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Graph topologies for function spaces, II

1. Introduction. A new function space topology (called "graph topology") on the set of all functions on a topological space X to a topological space Y, was studied by the first author in [5]. This topology, which we denote by Γ_2 in this paper, is defined by using the graphs of the functions and the product topology on $X \times Y$. The definition of the graph topology Γ_2 is similar to that of the hyperspace topology of Ponomarev [6] on the closed subsets of $X \times Y$. But in spite of this similarity they differ considerably. Firstly, although Ponomarev's topology is rather pathological in its separation properties, the topology Γ_2 inherits some of the decent separation properties from X and Y. Secondly, one generally considers only closed or compact subsets as elements of a hyperspace but the graphs of the functions need not be closed or compact subsets of $X \times Y$. We also note here that unlike the other known functions space topologies which are mostly studied on the set of *continuous* functions, the graph topology Γ_2 is quite useful for the *almost continuous* functions of Stallings [8].

Subsequently another graph topology (called Γ_1 in this paper) was studied by Poppe in [7]. In this case too the definition was motivated by Ponomarev's topology. Poppe showed that Γ_1 is coarser than Γ_2 but otherwise the two topologies share many common properties.

The above observations suggest that one might construct "graph topologies", the definitions of which are similar to those of the other known hyperspace topologies. This is indeed the case, as we shall explain in this paper. A unified approach to these problems was motivated by an axiomatic method of topologizing the hyperspace of closed subsets discovered by Marjanovič [4]. In the next section we shall show that this yields a general method of constructing several "graph topologies" including Γ_1 and Γ_2 as special cases.

In this paper we study four graph topologies Γ_i $(i = 1, 2, 3, 4)$ — their separation properties, their relationships among themselves and with the other well-known function space topologies such as the com-

pact-open (or k-) topology, the uniform convergence (or u.c.) topology and the σ-topologies of Arens and Dugundji [1]. For the sake of completeness, we also include a few known results.

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2. Preliminaries. Any terms and notations not explained here will be found in Kelley [3].

2.1. Let X and Y be topological spaces and let $X \times Y$ be assigned the usual product topology. If f is a function on X to Y, the graph of f denoted by $G(f)$, is the set $\{(x, f(x)) : x \in X\} \subset X \times Y$.

2.2. $\boldsymbol{F} = \{f : f$ a function on X to $Y\}$.

$\boldsymbol{C} = \{f \in \boldsymbol{F} : f$ is continuous$\}$.

$\boldsymbol{K} = \{f \in \boldsymbol{F} : G(f)$ is compact$\}$.

$\boldsymbol{P} = \{f \in \boldsymbol{F} : G(f)$ is a paracompact subset of $X \times Y\}$.

2.3. A collection $\boldsymbol{U}$ of open subsets of $X \times Y$ is called a *clover* of $G(f)$ (where $f \in \boldsymbol{F}$) if and only if $G(f) \cap U \neq \varphi$ for each $U \in \boldsymbol{U}$ and $\boldsymbol{U}$ covers $G(f)$. (Marjanovič [4] uses the word "cover" in place of what we call "clover". However, since "cover" is already used in a different sense, we prefer to use another word.)

2.4. If $\boldsymbol{U}$, $\boldsymbol{U}'$ are two clovers of $G(f)$, then we define (a partial order) $\ll$: $\boldsymbol{U}' \ll \boldsymbol{U}$ if and only if each $U \in \boldsymbol{U}$ contains some $U' \in \boldsymbol{U}'$ and $\bigcup \{U' : U' \in \boldsymbol{U}'\} \subset \bigcup \{U : U \in \boldsymbol{U}\}$. (Marjanovič [4] says "$\boldsymbol{U}'$ refines $\boldsymbol{U}$" but again we avoid this word which is usually used in another sense; see for example Kelley [3], p. 128.)

Following Marjanovič [4], we shall suppose that in each case *the family of clovers under consideration is a directed set with respect to* $\ll$. It then follows, as in Marjanovič's Proposition 1, that each such family of clovers gives a "graph topology" on $\boldsymbol{F}$. We now consider four special cases, the last three of which correspond to Examples 1, 2, 5 of Marjanovič [4]. It is possible to define graph topologies corresponding to his other examples but we do not consider them here.

2.5. Γ_1. Let the family of clovers consist of sets of the form $\bigcap_{i=1}^{n} [X \times Y - A_i \times B_i]$, where A_i, B_i are respectively closed subsets of X, Y. The resulting topology Γ_1 on $\boldsymbol{F}$ was introduced by Poppe [7] and it is clear that Γ_1 is generated by a subbasis consisting of sets of the form $(A, U) = \{f \in \boldsymbol{F} : f(A) \subset U\}$, A being a closed subset of X and U being an open subset of Y.

2.6. Γ_2. If each clover of the family consists of *one* open set, then the resulting topology Γ_2 on $\boldsymbol{F}$ is precisely the "graph topology Γ" of [5].

2.7. Γ_3. This topology on $\boldsymbol{F}$ is obtained by requiring each clover of the family to have only a *finite* number of open subsets of $X \times Y$.

2.8. Γ_4. This topology on $\boldsymbol{F}$ is generated by the family of *locally finite* clovers of the graphs of the functions.

If $\boldsymbol{U}$ is a collection of open subsets of $X \times Y$, we shall use the notation $\boldsymbol{U}^*$ to denote the set $\{f \in \boldsymbol{F}\colon \boldsymbol{U}$ is a clover of $G(f)\}$.

3. Comparison of graph topologies. Poppe [7] proved that $\Gamma_1 \subset \Gamma_2$ and showed that, in general, $\Gamma_1 \neq \Gamma_2$. It is easily seen that $\Gamma_i \subset \Gamma_{i+1}$ for $i = 1, 2, 3$. We now construct examples to show strict inclusion for $i = 2, 3$.

3.1. EXAMLPE. $\Gamma_2 \neq \Gamma_3$.

Let $X = \{a, b\}$ with the topology consisting of $\varphi, X, \{a\}$. Let $Y = \{c, d\}$ with the discrete topology. Let $f, g \in \boldsymbol{F}$ be defined by $f(a) = d$, $f(b) = c$, $g(a) = g(b) = c$. Let $U_1 = \{(a, d)\}$, $U_2 = \{(a, c), (b, c)\}$. Then $\{U_1, U_2\}$ is a clover of $G(f)$. Now every open set in $X \times Y$ which contains (b, c) also contains (a, c). Hence if an open set U contains $G(f)$, then $G(g) \subset U$. But $G(g) \cap U_1 = \varphi$ which shows that $\Gamma_2 \neq \Gamma_3$.

In the above example X is not a T_1-space; this is not a coincidence. In fact, whenever X is not T_1 we can similarly prove that $\Gamma_2 \neq \Gamma_3$ and we shall show below (Theorem 3.3) that $\Gamma_2 = \Gamma_3$ if X is T_1.

3.2. EXAMPLE. $\Gamma_3 \neq \Gamma_4$.

Let N be the set of all natural numbers, $\tau_1 =$ the cofinite topology on N and $\tau_2 =$ the discrete topology on N. Let $f\colon (N, \tau_1) \to (N, \tau_2)$ be the identity function, i.e. $f(x) = x$ for all $x \in N$. Then $\boldsymbol{U} = \{N \times \{x\}\colon x \in N\}$ is a locally finite clover of $G(f)$. Let $\boldsymbol{U}^*$ be the corresponding Γ_4-neighbourhood of f. Let $\boldsymbol{V} = \{U_k\colon k = 1, \ldots, n\}$ be any finite clover of $G(f)$ and consider $\boldsymbol{V}^*$ the corresponding Γ_3-neighbourhood of f. Choose $(x_k, x_k) \in G(f) \cap U_k$, $k = 1, \ldots, n$, and let U_1 contain a basis element $(N - M) \times \{y\}$ of $\tau_1 \times \tau_2$, where M is a finite subset of N and $y \in N$. Define $g\colon N \to N$ by

$$g(x) = \begin{cases} x & \text{for } x \in M \cup \{x_k\colon k = 1, \ldots, n\}, \\ y & \text{otherwise.} \end{cases}$$

Then $g \in \boldsymbol{V}^*$ but $g \notin \boldsymbol{U}^*$, since $G(g) \cap (N \times \{z\}) = \varphi$ if $z \notin M \cup \{x_k\colon k = 1, \ldots, n\}$.

Although $\Gamma_i \neq \Gamma_{i+1}$ $(i = 1, 2, 3)$ in general, it is interesting to find out the conditions under which $\Gamma_i = \Gamma_{i+1}$. Poppe [7] has shown that $\Gamma_1 \neq \Gamma_2$ (on $\boldsymbol{F}$) even when X is compact Hausdorff but that, in this case, they are equal on the subspace $\boldsymbol{C}$ of continuous functions. (This cor-

rects the last part of Theorem 4.2 of [5], where one should add "on $\mathbf{C}$" to get a correct statement.) He also showed that if Y is the real line and $\Gamma_1 = \Gamma_2$, then X is necessarily countably compact.

Ponomarev's hyperspace topology (to which Γ_2 corresponds) is rather pathological and, except in trivial cases, is never equal to the Vietoris finite topology (to which Γ_3 corresponds). In view of this well-known result, the following theorem is rather surprising:

3.3. THEOREM. *On* $\boldsymbol{F}$, $\Gamma_2 = \Gamma_3$ *if and only if* X *is* T_1.

Proof. In view of the remarks following Example 3.1, it is sufficient to show that if X is T_1, then $\Gamma_2 = \Gamma_3$. Let $\boldsymbol{U} = \{U_i\colon i = 1, \ldots, n\}$ be any finite clover of $f \in \boldsymbol{F}$. Choose $p_i \in X$ such that $(p_i, f(p_i)) \in U_i$, $i = 1, \ldots, n$. For each $x \in X - P$ (where $P = \bigcup \{p_i\colon i = 1, \ldots, n\}$) there exists an open set W_x containing $(x, f(x))$ and contained in $\bigcup \{U\colon U \in \boldsymbol{U}\} - P \times Y$. Also for each $i = 1, \ldots, n$ there exists an open set W_{p_i} containing $(p_i, f(p_i))$ and contained in U_i. Let $W = \bigcup \{W_x\colon x \in X\}$. Then $\boldsymbol{W} = \{W\}$ is a clover of f and $\boldsymbol{W}^* \subset \boldsymbol{U}^*$, showing $\Gamma_3 \subset \Gamma_2$.

Example 4.6 below shows that on $\boldsymbol{F}$, $\Gamma_3 \neq \Gamma_4$ even when X and Y are "nice" spaces such as $X = Y = [-1, 1]$. One might conjecture that the equality will result when one considers the subspace $\boldsymbol{C}$ of continuous functions; but the following example shows that this is not true.

3.4. EXAMPLE. Let $X = Y = (0, 1]$ and let f be the identity function. Consider the locally finite clover of f, namely $\boldsymbol{U}$ consisting of $(0, 1] \times (0, 1]$ and $S\big((1/n, 1/n), 1/n^3\big)$, where $S\big((x, y), r\big) = \{(u, v) \in X \times Y\colon (x-u)^2 + (y-v)^2 < r^2\}$. By Theorem 3.3 $\Gamma_2 = \Gamma_3$. If V is any open set containing $G(f)$, then there exists an $\varepsilon > 0$ such that 2ε-spheres about each point of $G(f)$ are contained in V. If $g\colon X \to Y$ is defined by

$$g(x) = \varepsilon + (1 - 2\varepsilon)x,$$

then $G(g) \subset V$ but $g \notin \boldsymbol{U}^*$.

4. Almost continuous functions. Γ_2-almost continuous functions were introducted by Stallings [8] and were called simply "almost continuous". In this section we consider Γ_i-almost continuous functions.

4.1. DEFINITION. A function $f \in \boldsymbol{F}$ is called *Γ_i-almost continuous* ($i = 1, 2, 3, 4$) if and only if each Γ_i-neighbourhood of f contains a $g \in \boldsymbol{C}$. The set of all such functions will be denoted by $\boldsymbol{A}_i$.

The following result is analogous to Theorem 2.6 of [5] and is easily proved.

4.2. THEOREM. $\boldsymbol{A}_i$ *is a closed subset of* $(\boldsymbol{F}, \Gamma_i)$; *in fact it is the* Γ_i*-closure of the set* $\boldsymbol{C}$ *of all continuous functions,* $i = 1, 2, 3, 4$.

A topological space X has *fixed point property* if and only if for each continuous self-map f of X, there exists a $p \in X$ such that $f(p) = p$.

Proof of the following theorem is similar to that of Theorem 3 of Stallings [8]:

4.3. THEOREM. *If X is a Hausdorff space with fixed point property, then every Γ_i-almost continuous self-map ($i = 2, 3, 4$) of X also has a fixed point.*

We do not know whether or not the above result is true for $i = 1$.

Since $\Gamma_i \subset \Gamma_{i+1}$ $(i = 1, 2, 3)$ it follows that a Γ_{i+1}-almost continuous function is always Γ_i-almost continuous. We now give examples of Γ_i-almost continuous functions which are not Γ_{i+1}-almost continuous for $i = 1, 2, 3$.

4.4. EXAMPLE. Consider the example, constructed by Cornette [2], of a connectivity function on $[0, 1]$ into itself which is not Γ_2-almost continuous. By Theorem 2 of Stallings [8], such a function is polyhedrally almost continuous. This implies that it is Γ_1-almost continuous.

4.5. EXAMPLE. Let X be the set of all natural numbers with the "Siamese Twin" topology with a base consisting of φ, X and $\{2n-1, 2n\}$, $n = 1, \ldots$, and let Y be the real line with the usual topology. Let $f(m) = 1/m$; then f is not continuous but is Γ_2-almost continuous. For if $G(f) \subset U$, where U is an open subset of $X \times Y$, then $g\colon X \to Y$ defined by $g(2n-1) = g(2n) = 1/(2n-1)$, $n = 1, \ldots$, is a continuous function and $G(g) \subset U$. But if $U_{2n-1} = \{2n-1, 2n\} \times V_{2n-1}$, where V_{2n-1} is an open neighbourhood (in Y) of $1/(2n-1)$ not containing $1/2n$, $U_{2n} = \{2n-1, 2n\} \times V_{2n}$, where V_{2n} is an open neighbourhood of $1/2n$ not containing $1/(2n-1)$, $n = 1, \ldots, m$, and $U_{2m+1} = \{n\colon n \geqslant 2m\} \times \{x\colon x > 2m\}$, then $\boldsymbol{U} = \{U_i\colon i = 1, \ldots, 2m+1\}$ is a clover of $G(f)$. However, if $h \in \boldsymbol{U}^*$, then h cannot be continuous and so f is not Γ_3-almost continuous.

4.6. EXAMPLE. Let $X = Y = [-1, 1]$ with the usual topology and let $f \in F$ be defined by

$$f(x) = \begin{cases} \sin\dfrac{\pi}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is Γ_i-almost continuous $(i = 1, 2, 3)$. We now show that f is not Γ_4-almost continuous. Consider the closed set A in $X \times Y$ defined by

$$A = \left\{\left(\frac{1}{n}, 0\right), (0, 0)\right\} \cup \left\{\left(\frac{2}{2n+1}, 1\right), (0, 1)\right\},$$

where n runs through all integers. For each $(x, f(x)) \notin A$, there exists an open neighbourhood U_x such that $U_x \cap A = \varphi$. If $(x, f(x)) \in A$, we select the following open neighbourhoods:

(i) $S\left(\left(\frac{1}{n}, 0\right), \frac{1}{2}\right)$ if $x = \frac{1}{n}$;

(ii) $S\left(\left(\frac{2}{2n+1}, 1\right), \frac{4}{(2n+1)^2}\right)$ if $x = \frac{1}{2n+1}$;

(iii) $S\left((0, 0), \frac{1}{100}\right)$ if $x = 0$.

Then $\boldsymbol{U}$ consisting of (i), (ii), (iii) and U_x for $x \notin A$, is a clover of $G(f)$. Since $G(f)$ is paracompact (it is in fact metrizable), there is a locally finite refinement $\boldsymbol{V}$ of $\boldsymbol{U}$ which is also a clover of $G(f)$. If $g \in \boldsymbol{V}^*$, then

$$G(g) \cap S\left(\left(\frac{2}{2n+1}, 1\right), \frac{4}{(2n+1)^2}\right) \neq \varphi$$

and so contains a point (x_n, y_n). Then $x_n \to 0$ but $y_n = g(x_n)$ does not tend to zero and so g is not continuous.

A Γ_4-almost continuous function is always Γ_3-almost continuous function and the latter can at worst have oscillatory discontinuities when $X = Y = [-1, 1]$. Proceeding as above one can in fact show that on such spaces every Γ_4-almost function is always continuous.

5. Comparison of graph topologies with other function space topologies. Most function space topologies known in the literature are useful only for the set of continuous functions. So in this section, for most part, we restrict ourselves to continuous functions. Also in order to get meaningful results we assume X to be at least T_1 and Y to be non-degenerate (i.e. there exists an open set V in Y such that $\varphi \neq V \neq Y$).

Poppe [7] has proved that if X is T_1, the pointwise convergence (or p.c.) topology is smaller than Γ_1. Hence we have the following result:

5.1. THEOREM. *If X is a T_1-space, then the p.c. topology on $\boldsymbol{F}$ is contained in Γ_i $(i = 1, 2, 3, 4)$.*

5.2. THEOREM. *Let X be a T_1-space and let Y be non-degenerate. Then the p.c. topology on $\boldsymbol{F}$ equals Γ_1 if and only if X is finite with discrete topology.*

Proof. Let X be a finite discrete space and let $f \in (A, U)$, where A is closed in X and U is open in Y. Then (A, U) is a subbasis element of Γ_1. If

$$g \in \bigcap_{i=1}^{n} (x_i, U), \quad \text{where } A = \bigcup \{x_i : i = 1, \ldots, n\},$$

then $g \in (A, U)$ and so $\Gamma_1 =$ the p.c. topology, in view of Theorem 5.1.

If X is not finite, we may suppose that X has cofinite topology. Let V be a non-trivial open subset of Y. If $y \in V$, then the constant func-

tion f which maps X onto $\{y\}$, is in (X, V) but for every open set $\bigcap_{i=1}^{n}(x_i, U_i)$ in the p.c. topology, we can find a function g such that $g(x_i) \in U_i$, $i = 1, \ldots, n$, and $g(x) \in Y - V$ for some $x \neq x_i$, $i = 1, \ldots, n$. Hence the p.c. topology $\neq \Gamma_1$.

Poppe [7] has shown that if X is Hausdorff, the compact-open (or k-) topology is smaller than Γ_1 and that the two are equal when X is also compact. Further if Y is not degenerate, X Hausdorff and $\Gamma_1 =$ the k-topology, then X is compact. The following result summarizes this information:

5.3. THEOREM. *If X is Hausdorff, then the k-topology is contained in Γ_i $(i = 1, 2, 3)$ on $\boldsymbol{F}$. Further the k-topology $= \Gamma_i$ on $\boldsymbol{C}$ $(i = 1, 2, 3)$ if and only if X is compact.*

Poppe [7] has constructed an example of a non-Hausdorff space X such that $\Gamma_1 \neq$ the k-topology even on $\boldsymbol{C}$.

We next consider the *σ-topologies* introduced by Arens and Dugundji [1]. Let σ be an open cover of X and let $\{A_i : i \in I\}$ be a collection of closed subsets of X such that each A_i is contained in some member of σ. Then the topology generated by the subbasis $\{(A_i, U) : i \in I, U \text{ open in } Y\}$, is called the σ-topology on $\boldsymbol{F}$. (Arens and Dugundji consider only continuous functions but we do not restrict ourselves to $\boldsymbol{C}$.) It is obvious that when $X \in \sigma$, Γ_1 coincides with the σ-topology. The next theorem follows from a remark of Poppe [7] that Γ_1 is the finest σ-topology.

5.4. THEOREM. *Any σ-topology is contained in Γ_i $(i = 1, 2, 3, 4)$.*

We now give an example to show proper inclusion.

5.5. EXAMPLE. Let $X = Y =$ the set of all real numbers with the usual topology. Let $\sigma = \bigcup_{n=-\infty}^{\infty}(n, n+2)$ be a cover of X. Let $f \in \boldsymbol{F}$ be the identity function. Then

$$f \in V = \left(J, \bigcup_{n \in J}\left(n - \frac{1}{10}, n + \frac{1}{10}\right)\right),$$

where J is the set of all integers. If $\bigcap_{i=1}^{n}(A_i, U_i)$ is a σ-neighbourhood of f, then there exists an $m \in J$ such that

$$(m, m+2) \subset X - \bigcup_{i=1}^{n} A_i.$$

Define $g: X \to Y$ as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x \leqslant m, \\ \sqrt{3}x + m(1 - \sqrt{3}) & \text{for } x \geqslant m. \end{cases}$$

Then $g \in \bigcap_{i=1}^{n}(A_i, U_i)$ but $g \notin V$. Hence the σ-topology $\neq \Gamma_1$.

Arens and Dugundji [1] have proved that on C, the subspace of continuous functions,

(i) when X is T_3, every σ-topology is admissible (Theorem 4.1);

(ii) the k-topology is proper (Theorem 4.21);

(iii) every proper topology is smaller than every admissible topology (Theorem 3.7).

The results show that when X is T_3, the k-topology on C is smaller than any σ-topology. Hence using our Theorem 5.3 we get the following:

5.6. THEOREM. *If X is compact Hausdorff, then on C, the k-topology, any σ-topology and Γ_i $(i = 1, 2, 3)$ all coincide.*

We next consider the uniform convergence (or u.c.) topology and so we assume that V is a uniformity on Y. For obvious reasons we also restrict ourselves to the subspace C of continuous functions. The following theorem and the subsequent example are perhaps known in the literature but we are unable to locate them.

5.7. THEOREM. *On C the k-topology is contained in the u.c. topology.*

Proof. Let (K, U) be a subbasis element of the k-topology (where K is a compact subset of X and U is an open subset of Y) and let $f \in (K, U) \cap C$. Then $f(K)$ is compact and so there is a $V \in V$ such that $V[f(K)] \subset U$ (Theorem 33, p. 119, Kelley [3]). Now if $g \in W(V)[f]$, then $\big(f(x), g(x)\big) \in V$ for all $x \in X$. Thus $g(K) \subset V[f(K)] \subset U$. This shows that $g \in (K, U)$.

We now give an example to show strict inclusion.

5.8. EXAMPLE. Let $X = Y =$ the set of all real numbers with the usual topology and f be the identity function. Let

$$N_\varepsilon = \{g \in C\colon |f(x) - g(x)| < \varepsilon \text{ for all } x \in X, \varepsilon > 0\}.$$

Let

$$f \in \bigcap_{i=1}^{n}(K_i, U_i),$$

K_i a compact subset of X, U_i an open subset of Y for $i = 1, \ldots, n$. Then there exists a $p \in \bigcup_{i=1}^{n} K_i$ such that $p \geqslant x$ for each $x \in \bigcup_{i=1}^{n} K_i$. If we define $h\colon X \to Y$,

$$h(x) = \begin{cases} f(x) & \text{for } x \leqslant p, \\ p & \text{for } x \geqslant p, \end{cases}$$

then $h \in \bigcap_{i=1}^{n}(K_i, U_i)$ but $h \notin N_\varepsilon$.

It is well known that on $\boldsymbol{C}$, the k-topology equals the u.c. topology when X is compact. Hence by our Theorem 5.3 we get the following result:

5.9. THEOREM. *If X is compact Hausdorff, then, on $\boldsymbol{C}$ the k-topology, the u.c. topology and Γ_i $(i = 1, 2, 3)$ all coincide.*

The following example shows that in general the u.c. topology is not comparable with Γ_1 (also see Poppe [7]).

5.10. EXAMPLE. Let $X = Y =$ the real numbers with the usual topology and let f be the identity function. Let

$$U = \bigcup_{n \in Z} \left(n - \frac{1}{|n|}, n + \frac{1}{|n|} \right),$$

where Z is the set of all integers. Then $f \in (Z, U)$ but for every $\varepsilon > 0$, $g(x) = x + \varepsilon/2$ is such that $d(f, g) < \varepsilon$ but $g \notin (A, U)$. So Γ_1 is not contained in the u.c. topology.

On the other hand consider an ε-neighbourhood $(\varepsilon > 0)$ of f in the u.c. topology. Then for every Γ_1-neighbourhood $\bigcap_{i=1}^{n} (A_i, U_i)$, A_i closed in X, U_i open in Y, either (i) one of U_i's contains arbitrarily large intervals, or (ii) $Y - \bigcup_{i=1}^{n} U_i$ contains arbitrarily large intervals. In case (i) let U_1 contain arbitrarily large intervals. Take a closed interval $[\alpha, \beta]$ in U_1 such that $A_1 \subset [\alpha, \beta]$ and $\beta - \alpha > 2\varepsilon$. Define $g\colon X \to Y$ by,

$$g(x) = \begin{cases} f(x) & \text{for } x \in X - [\alpha, \beta], \\ 2x - \alpha & \text{for } x \in \left[\alpha, \dfrac{\alpha + \beta}{2} \right], \\ \beta & \text{for } x \in \left[\dfrac{\alpha + \beta}{2}, \beta \right]. \end{cases}$$

Then $g \in \bigcap_{i=1}^{n} (A_i, U_i)$ and yet $d(f, g) > \varepsilon$. We can similarly consider the other case to show that on $\boldsymbol{C}$, the u.c. topology is not contained in Γ_1.

6. Separation axioms. In [5] and [7] separation axioms T_1, T_2 are discussed for $(\boldsymbol{F}, \Gamma_i)$, $i = 1, 2$. Supposing that Y has at least two points the results are

(a) $(\boldsymbol{F}, \Gamma_i)$ $(i = 1, 2)$ is T_1 if and only if X and Y are both T_1,

(b) $(\boldsymbol{F}, \Gamma_i)$ $(i = 1, 2)$ is T_2 if and only if X is T_1 and Y is T_2.

One can easily prove that the above results are also true for $i = 3, 4$.

6.1. THEOREM. *If X is T_1, Y is normal and $\boldsymbol{H} = \{f \in \boldsymbol{F}\colon f(A)$ is closed for each closed subset A of $X\}$, then $(\boldsymbol{H}, \Gamma_1)$ is regular.*

Proof. Let $f \in \boldsymbol{H} \cap (A, U)$, A closed in X, U open in Y. Since $f(A)$ is a closed subset of Y and is contained in the open subset U of Y and Y is normal, there is an open set V in Y such that $f(A) \subset V \subset \overline{V} \subset U$. If $g \notin (A, U)$, then there is an $a \in A$ such that $g(a) \in Y - U \subset Y - \overline{V}$. Then $g \in (\{a\}, Y - \overline{V})$ and no function in (A, V) can belong to this Γ_1-neighbourhood of g. Thus g is not in the Γ_1-closure of (A, V), thus showing that $(\boldsymbol{H}, \Gamma_1)$ is regular.

6.2. THEOREM. *If $X \times Y$ is T_3 and $\boldsymbol{K} = \{f \in \boldsymbol{F}: G(f)$ is compact$\}$, then $(\boldsymbol{K}, \Gamma_i)$, $i = 2, 3$ is T_3.*

The proof of the above theorem is similar to but simpler than that of the following and so we omit it.

6.3. THEOREM. *If $X \times Y$ is T_3 and $\boldsymbol{P} = \{f \in \boldsymbol{F}: G(f)$ is a paracompact subset of $X \times Y\}$, then $(\boldsymbol{P}, \Gamma_4)$ is T_3.*

Proof. Let $\boldsymbol{U} = \{U_i: i \in I\}$ be a locally finite clover of $G(f)$, $f \in \boldsymbol{P}$. Since $X \times Y$ is T_3 and $G(f)$ is paracompact, there exists a locally finite clover $\boldsymbol{V} = \{V_j: j \in J\}$ such that $\boldsymbol{V} \ll \boldsymbol{U}$ and $\bigcup \{\overline{V}_j: j \in J\} \subset \bigcup \{U_i: i \in I\}$. We claim that $\overline{\boldsymbol{V}}^* \subset \boldsymbol{U}^*$, where the bar over $\boldsymbol{V}^*$ denotes the Γ_4-closure. If $g \notin \boldsymbol{U}^*$, then either (i) there exists a $p \in X$ such that $(p, g(p)) \notin \bigcup \{U_i: i \in I\}$, or (ii) $G(g) \cap U_i = \varphi$ for some $i \in I$. In the first case $(p, g(p))$ has an open neighbourhood N_p and $\bigcup \{\overline{V}_j: j \in J\}$ has an open neighbourhood N such that $N \cap N_p = \varphi$. We then construct a locally finite clover $\{W_k\}$ of $G(g)$ such that one of the sets is N_p. Clearly $g \notin \overline{\boldsymbol{V}}^*$. In the second case there is an open neighbourhood V_l containing a point of $G(f)$ and such that $\overline{V}_l \subset U_i$. We then construct a locally finite clover of $G(g)$ such that none of the members of this clover intersects $\overline{V}_l$. In this case too $g \notin \overline{\boldsymbol{V}}^*$ thus showing that $(\boldsymbol{P}, \Gamma_4)$ is T_3.

Following is an immediate consequence of our Theorem 5.9.

6.4. THEOREM. *If X is compact Hausdorff and Y is completely regular, then $(\boldsymbol{C}, \Gamma_i)$, $i = 1, 2, 3$ is completely regular.*

It was proved in [5] that Γ_2 coincides with the usual "sup" metric topology on $\boldsymbol{C}$ whenever X is compact metric and Y metric (Theorem 4.8). From our Theorem 3.3 and the remarks preceeding it, we get the following result:

6.5. THEOREM. *If X is a compact metric space and Y a metric space, then $(\boldsymbol{C}, \Gamma_i)$, $i = 1, 2, 3$ is metrizable and the Γ_i-topologies coincide with the usual sup metric topology on $\boldsymbol{C}$.*

Note added in proof. Professor T. D. Hansard, in a paper *Function space topologies,* has shown that if X is connected, then the u.c. topology is smaller than Γ_1 on $\boldsymbol{C}$, thus contradicting the second part of (5.10). We are also grateful to Professor N. Noble for pointing out an error in the earlier version of (3.4).

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