ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XIII (1970)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XIII (1970)

Ministerstwo Nauki i Szkolnictwa Wyższego

J. RADECKI (Poznań)

Schauder bases in the space of continuous functions

Denote by $\{h_n(t)\}$ $(n = 1, 2, \ldots)$ the orthonormal Haar system. Ciesielski [1] proved that functions 1 and $\int_0^t h_n(s)\,ds$ $(0 \leqslant t \leqslant 1$ and $n = 1, 2, \ldots)$ form a Schauder basis in $C\langle 0, 1\rangle$ and that

$$F(t) = F(0) + \sum_{n=1}^{\infty} \int_0^1 h_n(s)\,dF(s) \int_0^t h_n(u)\,du$$

for any function $F \in C\langle 0, 1\rangle$.

The following theorem generalizes this result:

THEOREM. *If* $g \in C\langle 0, 1\rangle$, $g(t) \neq 0$ *for* $t \in \langle 0, 1\rangle$, *and* g *is of bounded variation on* $\langle 0, 1\rangle$, *then the functions*

$$\varphi_0(t) = 1, \quad \varphi_n(t) = \int_0^t g(u) h_n(u)\,du \qquad (n = 1, 2, \ldots)$$

form a basis in the space of continuous functions on $\langle 0, 1\rangle$, *and for every* $F \in C\langle 0, 1\rangle$ *there holds*

$$F(t) = F(0) + \sum_{n=1}^{\infty} \varphi_n(t) \int_0^1 \frac{h_n(s)}{g(s)}\,dF(s), \tag{1}$$

where the series is uniformly convergent over the interval $\langle 0, 1\rangle$.

Proof. It is well known that every positive integer $n > 1$ can be written as $n = 2^m + k$, where $m \geqslant 0$, $0 \leqslant k \leqslant 2^m$, and m, k are uniquely determined by n. Let us set $t_{1,0} = 0$, $t_{1,1} = 1$ and for $n > 1$

$$t_{n,i} = \begin{cases} \dfrac{i}{2^{m+1}} & \text{for } i = 0, 1, \ldots, 2k, \\ \dfrac{i-k}{2^m} & \text{for } i = 2k+1, \ldots, n. \end{cases}$$

Evidently, $\{t_{n,i}\}$ $(i = 0, 1, \ldots, n)$ is a sequence of normal partitions of the interval $\langle 0, 1\rangle$. Let us write $P_{n,i} = \{t\colon t \in \langle t_{n,i-1}, t_{n,i})\}$ for $i = 1, 2, \ldots, n-1$ and $P_{n,n} = \{t\colon t \in \langle t_{n,n-1}, t_{n,n}\rangle\}$. If $f \in L\langle 0, 1\rangle$, then $f/g \in L\langle 0, 1\rangle$ and for $t \in P_{n,i}$ the following equality

$$\sum_{j=1}^{n} h_j(t) \int_0^1 \frac{f(s)}{g(s)} h_j(s)\, ds = \frac{1}{|P_{n,i}|} \int_{P_{n,i}} \frac{f(s)}{g(s)}\, ds$$

holds [2], where $|P_{n,i}| = t_{n,i} - t_{n,i-1}$. Therefore, for $F(t) = F(0) + \int_0^t f(s)\, ds$ we have

$$\sum_{j=1}^{n} h_j(t) \int_0^1 \frac{h_j(s)}{g(s)}\, dF(s) = \frac{1}{|P_{n,i}|} \int_{P_{n,i}} \frac{dF(s)}{g(s)}, \tag{2}$$

where $t \in P_{n,i}$. Since absolutely continuous functions are dense in $C\langle 0, 1\rangle$, so the equality (2) holds for any $F \in C\langle 0, 1\rangle$ and $n = 1, 2, \ldots$ Multiplying both sides of this equality by g and integrating we obtain

$$\begin{aligned} \sigma_n(F; t) &\equiv \sum_{j=1}^{n} \int_0^1 \frac{h_j(s)}{g(s)}\, dF(s) \int_0^t g(u) h_j(u)\, du \\ &= \sum_{j=1}^{i-1} \frac{1}{|P_{n,j}|} \int_{P_{n,j}} \frac{dF(s)}{g(s)} \int_{P_{n,j}} g(u)\, du + \frac{1}{|P_{n,i}|} \int_{P_{n,i}} \frac{dF(s)}{g(s)} \int_{t_{n,i-1}}^{t} g(u)\, du, \end{aligned} \tag{3}$$

where $t \in P_{n,i}$.

Let us write $S_n(t) = \frac{1}{|P_{n,j}|} \int_{P_{n,j}} g(u)\, du$ for $t \in P_{n,j}$, $j = 1, 2, \ldots, n$, and $g_n(t) = S_n(t)/g(t)$. Then (3) can be statet in the form

$$\begin{aligned} F(0) + \sigma_n(F; t) - F(t) &= F(t_{n,i-1})[g_n(t_{n,i-1}) - 1] + \\ &+ F(t_{n,i-1}) - F(t) - F(0)[g_n(0) - 1] - \int_0^t F(s)\, dg_n(s) + \int_{t_{n,i-1}}^{t} F(s)\, dg_n(s) + \\ &+ \frac{1}{|P_{n,i}|} \int_{t_{n,i-1}}^{t} g(s)\, ds \left[\frac{F(t_{n,i})}{g(t_{n,i})} - \frac{F(t_{n,i-1})}{g(t_{n,i-1})} - \int_{P_{n,i}} F(s)\, d\left(\frac{1}{g(s)}\right)\right]. \end{aligned} \tag{4}$$

Let $\bigvee_a^b (H)$ denote the total variation of H over $\langle a, b\rangle$ and let $M(H) = \sup_{t \in \langle 0,1\rangle} |H(t)|$. Note that for $0 \leqslant a < b \leqslant 1$ there holds

$$\bigvee_a^b (g_n) \leqslant M(S_n) \bigvee_a^b \left(\frac{1}{g}\right) + M\left(\frac{1}{g}\right) \bigvee_a^b (S_n) \leqslant M(g) \bigvee_a^b \left(\frac{1}{g}\right) + M\left(\frac{1}{g}\right) \bigvee_a^b (g). \tag{5}$$

Since the sequence $\{S_n(t)\}$ $(n = 1, 2, \ldots)$ converges uniformly to $g(t)$ on $\langle 0, 1\rangle$, so $\lim_{n\to\infty} g_n(t) = G(t) = 1$ uniformly for $t \in \langle 0, 1\rangle$. By little modifications in the proof of Helly Theorem we get: If $a \in C\langle a, b\rangle$, functions $\beta_n(t)$ $(n = 1, 2, \ldots)$ are of uniformly bounded variation on $\langle a, b\rangle$ and if $\beta_n(t)$ converges uniformly to $\beta(t)$ on $\langle a, b\rangle$ as $n \to \infty$, then $\int_a^t a(s)\,d\beta_n(s)$ converges uniformly to $\int_0^t a(s)\,d\beta(s)$ on $\langle a, b\rangle$. So we obtain

$$\lim_{n\to\infty} \int_0^t F(s)\,dg_n(s) = \int_0^t F(s)\,dG(s) = 0$$

uniformly over $\langle 0, 1\rangle$. Functions F, F/g, $\bigvee_0^t(g)$, $\bigvee_0^t(1/g)$ are uniformly continuous on $\langle 0, 1\rangle$, so for given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for $0 \leqslant x \leqslant y \leqslant 1$ and $y - x < \delta$ there holds $|F(x) - F(y)| < \varepsilon$,

$$\left|\frac{F(x)}{g(x)} - \frac{F(y)}{g(y)}\right| < \varepsilon, \quad \bigvee_x^y(g) < \varepsilon, \quad \bigvee_x^y\left(\frac{1}{g}\right) < \varepsilon.$$

Moreover, for sufficiently large n we have $\sup_{1\leqslant i\leqslant n} |P_{n,i}| < \delta$, $|g_n(t) - 1| < \varepsilon$, $|\int_0^t F(s)\,dg_n(s)| < \varepsilon$ for all $t \in \langle 0, 1\rangle$. So, from (4) and (5) we obtain

$$\begin{aligned}
&|F(0) + \sigma_n(F; t) - F(t)| \\
&< M(F)\varepsilon + \varepsilon + M(F)\varepsilon + \varepsilon + M(F)[M(g)\varepsilon + M\left(\frac{1}{g}\right)\varepsilon] + M(g)[\varepsilon + M(F)\varepsilon] \\
&= \varepsilon[2M(F)M(g) + M(F)M\left(\frac{1}{g}\right) + 2M(F) + M(g) + 2].
\end{aligned}$$

Let now $F(t) = \sum_{n=0}^{\infty} a_n\varphi_n(t)$, where the series is uniformly convergent on $\langle 0, 1\rangle$. Obviously, $a_0 = F(0)$. By the continuity of the functionals $\int_0^1 \frac{h_n(s)}{g(s)}\,dF(s)$, we obtain

$$\int_0^1 \frac{h_m(s)}{g(s)}\,dF(s) = \sum_{n=0}^{\infty} a_n \int_0^1 \frac{h_m(s)}{g(s)}\,d\varphi_n(s) = \sum_{n=1}^{\infty} a_n \int_0^1 h_m(s)h_n(s)\,ds = a_m$$

for $m = 1, 2, \ldots$; that proves the uniqueness of (1).

Remark. If the continuous derivative $g'(t)$ exists for $g(t) \neq 0$ in $\langle 0, 1)$ and if the improper integral

$$\int_0^1 \frac{dF(s)}{g(s)} = \lim_{1>r\to 1} \int_0^r \frac{dF(s)}{g(s)}$$

exists for continuous $F(t)$ in $\langle 0, 1)$, then the equality (1) holds in $\langle 0, 1)$ and the convergence is almost uniform.

References

[1] Z. Ciesielski, *On Haar functions and on the Schauder basis of the space* $C\langle 0, 1\rangle$, Bull. Acad. Polon. Sci. 7 (1959), pp. 227-232.

[2] R. Sikorski, *Funkcje rzeczywiste*, tom II, str. 135, Warszawa 1959.

UNIWERSYTET im. ADAMA MICKIEWICZA W POZNANIU
KATEDRA MATEMATYKI II
UNIWERSITY OF ADAM MICKIEWICZ IN POZNAŃ