,dY$$
$$= a_{p,s}R^{2s}\Delta^{(s)}u(X) = -a_{p,s}R^{2s}[C_{p-1}\Delta^{(s-1)}u(\overline{X})+\ldots+C_0 u(\overline{X})],$$

where

$$H_{s-1}(r, R) = a_{n,p}\int_r^R \frac{[(R^2-t^2)(r^2-t^2)]^{s-1}}{t^{2s+p-3}}\,dt, \qquad \overline{X}\in\overline{K(R, X)}.$$

LEMMA 22. *Let the function $u(X)$ of class C^{2s} satisfy in each ball $\overline{K(R, X)}$ condition* (4.7); *then it satisfies equation* (4.1).

Proof. Writing Q_s in form (4.8) and dividing both sides of (4.7) by $a_{p,s}R^{2s}$ and passing to the limit as $R\to 0$, we obtain (4.1) in virtue of Lemma 20.

LEMMA 23. *Let $\{u_m(X)\}$ be a sequence of solutions of class C^{2s} of equation* (4.1) *converging almost uniformly in the ball $|X|<C$ to a function $u(X)$ of class C^{2s} and let the functions $\Delta u_m(X)$, $\Delta^2 u_m(X)$, ..., $\Delta^{(s-1)}u_m(X)$ converge almost uniformly to the functions $\Delta u(X)$, ..., $\Delta^2 u(X)$, ..., $\Delta^{(s-1)}u(X)$, respectively. If the functions $u_m(X)$ satisfy equation* (4.1), *then the function $u(X)$ satisfies equation* (4.1).

Proof. In virtue of Lemma 20

$$M(R, X, u_m) = \sum_{i=1}^{s-1} a_{p,i}R^{2i}\Delta^{(i)}u_m(X)+Q_s,$$

where

$$Q_s = (\omega)^{-1} \operatorname*{SSS}_{K(R,X)} [C_{p-1}\Delta^{(s-1)}u_m(Y)+\ldots+C_0(Y)]H_{s-1}(|Y-X|, R)\,dY.$$

Passing to the limit as $m\to\infty$, we obtain (4.6). By Lemma 21 the function $u(X)$ satisfies equation (4.1).

It follows from Lemma 23

THEOREM 7. *Let the series $\sum U_n(X)$ of functions defined by* (4.3) *be almost uniformly convergent in the ball $|X|<C$ to a function $u(X)$ of class C^{2s}, and let $\Delta\sum U_n(X) = \sum \Delta U_n(X) = \Delta u(X), \ldots, \Delta^{(s-1)}\sum U_n(X) = \sum \Delta^{(s-1)}U_n(X) = \Delta^{(s-1)}u(X)$ in the ball $|X|<C$; then the function $u(X)$ satisfies equation* (4.1) *in the ball $|X|<C$.*

An analogous theorem holds true for outside of the ball $|X|<C$.

5. In this section we shall consider properties of solutions of equation (5.1).

LEMMA 24 [3]. *Let a_i satisfy the system of equations*

$$\cdot\, 2\sum_{k=1}^{p} a_{ik}a_k + b_i = 0,$$

where $i = 1, 2, \ldots, p$, *and let the function* $u(X)$ *of class* C^2 *be a solution of equation* (5.1); *then the function*

$$v(X) = u(X)\exp\Big(-\sum_{i=1}^{p} a_i x_i\Big)$$

is a solution of the equation

$$\sum_{i,k=1}^{p} a_{ik} v''_{x_i x_i}(X) + C(X)v(X) = 0, \tag{5.2}$$

where

$$C(X) = \sum_{j,k=1}^{p} a_{jk} a_j a_k - p(X).$$

Conversely, if $v(X)$ *is a solution of equation* (5.2), *then the function*

$$u(X) = v(X)\exp\Big(\sum a_i x_i\Big) \tag{5.3}$$

is a solution of equation (5.1).

In the sequel we shall denote by E_R the inside of the ellipsoid defined by the equation

$$f(X) = \sum_{i,k=1}^{p} a^{ik}(x_i - x_i^0)(x_k - x_k^0) - R^2 = 0, \tag{5.4}$$

where a^{ik} are elements of the matrix inverse to (a_{ik}).

Let us write

$$M(R, X, v) = [a_p(R)]^{-1} \operatorname{SS}_{\partial E_R} \frac{v(Y)}{g(X)}\, dS,$$

where

$$a_p(R) = \operatorname{SS}_{\partial E_R} \frac{dS}{g(Y)}$$

and

$$g(X) = \Big\{\sum_{i=1}^{p}\Big[\sum_{k=1}^{p} a^{ik}(x_i - x_i^0)\Big]^2\Big\}^{\frac{1}{2}}. \tag{5.5}$$

In the two-dimensional case ($p = 2$) we have

$$M(R, X, v) = (a_2)^{-1} \int_{\partial E_2} \frac{v(Y)}{g(Y)}\, ds,$$

where

$$a_2 = \int_{\partial E_2} \frac{ds}{g(Y)}, \qquad Y = (y_1, y_2).$$

LEMMA 25 [6]. *Let the function* $v(X)$ *be of class* C^2 *in* E_R; *then*

$$\lim_{R\to 0} (2pR^{-2})[M(R, X, v) - v(X)] = Lv(X),$$

where

$$Lv = \sum_{i,k=1}^{p} a_{ik} v''_{x_i x_k}.$$

LEMMA 26 [6]. *Let $v(X)$ be a solution of equation (5.2) in E_R; then*

$$M(R, X, u) = v(X) - k_p \operatorname*{SSS}_{E_R} (r^{2-p} - R^{2-p}) v(Y) C(Y) dY, \tag{5.6}$$

where

$$k_p \operatorname*{SSS}_{E_R} (r^{2-p} - R^{2-p}) dY = \frac{R^2}{2p}.$$

In the two-dimensional case we have

$$M(R, X, v) = v(X) - k_2 \iint_{E_2} \log \frac{R}{r} C(Y) u(Y) dY, \tag{5.6a}$$

where

$$k_2 \iint_{E_2} \log \frac{R}{r} dY = \frac{R^2}{4}.$$

LEMMA 27. *Let a function $v(X)$ of class C^2 in a region G satisfy in an ellipsoid $\bar{E}_R \subset G$ condition (5.6); then the function $v(X)$ is a solution of equation (5.2) in G.*

Proof. Since $r^{2-p} - R^{2-p} > 0$ ($\log R/r > 0$), we obtain by the mean value theorem

$$M(R, X, v) - v(X) = -k_p C(P) v(P) \operatorname*{SSS}_{E_R} (r^{2-p} - R^{2-p}) dY,$$

where $P \subset \bar{E}_R$. By Lemma 26

$$M(R, X, v) - v(X) = -C(P) v(P) (2p)^{-1} R^2$$

whence

$$R^{-2} 2p M(R, X, v) - v(X) = -C(P) v(P).$$

By Lemma 25 we obtain (5.2) passing to the limit as $R \to 0$. Hence by Lemma 24 the function $u(X)$ defined by (5.3) satisfies equation (5.1).

LEMMA 28. *Let*

$$y_i = \sum_{j=1}^{p} t_{ij} x_j \quad (i = 1, \ldots, p), \tag{T}$$

and $TT^* = A^{-1}$, where $A = (a_{ik})$ be a positiv matrix, $A^{-1} = (a^{ik})$, $T = (t_{ij})$ and a^{ik}, t_{ij} are constants, and T^* denotes the transpose of the matrix T. If $U(Y) = u(X)|_{X=T^{-1}(Y)}$, $u(X)$ is of class C^2 in R^n; then:

$$1^\circ \sum_{j=1}^{p} y_i^2 = \sum_{k,i=1}^{p} a^{ik} x_i x_k = \varrho^2,$$

$$2^\circ \sum_{i=1}^{p} \frac{\partial^2 U(Y)}{\partial y_j^2} = \sum_{i,k=1}^{p} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k}.$$

We omit the simple proof.

It follows from Lemma 28.

LEMMA 29. *Let* $\sum_{i=1}^{p}(y_i^0)^2 = 1$, $Y = \varrho Y_0$, $\varrho^2 = y_1^2+\ldots+y_p^2$, *and let* $H_n(Y) = \varrho^n K_n(Y_0)$ *be a harmonic polynomial of degree* n; *then*

$$T_n(X) = H_n\big(T(X)\big) = \varrho^n S_n(X_0),$$

where $\sum_{i,k=1}^{p} a^{ik}x_i^0 x_k^0 = 1$ *and* $S_n(X_0) = K_n\big(T(X_0)\big)$, *forms a solution of the equation* $LT_n(X) = \sum_{i,k=1}^{p} a_{ik}(T_n'')x_i x_k = 0$.

Let $R_1(X), \ldots, R_s(X)$ $(s = N(p,n)-N(p,n-2))$ *be a maximal linearly independent system of harmonic polynomials of degree* n *and let* $Q_n(Y) = T^* R_n\big(T^{-1}(X)\big)$.

LEMMA 30 [2]. *The system* $Q_1, \ldots, Q_s$ *is a maximal linearly independent system of homogeneous polynomials of degree* n *satisfying the equation* $LQ(Y) = 0$ *(such polynomials are called quasi-harmonic).*

LEMMA 31. *Let* $\{v_n(X)\}$ *be a sequence of solutions of class* C^2 *of equation* (5.2) *converging almost uniformly in the ellipsoid* E_R *to the function* $v(X)$ *of class* C^2; *then the function defined by* (5.3) *satisfies equation* (5.1) *in* $\bar{E}_R$.

Proof. By Lemma 26

$$M(R, X, v_n) = v_n(X) - k_p \operatorname*{SSS}_{E_R} (r^{2-p}-R^{2-p})v_n(Y)C(Y)\,dY,$$

where

$$k_p \operatorname*{SSS}_{E_R} (r^{2-p}-R^{2-p})\,dY = (2p)^{-1}R^2.$$

In the two-dimensional case $(p = 2)$

$$M(R, X, v_n) = v_n(X) - k_2 \iint_{E_R} \log\frac{R}{r}\, C(Y)v_n(Y)\,dY,$$

where

$$k_2 \iint_{E_R} \log\frac{R}{r}\,dY = \frac{R^2}{4}.$$

Passing to the limit as $n \to \infty$ we obtain

$$M(R, X, v) = v(X) - k_p \operatorname*{SSS}_{E_R} (r^{2-p}-R^{2-p})v(Y)C(Y)\,dY.$$

Thus by Lemma 27 the function $v(X)$ satisfies equation (5.2) and by Lemma 24 it satisfies equation (5.1).

Lemmas 24 and 29 imply the

LEMMA 32. *Let the functions* $v_n(Y) = w_n(r)H_n(Y)$ *satisfy equation* (1.1); *then the functions*

$$u_n(X) = w_n(\varrho)T_n(X_0)\exp\Big(\sum \alpha_i x_i\Big)$$

satisfy equation (5.1).

Lemmas 30, 31, and 32 imply the

LEMMA 33. *Let the functions*

$$w_n^{(1)}(r)H_n^{(1)}(Y_0), \qquad w_n^{(2)}(r)H_n^{(2)}(Y_0) \tag{5.7}$$

are linearly independent solutions of equation (1.1); *then the functions*

$$w_n^{(1)}(\varrho)T_n^{(1)}(X_0)\exp\Big(\sum a_i x_i\Big), \qquad w_n^{(2)}(\varrho)T_n^{(2)}(X_0)\exp\Big(\sum a_i x_i\Big) \tag{5.8}$$

form linearly independent solutions of equation (5.1).

Lemma 33 implies easily the

LEMMA 34. *Transformation* (T) *is a bijection among functions* (5.7) *and* (5.8).

THEOREM 8. *Let the function* $u(X)$ *of class* C^2 *satisfy equation* (5.1) *in* E_R; *then there exists a sequence* $\{v_n(X)\}$ *such that* $u(X) = \sum u_n(X)$, *where* $u_n(X) = v_n(X)\exp(\sum a_i x_i)$ *and the series* $\sum u_n(X)$ *converges almost uniformly in* E_R.

Proof. By (1.1) and by Lemma 31 for each function $v(Y)$ of class C^2, $|Y| < C$ satisfying equation (1.1) there exists a sequence $\{K_n(Y)\}$ of polynomials such that the series $v(Y) = \sum w_n^{(1)}(r)K_n(Y_0)$ is almost uniformly convergent for $|Y| < C$. Let $u(X)$ be an arbitrary function of class C^2 satisfying equation (5.1) in E_R. Applying transformation (T) to the independent variables X and to the function $u(X)$ we obtain a function $v(Y)$ satisfying equation (5.7) in the ball $K(R, X)$. Next, applying the transformation T^* we obtain our conclusion, q.e.d.

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