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Countably modulared spaces connected with equisplittable families of measures

1. In paper [1], the notions of a countably modulared space X_ϱ, and a uniformly countably modulared space X_{ϱ_0}, are introduced as follows. Let ϱ_i, $i = 1, 2, \ldots$, be a sequence of pseudomodulars in a real linear space X, i.e. $0 \leqslant \varrho_i(x) \leqslant \infty$ for $x \in X$, $\varrho_i(0) = 0$, $\varrho_i(-x) = \varrho_i(x)$, $\varrho_i(\alpha x + \beta y) \leqslant \varrho_i(x) + \varrho_i(y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, and let $\varrho_i(x) = 0$ for all i imply $x = 0$. Two modulars ϱ and ϱ_0 are defined (see [1]) by formulae $\varrho(x) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x)\big(1 + \varrho_i(x)\big)^{-1}$, $\varrho_0(x) = \sup_i \varrho_i(x)$. Spaces X_ϱ and X_{ϱ_0} are just the modular spaces defined by means of the modulars ϱ and ϱ_0. In [1] the problem under which assumptions there holds $X_\varrho = X_{\varrho_0}$ is solved in case of pseudomodulars $\varrho_i(x) = \int_E \varphi_i\big(|x(t)|\big)\, d\mu$, where μ is an atomless finite measure on a σ-algebra $\mathscr{E}$ of subsets of an abstract set E, X is the space of all μ-measurable functions $x(t)$ defined on E, and $\varphi_i(u)$ is a sequence of s-convex φ-functions.

Here, we shall consider the more general case of pseudomodulars

$$\varrho_i(x) = \sup_\tau \int_E \varphi_i\big(|x(t)|\big)\, d\mu_\tau, \tag{1}$$

where $\{\mu_\tau\}$ is a family of non-negative measures on $\mathscr{E}$, and $\tau \in \mathscr{T}$, where $\mathscr{T}$ is an abstract set. However, the considerations require some additional assumptions on the family $\{\mu_\tau\}$ which will be introduced in the next Section.

2.1. Let $\{\mu_\tau\}$ be a family of non-negative measures on a σ-algebra $\mathscr{E}$ of subsets of a set E, where $\tau \in \mathscr{T}$ and $\mathscr{T}$ is an abstract set. We say that the family $\{\mu_\tau\}$ is

a. *uniformly bounded*, if there exists a constant $K \geqslant 0$ such that $\mu_\tau E \leqslant K$ for all $\tau \in \mathscr{T}$,

b. *equisplittable* (see also [2] and [5]), if there exists $\eta > 0$ such that for any sequence of numbers $\varepsilon_k \downarrow 0$ for which $\varepsilon_k \leqslant \eta$ and $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$

for all k, there exist constants $M \geqslant \delta > 0$ and a sequen ceof pairwise disjoint sets $A_k \in \mathscr{E}$ satisfying the condition

$$\delta\varepsilon_k \leqslant \sup_{\tau\in\mathscr{T}} \mu_\tau A_k \leqslant M\varepsilon_k.$$

2.2. Let $\varphi_i(u)$ be φ-functions. In the sequel we shall assume also the following conditions, to be satisfied (see [1], [3]):

(α) $\varphi_i(u)$ are equicontinuous at $u = 0$;

(β) for every index i there exist positive constants $\lambda_i, \beta_i, \vartheta_i$ such that for every $u \geqslant \vartheta_i$ and $k \geqslant i$ there holds the inequality $\varphi_i(\lambda_i u) \leqslant \beta_i \varphi_k(u)$;

(γ) there exist positive constants k, c, u_0 and an index i_0 such that $\varphi_i(cu) \leqslant k\varphi_{i_0}(u)$ for $u \geqslant u_0$ and $i \geqslant i_0$;

(δ) for every $\varepsilon > 0$ there exist numbers $u_\varepsilon > 0$ and $a_\varepsilon > 0$ depending on i, such that $\varphi_i(au) < \varepsilon\varphi_i(u)$ for $0 \leqslant a \leqslant a_\varepsilon$, $u \geqslant u_\varepsilon$.

This condition is certainly satisfied if $\varphi_i(u) = \Phi_i(|u|^s)$, where $0 < s \leqslant 1$ and Φ_i are convex φ-functions.

3. The following conditions for the equality $X_\varrho = X_{\varrho_0}$ will be proved now in case of pseudomodulars (1).

THEOREM 1. *If the family of measures $\{\mu_\tau\}$ is uniformly bounded and if the φ-functions φ_i satisfy* (α) *and* (γ), *then* $X_\varrho = X_{\varrho_0}$.

Proof. From (γ) it follows that there are $k > 0$ and an index i_0 such that for every u_0 there is a number $c_0 > 0$ satisfying the inequality $\varphi_i(u) \leqslant k\varphi_{i_0}(c_0 u)$ for $u \geqslant u_0$ and $i \geqslant i_0$. Let $x \in X_\varrho$, $\lambda > 0$, and let $E_1 = \{t: u_0 \leqslant |\lambda x(t)|\}$, $E_1' = E \setminus E_1$. Then

$$\begin{aligned}\varrho_i(\lambda x) &\leqslant k \sup_\tau \int_{E_1} \varphi_{i_0}\big(c_0\lambda|x(t)|\big)\,d\mu_\tau + \varphi_i(u_0) \sup_\tau \mu_\tau E \\ &\leqslant k\varrho_{i_0}(c_0\lambda x) + K\varphi_i(u_0)\end{aligned}$$

for $i \geqslant i_0$. The conclusion $X_\varrho = X_{\varrho_0}$ follows as in [1], 2.1.

THEOREM 2. *Let the family of measures $\{\mu_\tau\}$ be uniformly bounded and equisplittable and let the φ-functions φ_i satisfy* (β) *and* (δ). *If $X_\varrho = X_{\varrho_0}$, then φ_i satisfy* (γ).

Proof. Let us suppose the theorem does not hold, i.e. $X_\varrho = X_{\varrho_0}$ and for every $k, c, u_0 > 0$ and every i_0 there are a number $u \geqslant u_0$ and an index $i \geqslant i_0$ such that $\varphi_i(cu) \geqslant k\varphi_{i_0}(u)$. Given $k > 0$ and $c = 2^{-k}$, there exist $i_{n,m,k} \geqslant n$ and $u_{n,m,k} \geqslant m$ such that

$$\varphi_{i_{n,m,k}}(2^{-k}u_{n,m,k}) > 2^k\varphi_n(u_{n,m,k}) \tag{2}$$

for $n, m, k = 1, 2, \ldots$ We define an increasing sequence of indices m_k in the following manner. First, $m_1 \geqslant \vartheta_1$ and $\varphi_1(m_1) \geqslant \min(1, 1/\eta)$, where $\eta > 0$ is the number appearing in the definition of equisplittability. We

write $u_1 = u_{1,m_1,1}$. Let us suppose the numbers $m_1, m_2, \ldots, m_{k-1}$ are defined and satisfy the following inequalities: $\varphi_i(m_i) \geqslant 1$, $m_i \geqslant \vartheta_i$ and $\varphi_i(m_i) \geqslant \varphi_{i-1}(u_{i-1})$, where $u_i = u_{i,m_i,i}$, $i = 2, 3, \ldots, k-1$. Then we take m_k so large that $\varphi_k(m_k) \geqslant 1$, $m_k \geqslant \vartheta_k$ and $\varphi_k(m_k) \geqslant \varphi_{k-1}(u_{k-1})$ and we put $u_k = u_{k,m_k,k}$. Hence $\varphi_k(u_k) \geqslant \varphi_k(m_k) \geqslant \varphi_{k-1}(u_{k-1})$ for $k = 2, 3, \ldots$

We write

$$\varepsilon_k = \frac{1}{2^k \varphi_k(u_k)},$$

then $\varepsilon_k \downarrow 0$ and $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$. Hence there exist constants $M \geqslant \delta > 0$ and a sequence of pairwise disjoint sets $A_k \in \mathscr{E}$ such that 2.1.b holds. We define

$$x(t) = \begin{cases} u_k & \text{for } t \in A_k,\ k = 1, 2, \ldots, \\ 0 & \text{for } t \in E \setminus \bigcup_{i=1}^{\infty} A_i. \end{cases}$$

Obviously, $x \in X$. Taking λ_i and β_i from (β) we have

$$\begin{aligned} \varrho_i(\lambda_i x) &= \sup_\tau \sum_{k=1}^{\infty} \varphi_i(\lambda_i u_k)\, \mu_\tau A_k \\ &\leqslant \sup_\tau \sum_{k=1}^{i-1} \varphi_i(\lambda_i u_k)\, \mu_\tau A_k + \sup_\tau \sum_{k=i}^{\infty} \varphi_i(\lambda_i u_k)\, \mu_\tau A_k \\ &\leqslant K \sum_{k=1}^{i-1} \varphi_i(\lambda_i u_k) + \beta_i \sum_{k=i}^{\infty} \varphi_k(u_k)\, \mu_\tau A_k. \end{aligned}$$

But by 2.1.b and the definition of ε_k,

$$\sum_{k=i}^{\infty} \varphi_k(u_k)\, \mu_\tau A_k \leqslant M \sum_{k=i}^{\infty} \frac{1}{2^k} \leqslant M.$$

Hence $\varrho_i(\lambda_i x) < \infty$. Let $c > 0$ be arbitrary. Let us take $\varepsilon = c/2\varrho_i(\lambda_i x)$. We take $\lambda_c > 0$ so small that $\lambda_c < a_\varepsilon \lambda_i$ and $\varphi_i(\lambda_c u_\varepsilon/\lambda_i) < \eta/2K$. Then defining $E_1 = \{t\colon u_\varepsilon \leqslant \lambda_i x(t)\}$, $E_1' = E \setminus E_1$, we have for $0 < \lambda < \lambda_c$:

$$\begin{aligned} \varrho_i(\lambda x) &\leqslant \varepsilon \sup_\tau \int_{E_1} \varphi_i\big(\lambda_j x(t)\big)\, d\mu_\tau + \varphi_i\left(\frac{\lambda}{\lambda_i} u_\varepsilon\right) \sup_\tau \mu_\tau E \\ &\leqslant \varepsilon \varrho_i(\lambda_i x) + K \varphi_i\left(\frac{\lambda}{\lambda_i} u_\varepsilon\right) < c. \end{aligned}$$

Thus $x \in X_\varrho$. Now, by (2) we have

$$\begin{aligned} \varrho_{i_{k,m_k,k}}(2^{-k} x) &\geqslant \sup_\tau \int_{A_k} \varphi_{i_{k,m_k,k}}\big(2^{-k} x(t)\big)\, d\mu_\tau \\ &\geqslant \varphi_{i_{k,m_k,k}}(2^{-k} u_k) \sup_\tau \mu_\tau A_k \geqslant 2^k \varphi_k(u_k)\, \mu_{\tau_k} A_k \geqslant \tfrac{1}{2}\delta, \end{aligned}$$

where $\tau_k \in \mathscr{T}$ are chosen in such a manner that $\mu_{\tau_k} A_k > \frac{1}{2}\delta\varepsilon_k$ for $k = 1, 2, \ldots$, a contradiction.

Theorems 1 and 2 imply the following

THEOREM 3. *Let the family of measures $\{\mu_\tau\}$ be uniformly bounded and equisplittable and let the φ-functions φ_i satisfy conditions* (α), (β), (δ). *Then, $X_\varrho = X_{\varrho_0}$ if and only if φ_i satisfy* (γ).

Let us remark that Theorem 2.1 in [1] is a special case of our Theorem 3, in case if the family $\{\mu_\tau\}$ consists of one measure μ which is finite and atomless, whence equisplittable.

4. Now, we shall consider the special case of measures μ_τ defined on the σ-algebra of all Lebesgue measurable subsets of the interval $\langle t_0, \infty)$, which are absolutely continuous with respect to the Lebesgue measure. By the Radon-Nikodym theorem, there exists a function $a(t, \tau)$ defined for $t \geqslant t_0$, $\tau \in \mathscr{T}$, Lebesgue measurable in the variable t for every $\tau \in \mathscr{T}$, $a(t, \tau) \geqslant 0$ for all $t \geqslant t_0$, $\tau \in \mathscr{T}$, satisfying the condition

$$\mu_\tau A = \int_A a(t, \tau)\, dt. \tag{3}$$

Obviously, uniform boundedness of the family $\{\mu_\tau\}$ is equivalent to the condition

$$\sup_\tau \int_{t_0}^{\infty} a(t, \tau)\, dt < \infty.$$

The respective pseudomodulars ϱ_i defined by (1) were investigated in [6].

4.1. PROPOSITION 1. *Let us suppose, there exist $\eta > 0$ and constants $M \geqslant \delta > 0$ such that for every $0 < \varepsilon \leqslant \eta$, and $T \geqslant t_0$ there exist $\beta > \alpha \geqslant T$ satisfying the following conditions*:

$$\delta\varepsilon \leqslant \sup_\tau \int_\alpha^\beta a(t, \tau)\, dt \leqslant M\varepsilon. \tag{4}$$

Then the family of measures $\{\mu_\tau\}$ defined in (3) *is equisplittable.*

Proof. The sequence of sets A_k will be defined by induction. Given a sequence $\varepsilon_k \downarrow 0$, $\varepsilon_k \leqslant \eta$, $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$, $k = 1, 2, \ldots$, we define the set A_1 as the interval (α_1, β_1), where $\varepsilon = \varepsilon_1$, $T = t_0$, $\alpha_1 = \alpha$, $\beta_1 = \beta$. Obviously, condition 2.1.b is satisfied for $k = 1$. Let us suppose the sets $A_k = (\alpha_k, \beta_k)$ are defined for $k = 1, 2, \ldots, m-1$, $\beta_k \leqslant \alpha_{k+1}$, and condition 2.1.b is satisfied for $k < m$. Now, we take in the assumptions of Proposition 1 $\varepsilon = \varepsilon_m$, $T = \beta_{m-1}$, $\alpha_m = \alpha$, $\beta_m = \beta$ and we put $A_m = (\alpha_m, \beta_m)$. It is obvious that the sequence of sets $A_1, A_2, \ldots$ satisfies the required conditions.

4.2. *The following kernels* $a(t, \tau)$, *where* $t \geqslant 0$, τ *real,* $\tau \geqslant \tau^*$ ($\tau^* > 0$ *being fixed*) *define by means of* (3) *uniformly bounded and equisplittable families of measures* $\{\mu_\tau\}$:

(i) $$a(t, \tau) = \begin{cases} \dfrac{1}{\tau} & \text{for } 0 \leqslant t \leqslant \tau, \\ 0 & \text{for } t > \tau \end{cases}$$

(*M-means*).

(ii) $$a(t, \tau) = \begin{cases} \dfrac{k}{\tau}\left(1 - \dfrac{t}{\tau}\right)^{k-1} & \text{for } 0 \leqslant t \leqslant \tau, \\ 0 & \text{for } t > \tau \end{cases}$$

(*Riesz* (*Cesàro*) *means of order* k, $k > 1$).

(iii) $$a(t, \tau) = \frac{\varrho}{\tau}\left(1 + \frac{t}{\tau}\right)^{-\varrho-1}$$

(*Stieltjes means of order* ϱ, $\varrho > 0$).

Proof. It is sufficient to verify the assumptions of Proposition 1. Since (i) is a special case of (ii), we start with the proof of (ii). We have

(5) $$\int_0^\infty a(t, \tau)\, dt = 1 \quad \text{for all } \tau.$$

We take in Proposition 1, $M = \delta = \frac{1}{2}$ and we choose $\varepsilon > 0$ and $T > 0$ arbitrarily. Taking $\alpha \geqslant T$ we have

$$\int_\alpha^\beta a(t, \tau)\, dt = \begin{cases} 0 & \text{for } \tau \geqslant \beta, \\ \left(1 - \dfrac{\alpha}{\tau}\right)^k & \text{for } \alpha < \tau < \beta, \\ \left(1 - \dfrac{\alpha}{\tau}\right)^k - \left(1 - \dfrac{\beta}{\tau}\right)^k & \text{for } \tau \leqslant \alpha. \end{cases}$$

This integral as a function of variable τ assumes its maximum at $\tau = \beta$, and

$$\sup_\tau \int_\alpha^\beta a(t, \tau)\, dt = \left(1 - \frac{\alpha}{\beta}\right)^k.$$

Condition (4) will be satisfied if we choose α and β in such a manner that $(1 - \alpha/\beta)^k = \frac{1}{2}\varepsilon$, i.e. $\alpha/\beta = 1 - (\frac{1}{2}\varepsilon)^{1/k}$; this will be satisfied if we take for example $\alpha = T$, $\beta = T\left(1 - (\frac{1}{2}\varepsilon)^{1/k}\right)^{-1}$.

In case (iii), we have (5) and

$$\int_\alpha^\beta a(t, \tau)\, dt = \left(1 + \frac{\alpha}{\tau}\right)^{-\varrho} - \left(1 + \frac{\beta}{\tau}\right)^{-\varrho}.$$

Moreover,

$$\sup_{\tau}\int_{\alpha}^{\beta} a(t,\tau)\,dt = \left(1+\Theta\frac{1-\Theta^{1/(1+\varrho)}}{\Theta^{1/(1+\varrho)}-\Theta}\right)^{-\varrho} - \left(1+\frac{1-\Theta^{1/(1+\varrho)}}{\Theta^{1/(1+\varrho)}-\Theta}\right)^{-\varrho}, \tag{6}$$

where $\Theta = \alpha/\beta$, and this supremum is assumed at

$$\tau_0 = \beta\left(\left(\frac{\alpha}{\beta}\right)^{1/(1+\varrho)} - \frac{\alpha}{\beta}\right)\left(1-\left(\frac{\alpha}{\beta}\right)^{1/(1+\varrho)}\right)^{-1}$$

(see [6], 4.5). Given $\varepsilon > 0$, we choose $\Theta = \Theta_0$ in such a manner that $\int_{\alpha}^{\beta} a(t,\tau_0)\,dt = \varepsilon$; this is possible for small ε, because the right-hand side of formula (6) is a continuous function of Θ, $0 < \Theta < 1$, and tends to 0 as $\Theta \to 0_+$. Hence we may take $M = \delta = 1$, $\alpha = T$, $\beta = \alpha/\Theta_0$.

5. Let us now consider the case of measures μ_n defined on the σ-algebra $\mathscr{E}$ of all subsets of the set E of positive integers, where $n \in \mathscr{T}$ and $\mathscr{T}$ is also the set of positive integers. Then there exists a non-negative matrix $(a_{n\nu})$, $n, \nu = 1, 2, \ldots$, such that

$$\mu_n A = \sum_i a_{n\nu_i} \quad \text{for } A = \{\nu_i\} \in \mathscr{E},\ \mu_n \emptyset = 0. \tag{7}$$

Obviously, uniform boundedness of the family $\{\mu_n\}$ is equivalent to the condition

$$\sup_n \sum_{\nu=1}^{\infty} a_{n\nu} < \infty.$$

The respective pseudomodulars ϱ_i defined by (1) were investigated in [5].

5.1. Proposition 2. *If there are $\eta > 0$ and constants $M \geqslant \delta > 0$ such that for every $0 < \varepsilon < \eta$ and every positive integer N there exist indices $q > p \geqslant N$ satisfying the condition*

$$\delta\varepsilon \leqslant \sup_n \sum_{\nu=p}^{q} a_{n\nu} \leqslant M\varepsilon, \tag{8}$$

then the family of measures $\{\mu_n\}$ defined in (7) is equisplittable.

Proof. Let us denote by $[p, q]$ the set of indices $p, p+1, \ldots, q$ and let the sequence $\varepsilon_k \downarrow 0$, $\varepsilon_k \leqslant \eta$, $\varepsilon_{k+1}/\varepsilon_k \leqslant \frac{1}{2}$, be given. We take $A_1 = [p_1, q_1]$, where $\varepsilon = \varepsilon_1$, $N = 1$, $p_1 = p$, $q_1 = q$. Now, let $A_k = [p_k, q_k]$ be defined for $k = 1, 2, \ldots, m-1$, $q_k < p_{k+1}$, so that 2.1.b holds for $k < m$. Taking $\varepsilon = \varepsilon_m$, $N = q_{m-1}+1$, $p_m = p$, $q_m = q$, and writing $A_m = [p_m, q_m]$, the sequence $A_1, A_2, \ldots$ satisfies the requirements of the definition of equisplittability.

5.2. *As examples of matrices* (a_{nv}) *defining uniformly bounded and equisplittable families of measures* $\{\mu_n\}$ *one can take:*

(i) $$a_{nv} = \begin{cases} \dfrac{1}{n} & \text{for } v \leqslant n, \\ 0 & \text{for } v > n. \end{cases}$$

(ii) $$a_{nv} = \begin{cases} \dfrac{\binom{n+k-v-1}{k-1}}{\binom{n+k}{k}} & \text{for } v \leqslant n, \\ 0 & \text{for } v > n \end{cases}$$

(*Cesàro means* (C, k) *of orders* $k = 1, 2, \ldots$).

(iii) $$a_{nv} = \begin{cases} p_v/P_n & \text{for } v \leqslant n, \\ 0 & \text{for } v > n, \end{cases}$$

where $p_n > 0$ *for* $n = 0, 1, \ldots$; $P_n = p_0 + \ldots + p_n \to \infty$ *as* $n \to \infty$, *in case we suppose the infinite product* $\prod_{i=1}^{\infty} P_{i-1}/P_i$ *to be convergent* (*Riesz means* $(R, p, 1)$).

(iv) $$a_{nv} = \begin{cases} p_{n-v}/P_n & \text{for } v \leqslant n, \\ 0 & \text{for } v > n, \end{cases}$$

with the same assumptions as in (iii) (*Nörlund means* (N, p_n)).

Proof. It is sufficient to prove that the assumptions of Proposition 2 are satisfied. Concerning cases (i) and (ii), this follows from [4].

In case (iii), easy calculation shows that

$$\sup_n \sum_{v=p}^{q} a_{nv} = 1 - \frac{P_{p-1}}{P_q}.$$

Let $\delta = \frac{1}{2}$, $M = 1$, $0 < \varepsilon < 1$ and let N be given. Since the product $\prod_{i=1}^{\infty} P_{i-1}/P_i$ is convergent, the series $\sum_{i=1}^{\infty} \ln P_{i-1}/P_i$ is convergent, too. Hence there exists $N_1 \geqslant N$ such that

$$-\ln \frac{P_{p-1}}{P_q} = \left| \sum_{i=p}^{q} \ln \frac{P_{i-1}}{P_i} \right| < -\ln(1 - M\varepsilon),$$

for $q > p > N_1$, i.e. $P_{p-1}/P_q > 1 - M\varepsilon$ for $q > p > N_1$. Fixed $p > N_1$, we choose $q > p$ so large that $P_q > 2P_{p-1}$. Then $P_{p-1}/P_q < \frac{1}{2} < 1 - \delta\varepsilon$.

Hence

$$\delta\varepsilon < 1 - \frac{P_{p-1}}{P_q} < M\varepsilon,$$

and condition (8) follows. Let us still remark that if $\limsup_{n\to\infty}(P_{n-1}/P_n) < 1$, then it is easily verified that (8) does not hold.

In case (iv) we have

$$\sup_n \sum_{\nu=p}^{q} a_{n\nu} = \frac{P_{q-p}}{P_q}.$$

Taking $\delta = \frac{1}{2}$, $M = 1$, $0 < \varepsilon < 1$, and fixing N we obtain as in case (iii) that $P_{q-p}/P_q > \delta\varepsilon$ for $q - p > N_1$, where $N_1 > N$ is sufficiently large. We fix the difference $q-p$ and we choose p and $q > p$ so large that $P_{q-p}/P_q < M\varepsilon$. Hence condition (8) follows.

6. Let us notice the following remarks.

6.1. From the above it follows that Theorems 1-3 are valid if the family of measures $\{\mu_\tau\}$ is any one of those mentioned is Sections 4 and 5.

6.2. Examples of families of measures which are not equisplittable can be obtained easily applying δ-measures. For instance, one can take $\{\mu_n\}$ as in (7) with $a_{n\nu} = \delta_{n\nu}$, where $\delta_{n\nu}$ is Kronecker symbol. More generally, if $\mathscr{E}$ is the σ-algebra of all Lebesgue measurable subsets of the interval $\langle t_0, \infty)$ and $\mathscr{T}$ is an infinite set of real numbers, then the family of Dirac measures $\{\delta_\tau\}$ is not equisplittable. It is easily observed that in case of the above measures, Theorems 2 and 3 do not hold.

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