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On the differential equation $y''-a(x)y=0$

The purpose of this paper is to prove the following

THEOREM. *Suppose that*

(1) $a(x)\neq 0$, $|\operatorname{Arg} a(x)|\leqslant \pi-\delta$ *with some* $\delta\in(0,\pi)$ $(x\geqslant x_0)$,

(2) *there exists a continuous* $a'''(x)$ *for* $x\geqslant x_0$,

(3) $\left(a^{-1/4}(x)\right)''=O(x^{-3/2}\ln^{-1/2-\varepsilon}x)$, $\left(a^{-1/4}(x)\right)'=o(1)$ *as* $x\to\infty$,

with some $\varepsilon>0$.

Then the differential equation

$$y''-a(x)y=0 \tag{4}$$

has for $x\geqslant x_0$ *the solutions* $y_1(x)$ *and* $y_2(x)$ *such that*

$$y_1(x)\sim a^{-1/4}(x)\exp\int_{x_0}^{x}a^{1/2}(t)\,dt, \qquad |y_1(x)|\to\infty,$$

$$y_2(x)\sim a^{-1/4}(x)\exp\Big(-\int_{x_0}^{x}a^{1/2}(t)\,dt\Big)\to 0,$$

$$y_1'(x)\sim a^{1/2}(x)y_1(x), \qquad |y_1'(x)|\to\infty, \qquad y_2'(x)\sim -a^{1/2}(x)y_2(x)\to 0$$

as $x\to\infty$. (*We assume here* $\operatorname{Arg} a^{1/4}(x)=\{\operatorname{Arg} a^{1/2}(x)\}/2=\{\operatorname{Arg} a(x)\}/4$.)

In the proof of Theorem we shall use Theorem 1', 2 and 3 of [2], pp. 175, 177 and 180 in the modified form (see Theorems 1'', 2' and 3' below).

THEOREM 1''. *Let* $f(t)$, $F(t)$ *and* $\psi(t)$ *be almost uniformly bounded for* $t\geqslant x_0$ *and let* $N(x,t)$ *be defined for* $x\geqslant t\geqslant x_0$. *Suppose that*

(5) *there exist integrals* $\int_t^x f(s)\,ds$, $\int_t^x \psi(s)\,ds$, $\int_t^x N(x,s)\,ds$, $\int_t^x N(s,t)\,ds$

and that

$$\int_t^x |N(x,s)|\,|N(s,t)|^a\,ds \leqslant \lambda |N(x,t)|^a$$

with some $a \in \langle 0,1\rangle$ *and some* $\lambda < 1$, *for* $x \geqslant t \geqslant x_0$,

$$|N(x,t)| \leqslant F(t) \qquad (x \geqslant t \geqslant x_0), \tag{6}$$

$$\limsup_{x\to\infty} \int_{x_0}^x F^{1-a}(t)\,|N(x,t)|^a\,dt < \infty,$$

$$\limsup_{x\to\infty} \int_{x_0}^x |N(x,t)|\,dt + \int_{x_0}^\infty |\psi(t)|\,dt = \mu < 1, \tag{7}$$

$$\lim_{x\to\infty} f(x) = s \ (|s| < \infty), \quad \lim_{x\to\infty} \int_{x_0}^x N(x,t)\,dt = \lim_{x\to\infty} \int_{x_0}^{t_0} |N(x,t)|\,dt = 0 \tag{8}$$

for fixed $t_0 \geqslant x_0$.

We set $K_0(x,t) = N(x,t) + \psi(t)$ *for* $x \geqslant t \geqslant x_0$. *Then for every* $\varepsilon > 0$ *there exists an* $x^* \geqslant x_0$ *such that the unique solution* $\bar{y}(x)$ (*bounded for* $x^* \leqslant x < \infty$) *of the integral equation*

$$y(x) = f(x) + \int_{x^*}^x K_0(x,t)\,y(t)\,dt \qquad (x \geqslant x^*) \tag{9}$$

satisfies the relation $\lim_{x\to\infty} \bar{y}(x) = s_1$, *where* $|s - s_1| \leqslant \varepsilon$. *If* $f(x)$ *is continuous and if* $N(x,t)$ *is a continuous function of the variable* x, *uniformly for* $t \in \langle x^*, x\rangle$, *then* $\bar{y}(x)$ *is continuous for* $x \geqslant x^*$.

Proof. Under our hypotheses we have for $x \geqslant x_1$ (with some $x_1 \geqslant x_0$):

$$|f(x)| \leqslant C_1 = 1 + |s|, \quad \int_{x_1}^x F^{1-a}(t)\,|N(x,t)|^a\,dt \leqslant C_2,$$

$$\int_{x_1}^x \psi_1(x,t)\,dt = \int_{x_1}^x |\psi(s)| \int_{x_1}^s F^{1-a}(t)\,|N(s,t)|^a\,dt\,ds \leqslant C_2\mu,$$

$$\int_{x_1}^x |K_0(x,t)|\,dt \leqslant a,$$

where $\psi_1(x,t) = F^{1-a}(t) \int_t^x |N(s,t)|^a\,|\psi(s)|\,ds$ and a satisfies the inequality $\max(\lambda,\mu) < a < 1$.

We choose an arbitrary $x_2 \geqslant x_1$. The above inequalities remain true for $x \geqslant x_2$ if we replace x_1 by x_2 (since $F(t) \geqslant 0$).

As in the proof of Theorem 1′ ([2], p. 176) we set

$$K_n(x,t) = \int_t^x K_0(x,s)K_{n-1}(s,t)\,ds$$

for $x \geqslant t \geqslant x_2$, $n = 1, 2, \ldots$, and

$$R(x,t) = \sum_{n=0}^{\infty} K_n(x,t) \qquad (x \geqslant t \geqslant x_2);$$

we obtain, as in Theorem 1′:

$$\int_{x_2}^x |R(x,t)|\,dt \leqslant C_2/(1-a)+C_2\mu/(1-a)^2+\mu/(1-a) = C_3 \qquad (x \geqslant x_2).$$

We denote by $y_0(x)$ the unique solution (bounded for finite $x \geqslant x_2$) of the integral equation (9) with x_2 instead of x^*. We have

$$y_0(x) = f(x)+\int_{x_2}^x R(x,t)f(t)\,dt$$

and

$$|y_0(x)| \leqslant C_1+C_1C_3 = C \qquad (x \geqslant x_2).$$

Let us observe that C is independent of x_2 (if $x_2 \geqslant x_1$). For a given $\varepsilon > 0$ we choose a fixed $x^* \geqslant x_1$ such that

$$\int_{x^*}^{\infty} |\psi(x)|\,dx \leqslant \varepsilon/C.$$

Let $\bar{y}(x)$ be the bounded solution of the integral equation (9) with a chosen x^*. Obviously $|\bar{y}(x)| \leqslant C$ for $x \geqslant x^*$. As in the proof of Theorem 1′ ([2], p. 176) we find that $\lim_{x\to\infty} \bar{y}(x) = s_1$, where

$$|s_1 - s| \leqslant \int_{x^*}^{\infty} |\psi(x)\bar{y}(x)|\,dx \leqslant \varepsilon.$$

It is easy to prove that if $f(x)$ is continuous and if $N(x,t)$ is a continuous function of the variable x, uniformly for $t \in \langle x^*, x\rangle$, then $\bar{y}(x)$ is continuous for $x \geqslant x^*$.

THEOREM 2′. *Suppose that the hypotheses of Theorem* 2 ([2], p. 177) *are satisfied. Then, under the hypotheses* $\lim_{x\to\infty} |g(x)| = \infty$ *and* $\lim_{x\to\infty} f(x) = s$, *for every* $\varepsilon > 0$ *there exists* $x^* \geqslant x_0$ *such that the unique solution* $\bar{y}_1(x)$ *(bounded for finite* x, $x \geqslant x^*$*) of the integral equation* (9) *with*

$$K_0(x,t) = g'(t)\varphi(t)/g(x)+\psi(t),$$

satisfies the relation $\lim_{x\to\infty} \bar{y}_1(x) = s_1$, *where* $|s_1 - s| \leqslant \varepsilon$. *If* $f(x)$ *is continuous, then* $\bar{y}_1(x)$ *is also continuous for* $x \geqslant x^*$.

Proof. We prove, as in the proof of Theorem 2 ([2], p. 177) that the hypotheses of Theorem 1'' are satisfied, and we apply Theorem 1''.

THEOREM 3'. (Compare Theorem 3, [2], p. 180). *Suppose that*

(10) *there exist continuous* $A'''(x), a_2''(x), a_1''(x), a_0'(x)$ *and* $b(x)$ *for* $x \geqslant x_0$,

(11) $A \neq 0, a_2 \neq 0$ *and* $|A| \leqslant N|\mathrm{re}\, A|$ *with some constant* N $(x \geqslant x_0)$,

$$\int_{x_0}^{\infty} dx/|A(x)| = \infty, \tag{12}$$

$$\lim_{x\to\infty} \varphi(x) = 0, \tag{13}$$

$$\int_{x_0}^{\infty} \frac{A(x)b(x)}{a_2(x)}\,dx = s \quad (|s| < \infty), \tag{14}$$

$$\int_{x_0}^{\infty} |\psi(x)|\,dx < \infty, \tag{15}$$

where $\psi = A'' - (a_1 A/a_2)' + a_0 A/a_2$, $\varphi = A' + A(\psi - a_1/a_2) + 1$.

Then there exists an integral $\bar{y}(x)$ *of the differential equation*

$$a_2 y'' + a_1 y' + a_0 y = b \quad (x \geqslant x_0) \tag{16}$$

such that $\lim_{x\to\infty} \bar{y}(x) = 1$.

Proof. We set

$$f(x) = 1 - s + \int_{x_0}^{x} g'(t)B(t)\,dt/g(x),$$

where

$$g(x) = \exp \int_{x_0}^{x} dt/A(t), \qquad B(t) = \int_{x_0}^{t} \frac{A(u)b(u)}{a_2(u)}\,du.$$

It is easy to see that the functions $g(x)$ and $h(x) = 1/A(x)$ satisfy the hypotheses of 1.2 ([2], p. 170). Applying this Lemma and the l'Hospital's rule in the formulation of Theorem C ([1], p. 20), we obtain $\lim_{x\to\infty} f(x) = 1$ by (14).

Under the hypothesis $\mathrm{re}\, A < 0$ for $x \geqslant x_0$ it follows $\lim_{x\to\infty} g(x) = 0$

by (11) and (12). By Theorem 2 ([2], p. 177) we obtain that the integral equation

$$y(x) = f(x) + \int_x^\infty K_0(x, t)y(t)\,dt$$

with $f(x)$ defined as above and $K_0(x, t) = g'(t)\varphi(t)/g(x) + \psi(t)$ has for large $x \geqslant x_0$ a solution $\bar{y}(x)$ such that $\lim_{x\to\infty} \bar{y}(x) = 1$. By 2.1 ([2], p. 179) we obtain that $\bar{y}(x)$ satisfies differential equation (16) for large x. The function $\bar{y}(x)$ may be extended to the value x_0.

Under the hypothesis $\operatorname{re} A > 0$ we obtain $\lim_{x\to\infty} |g(x)| = \infty$. For a given $\varepsilon \in (0, 1/3)$ we find, applying Theorem 2′, that the integral equation (9) with $f(x)$ defined as above, with $K_0(x, t) = g'(t)\varphi(t)/g(x) + \psi(t)$ and sufficiently large $x^* = x_1^* \geqslant x_0$, has for $x \geqslant x_1^*$ a solution $y_1(x)$ such that $\lim_{x\to\infty} y_1(x) = s_1$, where $|s_1 - 1| \leqslant \varepsilon$. Similarly we obtain that the integral equation (9) with

$$f(x) = \int_{x_0}^x g'(t)B(t)\,dt/g(x) - s, \qquad K_0(x, t) = g'(t)\varphi(t)/g(x) + \psi(t)$$

and sufficiently large $x^* = x_2^* \geqslant x_0$, has for $x \geqslant x_2^*$ a solution $y_2(x)$ such that $\lim_{x\to\infty} y_2(x) = s_2$, where $|s_2| \leqslant \varepsilon$.

By 2.1 ([2], p. 179) we find that $y_1(x)$ and $y_2(x)$ satisfy differential equation (16) for sufficiently large x. The function $\bar{y}(x) = y_1(x) + (1 - s_1)\{y_2(x) - y_1(x)\}/(s_2 - s_1)$ satisfies also (16) for sufficiently large x and we have $\lim_{x\to\infty} \bar{y}(x) = 1$. The function $\bar{y}(x)$ may be extended to the value x_0. Let us notice that if $b(x) = 0$, then $f(x) = 0$ and $y_2(x) = 0$.

Proof of the theorem. We substitute

$$y(x) = w(x)a^{-1/4}(x)\exp\int_{x_0}^x a^{1/2}(t)\,dt$$

into (4) and we obtain the differential equation

$$(17) \qquad a^{-1/2}w''/2 + \{1 + (a^{-1/2})'/2\}w' + a^{-1/4}(a^{-1/4})''w/2 = 0 \qquad (x \geqslant x_0).$$

The functions $\psi(x)$ and $\varphi(x)$ (see Theorem 3′) in the case of differential equation (17) and for $A(x) = a_2(x) = a^{-1/2}(x)/2$ are of the form

$$\psi(x) = a^{-1/4}(a^{-1/4})''/2, \qquad \varphi(x) = a^{-3/4}(a^{-1/4})''/4.$$

Let us observe that using l'Hospital's rule we obtain from hypothesis (3)

$$(18) \qquad (a^{-1/4})' = O(x^{-1/2}\ln^{-1/2-\varepsilon}x) \qquad \text{and} \qquad a^{-1/4} = O(x^{1/2}\ln^{-1/2-\varepsilon}x)$$

for $x \to \infty$. It follows that $\psi(x)$ and $\varphi(x)$ satisfy the hypotheses of Theorem 3'.

By Theorem 3' for $A(x) = a_2(x) = a^{-1/2}(x)/2$ we obtain under our hypotheses that the differential equation (17) has for $x \geqslant x_0$ a solution $w_1(x)$ such that $\lim_{x\to\infty} w_1(x) = 1$. Then the function $y_1(x) = w_1(x)a^{-1/4}(x) \times$ $\times \exp\int_{x_0}^{x} a^{1/2}(t)\,dt$ satisfies differential equation (4) for $x \geqslant x_0$ and we have

$$y_1(x) \sim a^{-1/4}(x)\exp\int_{x_0}^{x} a^{1/2}(t)\,dt \quad \text{for } x \to \infty.$$

Let us observe that in virtue of (18) the functions $g(x) = a^{-1/4}(x) \times$ $\times \exp\int_{x_0}^{x} a^{1/2}(t)\,dt$ and $h(x) = a^{1/2}(x)$ satisfy the conditions of Lemma 1.2 ([2], p. 170). We have namely

$$g'/g = -a'/4a + a^{1/2} = a^{1/2}\{a^{-1/4}(a^{-1/4})' + 1\} \sim a^{1/2}$$

for $x \to \infty$. Moreover, $\lim_{x\to\infty} |g(x)| = \infty$ since $\operatorname{re} a^{1/2}(x) > 0$ (see the proof of 1.2, [2]).

It follows that $\lim_{x\to\infty} |y_1(x)| = \infty$. Next,

$$\lim_{x\to\infty} \frac{d}{dx}\left\{a^{1/4}(x)\exp\int_{x_0}^{x} a^{1/2}(t)\,dt\right\} \Big/ a^{3/4}(x)\exp\int_{x_0}^{x} a^{1/2}(t)\,dt \tag{19}$$

$$= \lim_{x\to\infty} (a^{1/4})' a^{-3/4} + 1 = 1$$

since

$$(a^{1/4})' a^{-3/4} = -(a^{-1/4})' a^{-1/4} \to 0 \quad \text{for } x \to \infty.$$

We prove, as above, that the function $g(x) = a^{1/4}(x)\exp\int_{x_0}^{x} a^{1/2}(t)\,dt$ satisfies the conditions of 1.2, [2] with $h(x) = a^{1/2}(x)$, and that $\lim_{x\to\infty} |g(x)|$ $= \infty$. We obtain, using 1.2, [2] for this $g(x)$ and applying the l'Hospital's rule in the formulation of Theorem C ([1], p. 20):

$$\lim_{x\to\infty} y_1' \Big/ a^{1/4}\exp\int_{x_0}^{x} a^{1/2}\,dt = \lim_{x\to\infty} y_1'' \Big/ \frac{d}{dx}\left\{a^{1/4}\exp\int_{x_0}^{x} a^{1/2}\,dt\right\}$$

$$= \lim_{x\to\infty} y_1'' \Big/ a^{3/4}\exp\int_{x_0}^{x} a^{1/2}\,dt = 1$$

by (19) and (4). It follows

$$y_1'(x) \sim a^{1/4}(x)\exp\int_{x_0}^{x} a^{1/2}(t)\,dt \sim a^{1/2}(x)y_1(x)$$

for $x \to \infty$, and $\lim_{x\to\infty} |y_1'(x)| = \infty$.

It is easy to see that there exists $x_1 \geqslant x_0$ such that $y_1(x) \neq 0$ for $x \geqslant x_1$. We shall prove the existence of the integral $\int_x^{\infty} y_1^{-2}(t)\,dt$ for $x \geqslant x_1$. We have namely,

$$(y_1^{-2})'/y_1^{-2} = -2y_1'/y_1 \sim -2a^{1/2} \quad \text{for } x \to \infty$$

(see the proof of 1.2, [2], p. 171, case b), for $g(x) = y_1^{-2}(x)$).

The function $y_2(x) = 2y_1(x)\int_x^{\infty} y_1^{-2}(t)\,dt$ is for $x \geqslant x_1$ the integral of differential equation (4), for which $y_1y_2' - y_1'y_2 = -2$. Then

$$\begin{aligned} y_2(x)/a^{-1/4}(x)\exp\Big(-\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) &= 2y_1(x)\int_x^{\infty} y_1^{-2}(t)\,dt/a^{-1/4}(x)\exp\Big(-\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) \\ &\sim 2\Big\{\int_x^{\infty} dt/y_1^2(t)\Big\}/\exp\Big(-2\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) \quad \text{for } x \to \infty. \end{aligned}$$

We get, using 1.2 ([2], p. 170) for $g(x) = \exp\big(-2\int_{x_0}^{x} a^{1/2}(t)\,dt\big)$,' $h(x) = -2a^{1/2}(x)$ and applying the l'Hospital's rule in the formulation of Theorem C ([1], p. 20):

$$\begin{aligned} \lim_{x\to\infty}\Big\{\int_x^{\infty} dt/y_1^2(t)\Big\}/\exp\Big(-2\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) &= \lim_{n\to\infty} 1/2y_1^2(x)a^{1/2}(x)\exp\Big(-2\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) = 1/2 \end{aligned}$$

and

$$y_2(x) \sim a^{-1/4}(x)\exp\Big(-\int_{x_0}^{x} a^{1/2}(t)\,dt\Big) \quad \text{for } x \to \infty.$$

We have $\lim_{x\to\infty} y_2(x) = 0$ since $y_2(x) \sim 1/y_1'(x)$. Moreover,

$$y_1y_2' = y_1'y_2 - 2 \to -1 \quad \text{and} \quad y_2'(x) \sim -1/y_1(x) \sim -a^{1/2}(x)y_2(x) \to 0$$

for $x \to \infty$. The function $y_2(x)$ may be extended to the value x_0.

References

[1] Z. Polniakowski, *Polynomial Hausdorff transformations, I*, Ann. Polon. Math. 5 (1958), pp. 1-24.

[2] — *Asymptotic properties of solutions of some integral equations and second order differential equations*, ibidem 16 (1965), pp. 169-183.

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