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On a simple case of asymptotic stability

Let us suppose that the function $f(x, y)$ appearing in the differential equation

$$y' = f(x, y) \tag{1}$$

is continuous in the plane set

$$D = \{(x, y)\colon x \in \langle a, +\infty), \ y \in (y_0 - a, y_0 + a)\}$$

(where $a > 0$; in particular, it may be $a = +\infty$), and that $f(x, y) \leqslant 0$ in the set

$$D_1 = \{(x, y)\colon x \in \langle a, +\infty), y \in (y_0, y_0 + a)\},$$

and $f(x, y) \geqslant 0$ in the set

$$D_2 = \{(x, y)\colon x \in \langle a, +\infty), \ y \in (y_0 - a, y_0)\}.$$

Then it is easily verified that $y = y_0$ is an integral of equation (1) defined for $x \in \langle a, +\infty)$ and stable in the sense of Lapunov. Let us observe that the stability of the integral $y = y_0$ is uniform. Indeed, given any $\varepsilon > 0$, for every straight line $x = x_0$ $(x_0 \geqslant a)$ one may choose on this straight line a segment

$$L = \{(x, y)\colon x = x_0, \ y \in (y_0 - \varepsilon, y_0 + \varepsilon)\} \quad (\text{for } 0 < \varepsilon \leqslant a)$$

such that any integral $y = \varphi(x)$ of equation (1) starting from this segment will satisfy the inequality $|\varphi(x) - y_0| < \varepsilon$ independently of the position of x_0; this follows from the fact that the integrals $y = \varphi(x)$ lying in D_1 have derivatives $\varphi'(x) \leqslant 0$, and in D_2, $\varphi'(x) \geqslant 0$.

One-side stability may also appear. For example, supposing $f(x, y)$ to be continuous in D as previously, $f(x, y) \leqslant 0$ in D_1, and $y = y_0$ to be a lower integral of equation (1) starting at the point (a, y_0), we have uniform stability of the integral $y = y_0$ "from above". Similarly, it may appear also uniform stability of the integral $y = y_0$ "from below".

EXAMPLE. We obtain the case of one-side stability if we take for example $f(x, y)$ continuous in D, satisfying the condition $f(x, y) \geqslant 0$

in D_2 (for $y_0 = 0$ and $a = +\infty$), and equal $f(x, y) = xy$ in D_1. Then the solution $y = 0$ will be uniformly stable, but only "from below", since for $y > 0$ the general form of the solution is $y = ce^{x^2/2}$ (for $c > 0$). Hence none of these solutions has a common point with the integral $y = 0$, and all tend to $+\infty$ as $x \to +\infty$.

In the following we shall give a simple sufficient condition in order that a stable integral $y = y_0$ of equation (1) be uniformly asymptotically stable. The following two lemmas will be needed.

LEMMA 1. *Let the function $f(x, y)$ be continuous in the set D, and let $y = \omega(x)$ be an upper (resp. lower) integral of equation* (1), *starting at the point $(x_0, y_1) \in D$ $(x_0 \geqslant a)$ and defined in $\Delta_{x_0} = \langle x_0, +\infty)$. Finally, let $y = \varphi(x)$ be an integral of equation* (1) *defined in Δ_{x_0} and satisfying the condition $\varphi(x_0) \leqslant y_1$ (resp. $\varphi(x_0) \geqslant y_1$). Then the inequality $\varphi(x) \leqslant \omega(x)$ (resp. $\varphi(x) \geqslant \omega(x)$) holds in the whole interval Δ_{x_0}.*

This lemma is a special case of Theorem 9.3 (Szarski [2], p. 25).

LEMMA 2. I. *If the function $\varphi(x)$ is defined and possesses a derivative $\varphi'(x)$ in Δ_{x_0}, and if $\lim_{x \to +\infty} \varphi'(x) = \delta$, where $-\infty \leqslant \delta < 0$, then $\lim_{x \to +\infty} \varphi(x) = -\infty$.*

II. *If the function $\varphi(x)$ is defined and possesses a derivative $\varphi'(x)$ in Δ_{x_0}, and if $\lim_{x \to +\infty} \varphi'(x) = \gamma$, where $0 < \gamma \leqslant +\infty$, then $\lim_{x \to +\infty} \varphi(x) = +\infty$.*

Proofs of the following three theorems are based on the above lemmas.

THEOREM 1. *Let us suppose that*

1. *the function $f(x, y)$ is defined and continuous in the set D,*

2. *$f(x, y) \leqslant 0$ in D_1,*

3. *for every $\bar{y} \in (y_0, y_0 + a)$ there exists the limit $\lim f(x, y) = \delta$ $(\delta < 0)$, where $x \to +\infty$ and $y \downarrow \bar{y}$,*

4. *$y = y_0$ is a lower integral of equation* (1) *starting at the point (a, y_0).*

Let the initial point of the integral $y = y_0$ lie on any fixed segment $L_1 = \{(x, y): x = x_0, y \in \langle y_0, y_0 + \beta \rangle\}$ (where $0 < \beta < a$ and $x_0 \geqslant a$). Then the integral $y = y_0$ is uniformly asymptotically stable "from above" with respect to initial conditions given at any fixed segment L_1.

Proof. It follows from the assumptions 1 and 2 that every integral $y = \varphi(x)$ of equation (1) starting at an arbitrary point of the segment L_1 is defined in Δ_{x_0}. Indeed, it possesses the derivative $\varphi'(x) \leqslant 0$; hence it is decreasing and cannot attain points below the straight line $y = y_0$, since by assumption 4, $y = y_0$ is a lower integral of equation (1) and, by Lemma 1, $y_0 \leqslant \varphi(x)$ in Δ_{x_0}. Next, it follows from assumptions 1, 2 and 4 that the upper integral $y = \omega(x)$ of equation (1) starting from the point $(x_0, y_0 + \beta)$ is also defined in Δ_{x_0}; this may be deduced also directly from Theorem 9.1 ([2], p. 23). From Lemma 1 it follows imme di-

ately, that the integrals starting at points of the segment L_1 (which, as we have shown, are defined in Δ_{x_0}), satisfy in Δ_{x_0} the inequality $y_0 \leqslant \varphi(x) \leqslant \omega(x)$. In order to conclude the proof of Theorem 1 it is sufficient to show that $\lim_{x\to+\infty} \omega(x) = y_0$. Let us suppose, this does not hold. Then from the fact that the function $\omega(x)$ is decreasing and $\omega(x) > y_0$ it follows that $\lim_{x\to+\infty} \omega(x) = \overline{y}$ and $\overline{y} > y_0$. But from assumption 3 and from the fact that $\omega(x)$ is an integral of equation (1) defined for all $x \in \langle x_0, +\infty)$ it would follow that $\lim_{x\to+\infty} \omega'(x) = \delta$ $(\delta < 0)$. Applying Lemma 2, this would imply $\lim_{x\to+\infty} \omega(x) = -\infty$, in contradiction to the inequality $y_0 \leqslant \omega(x)$ in Δ_{x_0}.

In a similar manner one may prove easily the following

THEOREM 2. *Let us suppose that*

1. *the function $f(x, y)$ is defined and continuous in the set D,*

2. $f(x, y) \geqslant 0$ *in* D_2,

3. *for every $\overline{y} \in (y_0 - a, y_0)$, the limit* $\lim f(x, y) = \gamma$ $(\gamma > 0)$ *as* $x \to +\infty$, $y \uparrow \overline{y}$ *exists,*

4. $y = y_0$ *is the upper integral of equation* (1) *starting at the point* (a, y_0).

Then the integral $y = y_0$ is uniformly asymptotically stable "from below" with respect to initial conditions given at any fixed segment $L_2 = \{(x, y)\colon x = x_0, y \in \langle y_0 - \beta, y_0\rangle\}$.

Putting together Theorems 1 and 2 we obtain Theorem 3, where assumption 4 from Theorems 1 and 2 need not be given any more.

THEOREM 3. *Let us suppose that*

1. *the function $f(x, y)$ is defined and continuous in the domain D,*

2. $f(x, y) \leqslant 0$ *in* D_1, *and* $f(x, y) \geqslant 0$ *in* D_2,

3. *there exist limits* $\lim f(x, y) = \delta$ $(\delta < 0)$ *as* $x \to +\infty$ *and* $y \downarrow \overline{y}$ *for every* $\overline{y} \in (y_0, y_0 + a)$ *and* $\lim f(x, y) = \gamma$ $(\gamma > 0)$ *as* $x \to +\infty$ *and* $y \uparrow \overline{y}$ *for every* $\overline{y} \in (y_0 - a, y_0)$.

Then $y = y_0$ is a solution of equation (1), *uniformly asymptotically stable with respect to the initial conditions given on every fixed segment* $K = L_1 + L_2$.

The proof of the above theorem is obtained from Theorems 1 and 2 directly. One needs only to remark that the integral $y = y_0$ satisfies assumption 4 in each of these theorems, but this follows from assumption 2 of Theorem 3, immediately. Strenghtening a little the assumptions of the above theorem we obtain Theorem 4 concerning asymptotic stability of the integral in a uniform manner. This will mean that

1° the integral $y = y_0$ is uniformly stable in Δ_a, and

2° for every $\varepsilon > 0$ there exists a number $A > 0$ depending only on ε, such that the inequality $x > x_0 + A$ implies $|\varphi(x) - y_0| < \varepsilon$ for all integrals $\varphi(x)$ starting at an arbitrary segment of the form

$$K = \{(x, y): x = x_0, y \in \langle y_0 - \beta, y_0 + \beta \rangle\}$$

for every $x_0 \geqslant a$ and a fixed value $\beta \in (0, a)$.

THEOREM 4. *Let the following conditions be satisfied:*

1. *the function $f(x, y)$ is defined and continuous in the set D,*

2. *$f(x, y) \leqslant 0$ in D_1 and $f(x, y) \geqslant 0$ in D_2, but $f(x, y) < 0$ inside D_1 and $f(x, y) > 0$ inside D_2,*

3. *for every $\bar{y} \in (y_0, y_0 + a)$ there exists the limit $\lim f(x, y) = \delta$ $(\delta < 0)$ as $x \to +\infty$ and $y \downarrow \bar{y}$, and for every $\bar{y} \in (y_0 - a, y_0)$ there exists the limit $\lim f(x, y) = \gamma$ $(\gamma > 0)$ as $x \to +\infty$ and $y \uparrow \bar{y}$,*

4. *the function $\varphi(x) = |f(x, \bar{y})|$ is non-decreasing in Δ_a for every value of $\bar{y} \in (y_0 - a, y_0 + a)$.*

Then $y = y_0$ is a solution of equation (1), *uniformly asymptotically stable in Δ_a.*

Proof. It follows from assumptions 1-3 that all assumptions of Theorem 3 are satisfied. Thus, in order to prove Theorem 4 it is sufficient to show that there holds condition 2° formulated above. This condition is satisfied for integrals starting at points of the segment K, as follows from the proofs of Theorems 1 and 2. Hence it remains only to prove that for the same values ε and A this condition will be satisfied also for the segment K lying on an arbitrary straight line $x = x_0$ (for $x_0 > a$). For this purpose it is sufficient to show that

I. The integral curve $y = \psi(x)$ of equation (1) corresponding to the upper integral starting at the point $P_1(x_0, y_0 + \beta)$ (for $x \geqslant x_0$), after translation from the initial point P_1 to the point $P_0(a, y_0 + \beta)$ parallely to the x-axis, will for none of the $x \geqslant a$ lie above the graph of the upper integral $y = \varphi(x)$ starting at the point P_0.

II. The integral curve $y = \psi(x)$ corresponding to the lower integral starting at the point $P_1'(x_0, y_0 - \beta)$, after a similar translation to the initial point at $P_0'(a, y_0 - \beta)$, will for none $x \geqslant a$ lie below the graph of the lower integral starting at the point P_0'.

We limit ourselves to an outline of the proof of I, part II being proved analogously. Denoting the translated curve by $y = \psi(x)$, as previously, we state that:

1′ $\varphi(a) = \psi(a)$;

2′ the derivatives $\varphi'(x)$ and $\psi'(x)$ are negative for $x > a$;

the second statement follows from the second part of assumption 2, for φ and ψ are integrals of equation (1) (the function $\varphi(x)$ may posses at a the derivative $\varphi'(a) = 0$, but this will not change the further part of the proof). Now, from 2′ it follows that the functions $\varphi(x)$ and $\psi(x)$ are invertible. Let us denote the inverse functions by $\varphi_1(y)$ and $\psi_1(y)$, respectively. Leaving the graphs of the curves $x = \varphi_1(y)$ and $x = \psi_1(y)$, let us choose a new Cartesian coordinate system $O'x'y'$ placing its origin at $O'(a, y_0+\beta)$ and choosing the y'-axis parallel to the y-axis but in a converse direction, and the x'-axis perpendicularly to the y'-axis and in the same direction as the x-axis (leaving the scales on the axes unchanged). In this coordinate system the curves $x = \varphi_1(y)$ and $x = \psi_1(y)$ become graphs of new functions which we shall denote by $\hat{\varphi}_1(y')$ and $\hat{\psi}_1(y')$, respectively. These functions have the following properties:

(a) are defined and continuous in the interval $\langle 0, \beta)$ on the y'-axis,

(b) are strictly increasing (this follows easily from 2′),

(c) satisfy the inequality $\hat{\psi}_1'(y') \leqslant \hat{\varphi}_1'(y')$ for $y' \in (0, \beta)$ (this follows from the assumption 4 and from the fact that $x' = x$, $y' = y_0+\beta-y$).

Since $\hat{\psi}_1(0) = \hat{\varphi}_1(0) = 0$, as follows from 1′, and since by (c), $[\hat{\psi}_1(y')-\hat{\varphi}_1(y')]' \leqslant 0$, we obtain $\hat{\psi}_1(y') \leqslant \hat{\varphi}_1(y')$ for $y' \in \langle 0, \beta)$. Hence it is clear that the graph of the function $\hat{\psi}_1(y')$ cannot lie above the graph of the function $\hat{\varphi}_1(y')$, for any $y' \in \langle 0, \beta)$. Thus in the coordinate system Oxy, the graph of the function $\psi_1(y)$ cannot lie above the graph of the function $\varphi_1(y)$, for any $y \in (y_0, y_0+\beta\rangle$. Hence there holds always

$$\psi_1(y) \leqslant \varphi_1(y) \quad \text{for } y \in (y_0, y_0+\beta\rangle.$$

Since the functions $\psi(x)$ and $\varphi(x)$ inverse to the above ones are strictly decreasing and continuous for $x \in \Delta_a$ and $\psi(a) = \varphi(a)$, they must satisfy the same inequality, i.e.

$$\psi(x) \leqslant \varphi(x) \quad \text{for } x \in \Delta_a,$$

and this was to be proved.

Remarks. I. The proof of Theorem 1 shows that all integrals starting from the segment L_1 are placed asymptotically with respect to the integral $y = y_0$; moreover, this holds uniformly with respect to initial conditions given on that segment. This could be obtained also applying a theorem of N. N. Krasowski given by Hahn ([1], p. 8). By this theorem, from the uniform stability of the integral $y = y_0$ and from the fact that every integral starting from L_1 is asymptotic with respect to $y = y_0$ it follows, that integrals starting at points of L_1 are asymptotic with respect to the straight line $y = y_0$ uniformly with regard to all initial points given on L_1.

II. One may prove a theorem a little more general than Theorem 4, obtained by a suitable weakening of the assumption 2 of this theorem, namely, replacing this assumption by the following one: $f(x, y) \leqslant 0$ in D_1 and $f(x, y) \geqslant 0$ in D_2.

III. Generally speaking, asymptoticity of the solutions and stability of the integral of differential equation (1) were not investigated under the kind of assumptions given in this paper. Hahn [1] gives a kind of classification of different types of problems connected with stability and asymptoticity (or the lack of them) of solutions of ordinary differential equations, and a summary of relations and differences concerning these notions.

IV. In case if $\delta = 0$ or $\gamma = 0$ or $\delta = \gamma = 0$, a stable solution $y = y_0$ may be not asymptotically stable, as follows from two simple examples 2 and 3 given below.

EXAMPLE 1. Let us consider the differential equation

$$y' = (1-x^2)(1+y)g(y) \tag{2}$$

in the set $D = \{(x, y): x \in \langle 1, +\infty), y \in (-1-a, -1+a)\}$ (for $a > 0$) under the assumption that the function $g(y)$ is defined and continuous in the interval $(-1-a, -1+a)$ and satisfies the inequality $0 < g(y)$ in this interval. It is easily verified that the function

$$f(x, y) = (1-x^2)(1+y)g(y)$$

satisfies the assumption of Theorem 4, for

1. the function $f(x, y)$ is continuous in the set D,

2. $f(x, y) \leqslant 0$ in D_1 and $f(x, y) \geqslant 0$ in D_2, and $f(x, y) < 0$ inside D_1 and $f(x, y) > 0$ inside D_2,

3. from the definition of the function $f(x, y)$ follows the existence of limits from Theorem 4, assumption 3, namely, $\lim f(x, y) = -\infty$ as $x \to +\infty$, $y \to \bar{y}$ (for $\bar{y} > -1$) and $\lim f(x, y) = +\infty$ as $x \to +\infty$, $y \to \bar{y}$ (for $\bar{y} < -1$),

4. the function $\varphi(x) = |(1-x^2)(1+\bar{y})g(\bar{y})|$ is non-decreasing in Δ_1 for every $\bar{y} \in (-1-a, -1+a)$.

Hence, applying Theorem 4 we state that the integral $y = -1$ of equation (2) is uniformly asymptotically stable in the interval $\langle 1, +\infty)$.

EXAMPLE 2. Let us consider the differential equation

$$y' = -y/x \tag{3}$$

and let us investigate the behaviour of its integrals in the set $D = \{(x, y): x \in \langle 1, +\infty), y \in (-\infty, +\infty)\}$. Equation (3) does not satisfy assumption 3 from Theorem 3, for in this case $\lim f(x, y) = 0$ as $x \to +\infty$ and

$y \to \bar{y}$ for every $\bar{y} \in (-\infty, +\infty)$, however, the integral $y = 0$ is uniformly asymptotically stable in Δ_1, but only with respect to initial conditions given on every fixed segment of the form K (from Theorem 3); this follows immediately from the form of the general integral of this equation, $y = c/x$.

EXAMPLE 3. In case of the differential equation

$$y' = -y/x^2 \tag{4}$$

considered in the same set D, the problem of stability of the solution $y = 0$ is solved otherwise. We have also $\lim f(x, y) = 0$ as $x \to +\infty$ and $y \to \bar{y}$, but the integral is stable uniformly only in the sense of Lapunov, and is not asymptotically stable, as follows immediately from the form of the general integral $y = ce^{1/x}$ of this equation.

References

[1] W. Hahn, *Über Typen des Stabilitätsverhaltens*, Monath. Math. 71 (1967), pp. 7-13.

[2] J. Szarski, *Differential inequalities*, Warszawa 1965.