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J. MUSIAŁEK (Kraków)

The Green's function and the solutions of the Neumann and Dirichlet problems

In this paper we construct by the method of symmetric images solutions of the Neumann or Dirichlet problem for the following regions: 1° the quarter E_2^+ ($y_1 > 0, y_2 > 0$) of the plane, with the Neumann conditions $\partial u/\partial n = 0$ (chapter I, section 1), 2° the region E_n^+ ($y_i > 0, i = 1, \ldots, n$) of the euclidean n-space, with the Neumann type conditions, (chapter I, section 2), 3° the region E_n^+ of the euclidean n-space $n \geqslant 3$, with the Dirichlet type conditions (chapter II, section 3), 4° the unbounded strip of the euclidean n-space $n \geqslant 4$, with Neumann type boundary conditions (chapter III, section 4), 5° the strip as in 4° with the Neumann condition an one component of the boundary, and the Dirichlet condition at the second one (chapter III, section 5), finally we solve the Dirichlet problem for the Poisson's equation in E_3^+ (chapter IV).

In the monograph [2], p. 265-272, 309, vol. I, the Green function is constructed by the method of symmetric images for the half plane, the half space, the quarter of the plane, and an unbounded strip in the 3-space, the imposed boundary conditions being of Dirichlet type. Also applications of Green functions to the boundary problems are given.

The solution of Dirichlet problem for the region E_2^+ was obtained in the monograph [3], p. 424, by the method of conformal mapping.

I. GREEN FUNCTION AND THE SOLUTION OF NEUMANN PROBLEM FOR THE REGION E_2^+ AND E_n^+

In this chapter we give the construction of Green function and the solution of Neumann problem for (a) the quarter of the plane E_2^+ ($y_1 > 0$, $y_2 > 0$), (b) the region E_n^+ ($y_1 > 0, \ldots, y_n > 0$) of the euclidean n-space ($n \geqslant 3$) with the Neumann boundary condition $\partial G/\partial n = 0$ on the set $y_i = 0$ ($i = 1, \ldots, n$).

1. Let us first consider the two dimensional case $n = 2$. Let two points $X = X_{11}(x_1, x_2)$ and $Q(y_1, y_2)$ be given in E_2^+ and let us write

$X_{-11}=(-x_1, x_2)$, $X_{1-1}=(x_1, -x_2)$ for the symmetric images of the point X with respect to the axes y_2 and y_1 respectively. The point $X_{-1-1}=(-x_1, -x_2)$ is symmetric to X_{-11} with respect to the axis y_1 or to X_{1-1} with respect to the axis y_2. Also let us write

$$r_{11}^{-1}=QX=[(y_1-x_1)^2+(y_2-x_2)^2]^{-1/2},$$
$$r_{-11}^{-1}=QX_{-11}=[(y_1+x_1)^2+(y_2-x_2)^2]^{-1/2},$$
$$r_{1-1}^{-1}=QX_{1-1}=[(y_1-x_1)^2+(y_2+x_2)^2]^{-1/2},$$
$$r_{-1-1}^{-1}=QX_{-1-1}=[(y_1+x_1)^2+(y_2+x^2)]_2^{-1/2}.$$

THEOREM 1. *The Green function for E_2^+ with the Neumann boundary condition*

$$\left.\frac{\partial G(X,Q)}{\partial y_1}\right|_{y_1=0}=\left.\frac{\partial G(X,Q)}{\partial y_2}\right|_{y_2=0}=0 \quad \text{for } Q(0,y_2)$$

or $Q(y_1, 0)$ is of form

$$G(X,Q)=\log r_{11}+\log r_{-11}+\log r_{1-1}+\log r_{-1-1}. \tag{1.1}$$

Proof. Let us check that the the function $G(X,Q)$ harmonic with respect to $Q(Y)$ is form $\log r_{11}+H(X,Q)$, where $H(X,Q)=\log r_{-11}+$ $+\log r_{1-1}+\log r_{-1-1}$ is a harmonic function of $Q(Y)$. Indeed, for $Q(y_1, 0)$ we have

$$\left.\frac{\partial G(X,Q)}{\partial y_1}\right|_{y_1=0}$$
$$=[r_{11}^{-2}(y_1-x_1)+r_{1-1}^{-2}(y_1-x_1)+r_{-11}^{-2}(y_1+x_1)+r_{-1-1}^{-2}(y_1+x_1)]_{y_1=0}$$
$$=r_{11}^{-1}(x_1-x_1)+r_{1-1}^{-2}(-x_1+x_1)=0,$$

since $r_{11}=r_{-11}$ and $r_{1-1}=r_{-1-1}$, thus $\left.\frac{\partial G(X,Q)}{\partial y_1}\right|_{y_1=0}=0$. Similarly we can show that $\left.\frac{\partial G(X,Q)}{\partial y_2}\right|_{y_2=0}=0$. Now let us verify that $\frac{\partial G(X,Q)}{\partial l}\to 0$ as $\overline{OQ}\to\infty$, for an arbitrary direction l (with cosini $\cos\alpha$, $\cos\beta$) and for an arbitrary fixed X. Indeed

$$\frac{\partial G}{\partial l}=\frac{\partial G}{\partial y_1}\cos\alpha+\frac{\partial G}{\partial y_2}\cos\beta=[r_{11}^{-2}(y_1-x_1)\cos\alpha+r_{11}^{-2}(y_2-x_2)\cos\beta+$$
$$+r_{-11}^{-2}(y_1+x_1)\cos\alpha+r_{-11}^{-2}(y_2+x_2)\cos\beta+$$
$$+r_{-1-1}^{-2}(y_1+x_1)\cos\alpha+r_{-1-1}^{-2}(y_2+x_2)\cos\beta+$$
$$+r_{1-1}^{-2}(y_1-x_1)\cos\alpha+r_{1-1}^{-2}(y_2-x_2)\cos\beta]\to 0 \quad \text{as } \overline{OQ}\to\infty,$$

since each term in the above sum is a product of two factors the first of which tends to zero as $\overline{OQ} \to \infty$ and the second one remains bounded.

LEMMA 1. *Let $f(s)$ and $h(t)$ be two even functions defined on the real axis and let the integrals*

$$I_1(X) = \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{f(s)\,(x-s)\,ds}{y^2+(x-s)^2}, \qquad I_2(X) = \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{h(t)\,x\,dt}{x^2+(y-t)^2}$$

be convergent. Then

$$I_1(X) = \frac{2}{\pi}\int_0^{\infty} f(s)\left[\frac{x-s}{(x-s)^2+y^2} + \frac{x+s}{(x+s)^2+y^2}\right] ds \tag{1.2}$$

and

$$I_2(X) = \frac{2}{\pi}\int_0^{\infty} h(t)\left[\frac{x}{x^2+(y-t)^2} + \frac{x}{x^2+(y+t)^2}\right] dt. \tag{1.3}$$

Proof. Representing the integral $I_1(X)$ as

$$I_1(X) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{f(s)\,(x-s)\,ds}{y^2+(x-s)^2} + \frac{1}{\pi}\int_0^{\infty} \frac{f(s)\,(x-s)\,ds}{y^2+(x-s)^2}$$

and replacing in the first integral s by $-u$ we obtain (1.2) since the function $f(s)$ is even. Similar computation applies to the second integrals.

Let us take into consideration the integrals

$$I_1^1(X) = \frac{2}{\pi}\int_{-\infty}^{+\infty} f(s)\log[(x-s)^2+y^2]\,ds,$$

$$I_2^1(X) = \frac{2}{\pi}\int_{-\infty}^{+\infty} h(t)\log[x^2+(y-t)^2]\,ds.$$

We shall say that the function $f(s)$ belongs to the class H if $f(s)$ is continuous in E_1^+ and the integral $\int_R^{\infty} \omega(r)\log r\,dr$, where $\omega(r) = \max_{|s|=r}|f(s)|$ exist and converge for $R > R_1 > 0$.

LEMMA 2. *Let $f(s)$ and $h(t)$ be of class H; then the integrals $I_1^1(X)$ and $I_2^1(X)$ exist and converge uniformly with respect to X belonging to the cylinder $D(a, A, B)$: $\{-B \leqslant x \leqslant B, 0 \leqslant a \leqslant y \leqslant A\}$ for any positive a, A, B.*

Proof. First let us show that the integral $I_1(X)$ exists. If $r_1 < s < r_2$, then

$$\left|\int_{r_1}^{r_2} f(s)\log[(x-s)^2+y^2]\,ds\right| < \infty.$$

The function $\log[(x-s)^2+y^2]$ is continuous in the interval $r_1 < s < r_2$ for each $(x, y) \in D(\alpha, A, B)$, since $0 < \alpha^2 \leqslant (x-s)^2+y^2$. Thus the function under the sign of the integral in $I_1^1(X)$ is continuous, whence the integral $I_1^1(X)$ exists. For R and s sufficiently large we have:

$$\varrho \leqslant \varrho_1 + r \leqslant (A^2+B^2)^{1/2} + |s| \leqslant \gamma + |s| \leqslant 2|s|,$$

where $\varrho^2 = (x-s)^2+y^2$, $\gamma = 2\max(A, B)$. Thus

$$\begin{aligned}\Big| \int\limits_{|s|>R} f(s)\log[(x-s)^2+y^2]\,ds \Big| &\leqslant \int\limits_{|s|>R} |f(s)|\,|\log \varrho_1^2|\,ds \leqslant \int\limits_{|s|>R} |f(s)|\,\big|\log(2|s|)^2\big|\,ds \\ &= 2\int\limits_{|s|>R} |f(s)|\,\big|\log(2|s|)\big|\,ds \leqslant 2\int\limits_{|s|>R} |f(s)|\,2\big|\log|s|\big|\,ds \\ &= 4\int\limits_{|s|>R} |f(s)|\,\big|\log|s|\big|\,ds \leqslant 4\int\limits_{|s|>R} \omega(|s|)\big|\log|s|\big|\,ds.\end{aligned}$$

So, for any $\varepsilon > 0$ there exists an $R(\varepsilon)$ such that $\int\limits_{|s|>R} \omega(|s|)\log|s| < \varepsilon$. Therefore the integral $I_1^1(X)$ converges uniformly for $X \in D(\alpha, A, B)$.

Analogously we can show that the integral $I_2^1(X)$ exists and converges uniformly.

Let

$$I_3(X) = \frac{\partial I_1^1(X)}{\partial x} = 2\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-2}(x-s)\,ds,$$

$$I_4(X) = \frac{\partial I_1^1(X)}{\partial y} = 2\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-2}y\,ds,$$

$$I_5(X) = \frac{\partial^2 I_1^1(X)}{\partial x^2} = 2\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-2}\,ds - 4\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-3}(x-s)^2\,ds,$$

$$I_6(X) = \frac{\partial^2 I_1^1(X)}{\partial y_2} = 2\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-2}\,ds - 4\int\limits_{-\infty}^{+\infty} f(s)\,\varrho^{-3}y^2\,ds,$$

$$I_s(X) = \frac{\partial^p I_2^1(X)}{\partial x_i^p}, \qquad i, p = 1, 2,\ s = 7, 8, 9, 10,\ x_1 = x,\ x_2 = y.$$

LEMMA 3. *Let functions $f(s)$ and $h(t)$ be of class H; then the integrals $I_k(X)$, $k = 3, 4, 5, 6$ and $I_s(X)$ exist and are uniformly convergent for $X \in D(\alpha, A, B)$.*

Proof. We shall prove this only for the function $I_3(X)$. For $X \in D(a, A, B)$ we have $0 < a^2 \leqslant \varrho^2$ and

$$\left|\int_{r_1}^{r_2} f(s)\,\varrho^{-2}(x-s)\,ds\right| < \infty.$$

There exists a number $s_1 > 0$ such that $|\varrho^{-2}(x-s)| \leqslant 1$ for $s \geqslant s_1$ and

$$\left|\int_{|s|>R>s_1} f(s)\,\varrho^{-2}(x-s)\,ds\right| \leqslant \int_{|s|>R>s_1} |f(s)|\,ds < 2\int_{|s|>R>s_1} \omega(|s|)\log|s|\,ds.$$

Hence by the definition of the class H we infer that the integral $I_3(X)$ is uniformly convergent for $X \in D(a, A, B)$.

The proof for remain functions follows in the same way.

The Lemmas 2 and 3 imply

$$\Delta I_1^1(X) = \int_{-\infty}^{+\infty} f(s)\,\Delta\{\log[(x-s)^2+y^2]\}\,ds = 0$$

and

$$\Delta I_2^1(X) = \int_{-\infty}^{+\infty} h(t)\,\Delta\{\log[x^2+(y-t)^2]\}\,dt = 0.$$

Therefore the function $u(X) = I_1^1(X)+I_2^1(X)$ is harmonic for $X \in E_2^+$. Applying Lemma 1 we infer, in turn, that the function

$$(1.4)\qquad u(X) = \frac{1}{\pi}\int_0^{\infty} f(s)\log[(x-s)^2+y^2]\,ds + \frac{1}{\pi}\int_0^{\infty} f(s)\log[(x+s)^2+y^2]\,ds + \frac{1}{\pi}\int_0^{\infty} h(t)\log[x^2+(y-t)^2]\,dt + \frac{1}{\pi}\int_0^{\infty} h(t)\log[x^2+(y+t)^2]\,dt$$

is also harmonic in E_2^+.

Lemma 4 ([2], p. 270).

$$\frac{x}{\pi}\int_{-\infty}^{+\infty} [x^2+(y-t)^2]^{-1}\,dt = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{ds}{1+s^2} = 1.$$

Remark. Lemma 4 implies that

$$(1.5)\qquad \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{xh(y)}{x^2+(y-t)^2}\,dt = h(y).$$

THEOREM 2. *Suppose that the functions $f(s)$ and $h(t)$ of class H satisfy the assumptions of Lemma* 1 *and are bounded on E_1; then the Neumann problem in E_2^+ with boundary conditions*:

$$1^\circ \lim \frac{\partial u}{\partial x} = h(y_0) \text{ as } X \to (0, y_0),\ y_0 > 0,$$

$$2^\circ \lim \frac{\partial u}{\partial y} = f(x_0) \text{ as } X \to (x_0, 0),\ x_0 > 0,$$

admits the solution

$$u(X) = \frac{1}{\pi} \int_0^\infty f(s) G(X, Y)|_{y_2=0} ds + \frac{1}{\pi} \int_0^\infty h(t) G(X, Y)|_{y_1=0} dt, \tag{1.6}$$

the function $G(X, Y)$ being defined by (1.1).

Proof. From formulae (1.1) and (1.6) we get (1.4). Thus by Lemma 1

$$\frac{\partial u}{\partial x} = \frac{2}{\pi} \int_{-\infty}^{+\infty} f(s)\,(x-s)\,[(x-s)^2+y^2]^{-1} ds + \tag{1.7}$$

$$+ \frac{2}{\pi} \int_{-\infty}^{+\infty} h(t) x [x^2+(y-t)^2]^{-1} dt$$

and

$$\frac{\partial u}{\partial y} = \frac{2}{\pi} \int_{-\infty}^{+\infty} f(s) y [y^2+(x-s)^2]^{-1} ds + \frac{2}{\pi} \int_{-\infty}^{+\infty} h(t)\,(y-t)\,[x^2+(y-t)^2]^{-1} dt.$$

We shall prove that condition 1° is satisfied (a proof for condition 2° is similar). It is enough to prove that

$$\lim_{x \to 0} \frac{2}{\pi} \int_{-\infty}^{+\infty} f(s)\,(x-s)\,[(x-s)^2+y^2]^{-1} ds = 0 \tag{a}$$

and

$$\lim_{x \to 0} \frac{2}{\pi} \int_{-\infty}^{+\infty} h(t) x [x^2+(y-t)^2]^{-1} dt = h(y). \tag{b}$$

In view of the uniform convergency of the integral

$$J_1(X) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(s)\,(x-s)\,[(x-s)^2+y^2]^{-1} ds$$

for $y > 0$ the function $J_1(X)$ is continuous for all x. Hence we have

$$\lim_{x \to 0^+} J_1(x, y) = J_1(0, y) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} f(s) s (s^2+y^2)^{-1} ds = 0$$

since the function under the integral sign is odd with respect to s.

Proof of conditions (b) will be carried out similarly as in the paper [2], p. 270-271. For every $\varepsilon > 0$ there exist $\delta > 0$ such the inequality $|t-y_0| < \delta$, implies the inequality $|h(t)-h(y_0)| < \frac{1}{2}\varepsilon$.

Let us write $\partial u/\partial x$ in form

$$\frac{\partial u}{\partial x} = \frac{1}{\pi}\left[h(y_0)\int_{-\infty}^{+\infty}\frac{x\,dt}{x^2+(y-t)^2} + x\int_{|t-y_0|\leqslant\delta}\frac{h(t)-h(y_0)}{x^2+(y-t)^2}\,dt + \right. \tag{1.8}$$

$$\left. + x\int_{|t-y_0|>\delta}\frac{h(t)-h(y_0)}{x^2+(y-t)^2}\,dt\right].$$

In view of Lemma 4 we have

$$\left|\frac{x}{\pi}\int_{|t-y_0|\leqslant\delta}\frac{h(t)-h(y_0)}{x^2+(y-t)^2}\,dt\right| \leqslant \frac{\varepsilon}{2\pi}\int_{-\infty}^{+\infty}\frac{d\sigma}{1+\sigma^2} = \frac{\varepsilon}{2}. \tag{1.9}$$

If we assume that $|t-y_0| < \delta/2$, then we obtain the inequality:

$$\left|\frac{x}{\pi}\int_{|t-y_0|>\delta}\frac{h(t)-h(y_0)}{x^2+(y-t)^2}\,dt\right| \leqslant \frac{2Mx}{\pi}\int_{|t-y_0|>\delta}\frac{dt}{x^2+(y-t)^2}$$

$$\leqslant \frac{2Mx}{\pi}\int_{|t-y_0|>\delta/2}\frac{dt}{x^2+(y-t)^2} = \frac{4M}{\pi}\int_{\delta/2}^{+\infty}\frac{d\sigma}{1+\sigma^2},$$

where $M = \sup_{-\infty<x<+\infty}|h(x)|$. If η is sufficiently small the inequality $0 < y < \eta$ implies the following

$$\int_{\delta/2y}^{+\infty}\frac{d\sigma}{1+\sigma^2} < \frac{\pi\varepsilon}{8M}$$

which implies successively

$$\left|\frac{x}{\pi}\int_{|t-y_0|\geqslant\delta}\frac{h(t)-h(y_0)}{x^2+(y-t)^2}\,dt\right| < \frac{\varepsilon}{2} \tag{1.10}$$

for

$$0 < y < \eta \quad \text{and} \quad |y-y_0| < \delta/2. \tag{1.11}$$

Finally we obtain in view of (1.8), (1.9), (1.10) that (1.11) implies

$$|u(x, y)-h(y_0)| < \varepsilon$$

which finishes proof of condition (b).

2. In this paragraph we shall construct the Green function and the solution of the Neumann problem for the region E_n^+, the boundary condi-

tions being $\partial G/\partial n = 0$ for $Q(X, Y)$ belonging to the hyperplane $y_i = 0$ $(i = 1, \ldots, n)$.

Let $X = (x_1, \ldots, x_n)$ and $Q = (y_1, \ldots, y_n)$ be two different points in E_n^+. Let $X_{\varepsilon_1,\ldots,\varepsilon_n} = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$; denote all possible points which are symmetric images of the point $X = (x_1, \ldots, x_n)$ with respect to the coordinate hyperplanes or are iterations of those symmetric images. Let

$$r_{\varepsilon_1,\ldots,\varepsilon_n} = \overline{QX}_{\varepsilon_1,\ldots,\varepsilon_n} = [(y_1 + \varepsilon_1 x_1)^2 + \ldots + (y_n + \varepsilon_n x_n)^2]^{1/2}.$$

THEOREM 3. *The Green function for E_n^+ with the Neumann boundary conditions is*

$$G(X, Q) = \sum_{\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_n} (r^{2-n}_{\varepsilon_1,\ldots,\varepsilon_{i-1},1,\varepsilon_{i+1},\ldots,\varepsilon_n} + r^{2-n}_{\varepsilon_1,\ldots,\varepsilon_{i-1},-1,\varepsilon_{i+1},\ldots,\varepsilon_n}). \tag{2.1}$$

Proof. By (2.1)

$$\left.\frac{\partial G(X, Q)}{\partial y_i}\right|_{y_i=0} = (x_i - x_i) \sum_{\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_n} r^{2-n}_{\varepsilon_1,\ldots,\varepsilon_{i-n},1,\varepsilon_{i+1},\ldots,\varepsilon_n} = 0,$$

since $r_{\varepsilon_1,\ldots,\varepsilon_{i-1},1,\varepsilon_{i+1},\ldots,\varepsilon_n} = r_{\varepsilon_1,\ldots,\varepsilon_{i-1},-1,\varepsilon_{i+1},\ldots,\varepsilon_n}$.

It is easy to check that $\partial G/\partial l \to 0$ as $\overline{OQ} \to \infty$ for each fixed x.

LEMMA 1a. *Let the functions $f_i(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) = f_i(Y_i)$ $(i = 1, \ldots, n)$ be even with respect to each of the variables y_i and let the integrals*

$$J_i = \int_0^\infty \ldots \int_0^\infty f_i(Y_i) \sum_{\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_n} [(x_1 + \varepsilon_1 y_1)^2 + \ldots + (x_{i-1} + \varepsilon_{i-1} y_{i-1})^2 + x_i^2 + (x_{i+1} + \varepsilon_{i+1} y_{i+1})^2 + \ldots + (x_n + \varepsilon_n y_n)^2]^{\frac{1}{2}(2-n)} dY_i,$$

i.e.

$$J_i = \sum_{\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_n} \int_0^\infty \ldots \int_0^\infty f_i(Y_i)[(x_1 + \varepsilon_1 y_1)^2 + \ldots + (x_{i-1} + \varepsilon_{i-1} y_{i-1})^2 + x_i^2 + (x_{i+1} + \varepsilon_{i+1} y_{i+1})^2 + \ldots + (x_n + \varepsilon_n y_n)^2]^{(2-n)/2} dY_i$$

exist. Then

$$J_i = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f_i(Y_i) K_i(X, Y_i) dY_i, \tag{2.2}$$

where

$$K_i(X, Y_i) = [(x_1 - y_1)^2 + \ldots + (x_{i-1} - y_{i-1})^2 + x_i^2 + (x_{i+1} - y_{i+1})^2 + \ldots + (x_n - y_n)^2]^{(2-n)/2} \tag{2.3}$$

and $dY_i = dy_1 \ldots dy_{i-1} dy_{i+1} \ldots dy_n$.

Proof. For $i = 1$

$$J_1 = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_1(Y_1) K_1(X, Y_1) dY_1$$

$$= J_1^1 + \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \int_{0}^{+\infty} f_1(Y_1) K_1(X, Y_1) dY_1,$$

where $J_1^1 = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \int_{-\infty}^{0} f_1(Y_1) K_1(X, Y_1) dY_1$. Replacing the variable y_2 by $-s_2$ and taking into account that the function $f_1(Y_1)$ is even with respect to y_2 we obtain

$$J_1 = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \int_{0}^{+\infty} f_1(Y_1) K_1^1(X, Y_1) dY_1 +$$

$$+ \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \int_{0}^{+\infty} f_1(Y_1) K_1(X, Y_1) dY_1,$$

where $K_1^1(X, Y_1) = [x_1^2 + (y_2 + x_2)^2 + (y_3 - x_3)^2 + \dots + (y_n - x_n)^2]^{\frac{1}{2}(2-n)}$.

Applying the same argument to the integrals J_i, $i = 2, \dots, n$, we obtain the these of Lemma 1a.

LEMMA 2a. *Let* $a_n(2-n) = \int_{E_{n-1}} \dots \int (1 + t_1^2 + \dots + t_{i-1}^2 + t_{i+1}^2 \dots + t_n^2)^{-\frac{1}{2}n} dT_i$ *and* $x_i > 0$; *then*

$$J = \frac{1}{a_n} \int_{E_{n-1}} \dots \int \frac{\partial K_i(X, Y_i)}{\partial x_i} dY_i = 1, \tag{2.4}$$

where $K_i(X, Y_i)$ *is given by* (2.3).

Proof. The transformation of the variables $y_1 = x_1 + t_1 x_1, \dots, y_{i-1} = x_{i-1} + t_{i-1} x_i$, $y_{i+1} = x_{i+1} + t_{i+1} x_i, \dots, y_n = x_n + t_n x_i$, with the Jacobian $|J| = x_i^{n-1}$, gives (2.4).

Remark. Multiplying both sides of (2.4) by $f_i(X_i)$, where $X_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, we obtain

$$\frac{1}{a_n} \int_{E_{n-1}} \dots \int f_i(X_i) \frac{\partial K_i(X, Y_i)}{\partial x_i} dY_i = f_i(X_i). \tag{2.5}$$

Let

$$\|T_i\| = \|t_1\| + \dots + \|t_{i-1}\| + \|t_{i+1}\| + \dots + \|t_n\|,$$

$$T_i^2 = t_1^2 + \dots + t_{i-1}^2 + t_{i+1}^2 + \dots + t_n^2.$$

LEMMA 5. *Let* $a \in [0, 1]$ *be a constant*; *then the integral*

$$\int_{E_{n-1}} \dots \int \frac{(\|T_i^2\|)^a dT_i}{(1 + T_i^2)^{\frac{1}{2}n}} \qquad (n \geqslant 2)$$

converges.

Proof. The integral we deal with is majorized by the convergent integral

$$\int_{E_{n-1}}\dots\int \frac{(T_i^2)^{\alpha/2}dT_i}{(1+T_i^2)^{\frac{1}{2}n}} \quad [1].$$

We shall say that the functions $f_i(Y_i)$ belongs to the class H_1 if $f_i(Y_i)$ are continuous in E_{n-1}^+ and the integral

$$\int_{C(K_R)}\dots\int \omega(r)r^{2-n}dY_i, \tag{2.6}$$

where $\omega(r) = \max\limits_{\substack{Y^2=r^2\\ i=1,\dots,n}} |f_i(Y_i)|$, and $C(K_R)$ is the set of the points satisfying $Y_i^2 \geqslant R^2$, is convergent for $R > R_1 > 0$.

LEMMA 6. *Let $f_i(Y_i) \in H_1$; then for R sufficiently large integral* (2.6) *majorizes the integral $\int_{C(K_R)}\dots\int f_i(Y_i)\varrho^{2-n}dY_i$ for $X \in D_i(a, A, B)$ as $0 < a \leqslant x_i \leqslant A$, $\varrho_1^2 = x_1 + \dots + x_n^2 \leqslant B^2$, $B > 0$, $\max \varrho_1^2 = B + A^2$, and $x_2^2 + x_3^2 \leqslant B$.*

Proof. Let $\varrho_1^2 = |X|^2 = \sum\limits_{i=1}^{n} x_i^2$, $r^2 = Y_i^2 \leqslant B^2$, $\varrho^2 = |\overline{XY_i}|^2 = (x_1 - y_1)^2 + \dots + (x_{i-1} - y_{i-1})^2 + x_i^2 + (x_{i+1} - y_{i+1})^2 + \dots + (x_n - y_n)^2$. Then $\sup\limits_{X \in D_i} \varrho_1 \leqslant \max(2A, 2B)$, and by the triangle inequality

$$\varrho \geqslant r - \varrho_1 > r - \max \varrho_1 \geqslant r - \gamma \geqslant r - \tfrac{1}{2}r = \tfrac{1}{2}r \quad \text{for } \tfrac{1}{2}r \geqslant \gamma.$$

This implies that

$$\int_{C(K_R)}\dots\int |f_i(Y_i)|\,\varrho^{2-n}dY_i \leqslant \int_{C(K_R)}\dots\int \omega_i(\tfrac{1}{2}r)^{2-n}dY_i = C\int_{C(K_R)}\dots\int \omega_i(r)r^{2-n}dY_i,$$

where $C = 2^{n-2}$, moreover,

$$\begin{aligned}\varrho^{-1} &= [(y_1 - x_1)^2 + \dots + (y_{i-1} - x_{i-1})^2 + x_i^2 + (y_{i+1} - x_{i+1})^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}} \\ &\leqslant [(y_1 - x_1)^2 + \dots + (y_{i-1} - x_{i-1})^2 + a^2 + (y_{i+1} - x_{i+1})^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}} \\ &\leqslant (a^2)^{-\frac{1}{2}} = a^{-1}.\end{aligned}$$

Thus the integral

$$\int_{K_R}\dots\int f_i(Y_i)\varrho^{2-n}dY_i,$$

where K_R: $|Y_i|^2 < R^2$, exists for large R.

Let

$$P_i^j(X) = \int_{E_{n-1}}\dots\int f_i(Y_i)\frac{\partial K_i(X, Y_i)}{\partial x}\,dY_i, \quad i, j = 1, \dots, n, \tag{2.7}$$

and

$$(2.8)\qquad Q_i^j(X) = \int_{E_{n-1}} \dots \int f_i(Y_i) \frac{\partial^2 K_i(X, Y_i)}{\partial x_j^2}\, dY_i, \qquad i, j = 1, \dots, n.$$

LEMMA 7. *Let $f_i(Y_i) \in H_1$; then each of the integrals defined by* (2.2) *and the integrals $P_i^j(X)$, $Q_i^j(X)$ defined by* (2.7) *and* (2.8) *respectively are uniformly convergent in each of the n-dimensional cylinders $D_i(a, A, B)$ and*

$$(2.9)\qquad P_i^j(X) = \frac{\partial J_i(X)}{\partial x_j}, \qquad Q_i^j(X) = \frac{\partial^2 J_i(X)}{\partial x_j^2}.$$

Proof. It is enough to check that for each $\varepsilon > 0$ there exists an $R(\varepsilon)$ such that

$$\left| \int_{C(K_{R(\varepsilon)})} \dots \int f_i(Y_i) K_i(X, Y_i)\, dY_i \right| < \varepsilon, \qquad \left| \int_{C(K_{R(\varepsilon)})} \dots \int f_i(Y_i) \frac{\partial K_i(X, Y_i)}{\partial x_j}\, dY_i \right| < \varepsilon$$

and

$$\left| \int_{C(K_{R(\varepsilon)})} \dots \int f_i(Y_i) \frac{\partial^2 K_i(X, Y_i)}{\partial x_j^2}\, dY_i \right| < \varepsilon$$

uniformly for $X \in D_i(a, A, B)$. Introducing the spherical coordinates in (2.2)

$$(2.10)\qquad y_1 = x_1 + r\cos\varphi_1 \dots \cos\varphi_{n-2}\sin\varphi_{n-1}, \dots, y_{n-1} = x_{n-1} + r\sin\varphi_1,$$

where $0 \leqslant \varphi_1 \leqslant 2\pi$, $0 \leqslant \varphi_j \leqslant \pi$, $j = 2, \dots, n-1$, $0 < r < B+R$, $|J| = r^{n-2}\Phi(\varphi_1, \dots, \varphi_{n-1})$, we get

$$\int_{C(K_R)} \dots \int \omega_i(r) r^{2-n}\, dY_i = C \int_R^\infty \omega_i(r)\, dr < \varepsilon \qquad \text{for } R > R_0(\varepsilon),$$

where C is positive constant.

Let now us now prove that the integrals representing $P_i^j(X)$ are uniformly convergent in the domains $D_i(a, A, B)$. We shall show this for the integrals $P_1^1(X)$ and $P_1^2(X)$, the proof for the remaining ones being similar. By (2.3)

$$K_1(X, Y_1) = [x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{\frac{1}{2}(2-n)},$$

whence

$$(2.11)\qquad \frac{\partial K_1}{\partial x_2} = (2-n) x_1 W_1(X, Y_1),$$

and

$$(2.12)\qquad \frac{\partial K_1}{\partial x_2} = (n-2)(y_2 - x_2) W_1(X, Y_1),$$

where $W_1(X, Y_1) = [x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}n}$.

By (2.7) and (2.11)

$$P_1^1(X) = \int_{E_{n-1}} \dots \int f_1(Y_1) \frac{\partial K_1(X, Y_1)}{\partial x_1} dY_1 \tag{2.13}$$

$$= (2-n) \int_{E_{n-1}} \dots \int f_1(Y_1) x_1 \varrho_1^{-1} \varrho_1^{1-n} dY_1$$

where $\varrho_1^2 = x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2$. Since $|\varrho_1^{-1} x_1| = |\cos(\varrho_1 x_1)| \leqslant 1$, we get by Lemma 6

$$|P_1^1(X)| \leqslant M \int_{E_{n-1}} \dots \int |f_1(Y_1)| \varrho_1^{1-n} dY_1 \leqslant M \int_{E_{n-1}} \dots \int \omega_1(r) r^{1-n} dY_1$$

$$\leqslant M \int_{C(K_R)} \dots \int \omega_1(r) r^{1-n} dY_1 \leqslant \varepsilon \quad \text{for } R > R_1(\varepsilon),$$

where $M = 2-n$. By (2.7) and (2.12)

$$P_1^2(X) = \int_{E_{n-1}} \dots \int f_1(Y_1) \frac{\partial K_1(X, Y_1)}{\partial x_2} dY_1$$

$$= N \int_{E_{n-1}} \dots \int f_1(Y_1) \varrho_1^{-1} (y_2 - x_2) \varrho_1^{1-n} dY_1,$$

where $N = 2-n$. Now $|\varrho_1^{-1}(y_2 - x_2)| \leqslant 1$, whence and by Lemma 6 $|P_1^2(X)| < \varepsilon$ for $R > R_1(\varepsilon)$.

It follows that the integrals $P_1^1(X)$ and $P_1^2(X)$ are uniformly convergent in the cylinders $D_i(a, A, B)$, $i = 1, \dots, n$.

The integrals $Q_i^j(X)$, $i, j = 2, \dots, n$, admit as majorants integrals of type $P_i^j(X)$. Therefore these integrals are also uniformly convergent.

Formula (2.9) results now by uniform convergence of the inetgrals $P_i^j(X)$ and $Q_i^j(X)$.

THEOREM 4. *Let the functions $f_i(Y_i)$ be bounded in E_n^+ and let the assumption of Lemmata* 1a, 2a, 6 *and* 7 *be satisfied. Then the solution of Neumann's problem in E_n^+ with the boundary conditions*

$$\lim \frac{\partial u}{\partial x_i} = f_i(X_i) \quad \text{as } X \to (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \tag{2.14}$$

is in form of:

$$u(X) = \frac{1}{a_n} \int_{E_{n-1}^+} \dots \int f_1(Y_1) G(X, Q) \Big|_{y_1=0} dY_1 + \dots +$$

$$+ \frac{1}{a_n} \int_{E_{n-1}^+} \dots \int f_n(Y_n) G(X, Q) \Big|_{y_n=0} dY_n,$$

i.e., in virtue of (2.1)

$$(2.15)\qquad u(X)=\frac{2}{a_n}\int_0^\infty\dots\int_0^\infty f_1(Y_1)\sum_{\varepsilon_2,\dots,\varepsilon_n}[x_1^2+(y_2+\varepsilon_2x_2)^2+\dots+$$

$$+(y_n+\varepsilon_nx_n)^2]^{(2-n)/2}dY_1+\dots+\frac{2}{a_n}\int_0^\infty\dots\int_0^\infty f_n(Y_n)\sum_{\varepsilon_1,\dots,\varepsilon_{n-1}}[(x_1+\varepsilon_1y_1)^2+$$

$$+\dots+(y_{n-1}+\varepsilon_{n-1}x_{n-1})^2+x_n^2]^{(2-n)/2}dY_n.$$

Proof. By Lemma 1a, $u(X)$ is of form

$$u(X)=\frac{1}{a_n}\sum_{i=1}^n\int_{E_{n-1}}\dots\int f_i(Y_i)K_i(X,Y_i)dY_i.$$

Let us check that the boundary conditions (2.14) are satisfied. For $i=1$ the boundary condition takes on the form

$$\lim_{x_1\to0^+}\frac{\partial u}{\partial x_1}=\lim_{x_1\to0^+}\int_{E_{n-1}}\dots\int f_1(Y_1)\frac{\partial K_1(X,Y_1)}{\partial x_1}dY_1=f_1(X_1),$$

i.e.

$$(2.14a)\qquad \lim_{x_1\to0^+}\frac{2-n}{a_n}\int_{E_{n-1}}\dots\int f_1(Y_1)x_1\varrho_1^{-n}dY_1=f_1(X_1)$$

and

$$(2.14b)\qquad \lim_{x_1\to0^+}\frac{2-n}{a_n}\int_{E_{n-1}}\dots\int f_i(Y_i)(y_1-x_1)\varrho_i^{-n}dY_i=0$$

for $i=2,3,\dots,n$.

Let us prove condition (2.14a). By Lemma 2a

$$\frac{x_1}{a_n}\int_{E_{n-1}}\dots\int f_1(X_1)\frac{\partial K_1(X,Y_1)}{\partial x_1}dY_1=f_1(X_1),$$

i.e.

$$(2.16)\qquad \frac{(2-n)x_1}{a_n}\int_{E_{n-1}}\dots\int f_1(X_1)\varrho_1^{-n}dY_1=f_1(X_1).$$

Writing by (2.16)

$$\frac{2-n}{a_n}\int_{E_{n-1}}\dots\int f_1(Y_1)x_1\varrho_1^{-n}dY_1=f_1(X_1)+C(X),$$

where

$$C(X)=\frac{2-n}{a_n}\int_{E_{n-1}}\dots\int[f_1(Y_1)-f_1(X_1)]x_1\varrho_1^{-n}dY_1.$$

It is enough to state that $\lim_{x_1 \to 0^+} C(X) = 0$. Let K_ϱ denote a ball $(n-1)$ dimensional at the centre in the point $X_1(x_2, \ldots, x_n)$ with the radius and let $C(K_\varrho)$ denote its complement to the space E_{n-1}, let also $M = \sup_{E_{n-1}} |f(Y_1)|$. Well let $C(X) = C_1(X) + C_2(X)$, where

$$C_1(X) = \frac{2-n}{a_n} \int_{K_\varrho} \ldots \int [f_1(Y_1) - f_1(X_1)] x_1 \varrho_1^{-n} dY_1$$

and

$$C_2(X) = \frac{2-n}{a_n} \int_{C(K_\varrho)} \ldots \int [f_1(Y_1) - f_1(X_1)] x_1 \varrho_1^{-n} dY_1.$$

Let ε denote and arbitrary positive number. From the continuity of the function $f(Y_1)$ in the point X_1 belonging to E_{n-1}^+, we get

$$|f(Y_1) - f(X_1)| < \tfrac{1}{2}\varepsilon$$

for Y_1 belonging to $K_{\varrho(\varepsilon)}$. In view of the above as well as (2.4) we have

$$|C_1(X)| \leqslant \frac{\varepsilon}{2} \frac{2-n}{a_n} \int_{E_{n-1}} \ldots \int x_1 \varrho_1^{-n} dY_1 = \tfrac{1}{2}\varepsilon$$

and

$$|C_2(X)| \leqslant \frac{2-n}{a_n} \int_{C(K_\varrho)} \ldots \int |f(Y_1) - f(X_1)| x_1 \varrho_1^{-n} dY_1$$

$$\leqslant 2M \frac{2-n}{a_n} \int_{C(K_\varrho)} \ldots \int x_1 \varrho_1^{-n} dY_1.$$

By transformation (2.10) we get

$$\int_{C(K_\varrho)} \ldots \int x_1 \varrho_1^{-n} dY_1 = x_1 \int_{C(K_\varrho)} \ldots \int \varrho_1^{-n} dY_1 = N_1 x_1 \int_\varrho^{+\infty} r^{n-2} (x_1^2 + r^2)^{-\frac{1}{2}n} dr \to 0,$$

as $x_1 \to 0^+$, where N_1 being certain constant.

Let us prove now the boundary condition (2.14b). Let us notice that the integrals

$$A_i(X) = \frac{n-2}{a_n} \int_{E_{n-1}} \ldots \int f_i(Y_i)(y_i - x_i) \varrho_i^{-n} dY_i$$

for $i = 2, \ldots, n$ are of (2.7) type, hence from Lemma 7 we infer that they are uniformely convergent in every set $D_1(\alpha, A, B)$ and as a result of this, functions $A_i(X)$ are continuous for every $X \in E_{n-1}$. Hence, we obtain

$$\lim_{x_1 \to 0^+} A_i(X) = A_i(0, x_2, \ldots, x_n) = \frac{n-2}{a_n} \int_{E_{n-1}} \ldots \int f_i(Y_i) y_1 [y_1^2 + \ldots +$$

$$+ (y_{i-1} - x_{i-1})^2 + x_i^2 + (y_{i+1} - x_{i+1})^2 + \ldots + (y_n - x_n)^2]^{-\frac{1}{2}n} dY_i = 0$$

because the function under the sign of the integral is odd with respect to y_1.

The proof of the boundary condition (2.14) for $i = 2, \ldots, n$ runs analogically.

The harmonicity of the function $u(X)$ determined with formula (2.15) in E_n^+ implies that derivatives of the first and the second order of the function $u(X)$ are integrals of (2.7) type as well as (2.8), hence, by Lemma 7 they are uniformly convergent towards X belonging to the sets $W_i(\alpha, A, B)$. Hence we have

$$\Delta u(X) = \sum_{i=1}^{n} \int_{E_{n-1}^+} \ldots \int f_i(Y_i) \frac{\partial \Delta_Y G(X, Y)}{\partial y_i} dY_i = 0.$$

II. GREEN FUNCTION AND THE SOLUTION OF DIRICHLET PROBLEM IN E_n^+

In this chapter we shall construct the Green function $G(X, Y)$ in E_n^+ $(n \geqslant 3)$, satisfying the boundary condition $G(X, Y) = 0$ for Y in the coordinate hyperplane, moreover, we give the solution of Dirichlet problem for E_n^+.

3. To the notation admitted in chapter I, section 2 we add the following ones. Let $\{\eta\}$: $\eta_1, \eta_2, \ldots, \eta_n, \ldots$ be any sequence with an even number of ones the remaing terms being equal -1. The sequence $\{-\eta\}$: $\eta_1, \eta_2, \ldots, \eta_{i-1}, -\eta_i, \eta_{i+1}, \ldots, \eta_n$ is called *of type* $\{\gamma\}$. The sequence $\{\varepsilon\}$: $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, n$, are either of type $\{\eta\}$ or $\{\gamma\}$. The number of sequences of both types is the same. Let us consider the sum

$$S = \sum_{\varepsilon_2, \ldots, \varepsilon_n} \pm (r^{2-n}_{1, \varepsilon_2, \ldots, \varepsilon_n} - r^{2-n}_{-1, \varepsilon_2, \ldots, \varepsilon_n})$$

and let the coefficient at $\{r_\eta\}$ be equal to $+1$ and this at $\{r_\gamma\}$ equal to -1, then

$$S = \sum_{\varepsilon_2, \ldots, \varepsilon_n} \pm (r^{2-n}_{1, \varepsilon_2, \ldots, \varepsilon_n} - r^{2-n}_{-1, \varepsilon_2, \ldots, \varepsilon_n}) = \sum_{\{\eta\}} r^{2-n}_{\{\eta\}} - \sum_{\{\gamma\}} r^{2-n}_{\{\gamma\}}, \tag{3.1}$$

i.e.

$$S = \sum_{\{\eta\}} (r^{2-n}_{\{\eta\}} - r^{2-n}_{\{-\eta\}}) = \sum_{\{\eta\}} \{[(y_1 + \eta_1 x_1)^2 + \ldots + (y_{i-1} + \eta_{i-1} x_{i-1})^2 +$$

$$+ (y_i + \eta_i x_i)^2 + (y_{i+1} + \eta_{i+1} x_{i+1})^2 + \ldots + (y_n + \eta_n x_n)^2]^{\frac{1}{2}(2-n)} - [(y_1 + \eta_1 x_1)^2 +$$

$$+ \ldots + (y_{i-1} + \eta_{i-1} x_{i-1})^2 + (y_i - \eta_i x_i)^2 + (y_{i+1} + \eta_{i+1} x_{i+1})^2 + \ldots +$$

$$+ (y_n + \eta_n x_n)]^{\frac{1}{2}(2-n)}\},$$

this means that

$$S = \sum_{\{\eta^i\}} \sum_{i=1}^{n} \{[(y_1+\eta_1 x_1)^2+\ldots+(y_{i-1}+\eta_{i-1}x_{i-1})^2+(y_i+\eta_i x_i)^2+ \\ +(y_{i+1}+\eta_{i+1}x_{i+1})^2+\ldots+(y_n+\eta_n x_n)^2]^{\frac{1}{2}(2-n)}-[(y_1-\eta_1 x_1)^2+\ldots+ \\ +(y_{i-1}+\eta_{i-1}x_{i-1})^2+(y_i-\eta_i x_i)^2+(y_{i+1}+\eta_{i+1}x_{i+1})^2+\ldots+ \\ +(y_n+\eta_n x_n)^2]^{\frac{1}{2}(2-n)}\},$$

where $\{\eta^i\}$ denotes the sequence with fixed η_i. Thus

$$(3.1a)\quad S = \sum_{\{\eta^i\}} \{[(y_1+\eta_1 x_1)^2+\ldots+(y_n+\eta_n x_n)^2]^{\frac{1}{2}(2-n)}-[(y_1-\eta_1 x_1)^2+ \\ +(y_2+\eta_2 x_2)^2+\ldots+(y_n+\eta_n x_n)^2]^{\frac{1}{2}(2-n)}\}+\sum_{\{\eta^i\}}\sum_{i=2}^{\eta}\{[(y_1+\eta_1 x_1)^2+\ldots+ \\ +(y_{i-1}+\eta_{i-1}x_{i-1})^2+(y_i+\eta_i x_i)^2+(y_{i+1}+\eta_{i+1})^2+\ldots+ \\ +(y_n+\eta_n x_n)^2]^{\frac{1}{2}(2-n)}-[(y_1+\eta_1 x_1)^2+\ldots+(y_{i-1}+\eta_{i-1}x_{i-1})^2+ \\ +(y_i+\eta_i x_i)^2+(y_{i+1}+\eta_{i+1}x_{i+1})^2+\ldots+(y_n+\eta_n x_n)^2]^{(2-n)/2}\}$$

and

$$(3.2)\quad \left.\frac{\partial S}{\partial y_i}\right|_{y_i=0} = (2-n)\sum_{\eta_i=\pm 1}\eta_i x_i \sum_{\{\eta^i\}}[(y_1+\eta_1 x_1)^2+\ldots+ \\ +(y_{i-1}+\eta_{i-1}x_{i-1})^2+x_i^2+(y_{i+1}+\eta_{i+1}x_{i+1})^2+\ldots+(x_n+\eta_n x_n)^2]^{-\frac{1}{2}n}.$$

THEOREM 5. *The Green function $G(X, Y)$ for the region E_n^+ with Dirichlet boundary condition at the hyperplane $y_i = 0$, $i = 1, \ldots, n$, is of form* (3.1), *i.e.*

$$(3.1b)\quad G_i(X, Y) \\ = \sum_{\varepsilon_1,\ldots,\varepsilon_{i-1},\varepsilon_{i+1},\ldots,\varepsilon_n} (r^{2-n}_{\varepsilon_1,\ldots,\varepsilon_{i-1},1,\varepsilon_{i+1},\ldots,\varepsilon_n} - r^{2-n}_{\varepsilon_1,\ldots,\varepsilon_{i-1},-1,\varepsilon_{i+1},\ldots,\varepsilon_n})$$

for $i = 1, \ldots, n$.

Proof. Let us check that $G(X, Y) = 0$ for Y on the coordinate hyperplanes. Indeed for $y_1 = 0$, we obtain

$$(r^{2-n}_{1,\varepsilon_2,\ldots,\varepsilon_n} - r^{2-n}_{-1,\varepsilon_2,\ldots,\varepsilon_n})|_{y_1=0} = [(y_1-x_1)^2+(x_2+\varepsilon_2 y_2)^2+\ldots+ \\ +(x_n+\varepsilon_n y)^2]^{(2-n)/2}|_{y_1=0}-[(x_1+y_1)^2+(x_2+\varepsilon_2 y_2)^2+\ldots+ \\ +(x_n+\varepsilon_n y_n)^2]^{\frac{1}{2}(2-n)}|_{y_1=0} = 0$$

for each system $\varepsilon_2, \ldots, \varepsilon_n$. It follows from (3.1b) that $G(X, Y)|_{y_i=0} = 0$.

In the three dimensional case ($n = 3$) the Green function takes on the form

$$G(X, Y) = (r_{111}^{-1} - r_{-111}^{-1}) - (r_{11-1}^{-1} - r_{-11-1}^{-1}) - (r_{1-11}^{-1} - r_{-1-11}^{-1}) - (r_{1-1-1}^{-1} - r_{-1-1-1}^{-1})$$

equivalently

$$G(X, Y) = (r_{111}^{-1} - r_{1-11}^{-1}) + (r_{-1-11}^{-1} - r_{-111}^{-1}) + (r_{1-1-1}^{-1} - r_{11-1}^{-1}) + (r_{-11-1}^{-1} - r_{-1-1-1}^{-1})$$

or

$$G(X, Y) = (r_{111}^{-1} - r_{11-1}^{-1}) + (r_{-11-1}^{-1} - r_{-111}^{-1}) + (r_{-1-11}^{-1} - r_{-1-1-1}^{-1}) + (r_{1-1-1}^{-1} - r_{1-11}^{-1}).$$

Let

$$P_i(X) = \int_{E_{n-1}^+} \dots \int f_i(Y_i) \sum_{\eta^i = \pm 1} \eta_j x_j \sum_{\{\eta^i\}} [(y_1 + \eta_1 x_1)^2 + \dots + (y_{i-1} + \eta_{i-1} x_{i-1})^2 + x_i^2 + (y_{i+1} + \eta_{i+1} x_{i+1})^2 + \dots + (y_n + \eta_n x_n)^2]^{-\frac{1}{2}n} dY_i,$$

where the summation $\sum_{\{\eta^i\}} \pm$ is extended over all sequences $\{\eta^i\} = \eta_1, \eta_2, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_n$ and $\eta_i = \pm 1$, the sign $+$ standing at the sequences $\{\eta^i\}$ of type $\{\eta\}$ and the sign $-$ at the sequences of type $\{\gamma\}$.

LEMMA 1b. *Let the functions $f_i(Y_i) = f_i(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$, be odd with respect to each of the variables and let the integrals $P_i(X)$ exist. Then*

$$(3.3)\quad P(X) = \sum_{i=1}^n P_i(X) = \sum_{i=1}^n \int_{E_{n-1}^+} \dots \int f_i(Y_i) \sum_{\eta = \pm 1} \eta_j x_j \sum_{\{\eta^i\}} [(y_1 + \eta_1 x_1)^2 + \dots + (y_{i-1} + \eta_{i-1} x_{i-1})^2 + x_i^2 + (y_{i+1} + \eta_{i+1} x_{i+1})^2 + \dots + (y_n + \eta_n x_n)^2]^{-\frac{1}{2}n} dY_i$$

$$= \sum_{i=1}^n \int_{E_{n-1}} \dots \int x_i f_i(Y_i) [(y_1 - x_1)^2 + \dots + (y_{i-1} - x_{i-1})^2 + x_i^2 + (y_{i+1} - x_{i+1})^2 + \dots + (y_n - x_n)^2]^{-n/2} dY_i.$$

Proof. It is enough to verify that

$$P_i(X) = \int_{E_{n-1}} \dots \int x_i f_i(Y_i) [(y_1 - x_1)^2 + \dots + (y_{i-1} - x_{i-1})^2 + x_i^2 + (y_{i+1} - x_{i+1})^2 + \dots + (y_n - x_n)^2]^{-n/2} dY_i.$$

We shall prove this, for example, in the case when $i = 1$. Since

$$P_1(X) = \int_{E_{n-1}} \dots \int x_1 f_1(Y_1) [x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}n} dY_1$$

$$= \int_{E_{n-1}} \dots \int \Big\{ \int_{-\infty}^{0} x_1 f_1(Y_1) [x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}n} dy_n + \int_0^{\infty} f_1(Y_1) x_1 [x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}n} dy_n \Big\} dY_1,$$

we obtain, writing $y_n = -t_n$ in the inner integral

$$P_1(X) = \int_{E_{n-1}} \dots \int \Big\{ \int_0^\infty f_1(Y_1) x_1^2 \big([x_1^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2]^{-\frac{1}{2}n} - \\ - [x_1^2 + (y_2 - x_2)^2 + \dots + (y_{n-1} - x_{n-1})^2 + (y_n + x_n)^2]^{-\frac{1}{2}n} \big) dy_n \Big\} dY_1,$$

because $f_1(Y_1)$ is odd. The same procedure applied to the variables y_{n-1}, $y_{n-2}, \dots, y_2$, gives our assertion.

Let us call the function $f_i(Y_i)$ *of class* H_2 if $f_i(Y_i) \in C$ for $Y_i \in E_{n-1}^+$ and if the integral $\int_{C(K_R)} \dots \int \omega(r) r^{-n} dY_i$, where $\omega(r) = \max_{\substack{|Y|^2 = r^2 \\ i=1,\dots,n}} |f_i(Y_i)|$ is convergent.

LEMMA 8. *Let* $f_i(Y_i) \in H_2$; *then for* R *sufficiently large the integral* $\int_{C(K_R)} \dots \int \omega(r) r^{-n} dY_i$ *majorizes the integral* $\int_{C(K_R)} \dots \int |f_i(Y_i)| \varrho^{-n} dY_i$ *for* $X \in W(\alpha, A, B)$, *where* $0 < \alpha < x_1 \leqslant A$, $\sum_{i=2}^n x_i^2 \leqslant B^2$; $\varrho_1^2 = \sum_{i=1}^n x_i^2 \leqslant B^2 + A^2$.

Proof. By the triangle inequality we have for sufficiently large r_1: $\frac{1}{2} r < r - \gamma \leqslant r - \varrho_1 < \varrho$, where $\varrho = \big[\sum_{i=1}^n (x_i - y_i)^2\big]^{\frac{1}{2}} = \overline{|XV|}$, $r > r_1$, $\gamma = (A^2 + 4B^2)^{\frac{1}{2}}$. Thus

$$\int_{C(K_R)} \dots \int |f_i(Y_i)| \varrho^{-n} dY_i \leqslant \int_{C(K_R)} \dots \int \frac{\omega(r)}{(\frac{1}{2}r)^n} dY_i = 2^{-n} \int_{C(K_R)} \dots \int \omega(r) r^{-n} dY_i.$$

By Lemma 8 the integrals $P_i(X)$ are uniformly convergent in each set of form: $W_n(\alpha, A, B) = W_1(\alpha, A, B) \cap W_2(\alpha, A, B) \cap \dots \cap W_n(\alpha, A, B)$, where $W_i(\alpha, A, B)$ denotes the cylinder defined by the inequalities:

$$0 < \alpha < x_i < A; \qquad x_1^2 + \dots + x_{i-1}^2 + x_{i+1}^2 + \dots + x_n^2 < B^2,$$

α, A, B being arbitrary positive numbers.

LEMMA 9. *Let* $f_i(Y_i) \in H_2$; *then the integrals*:

$$x_i^2 \int_{E_{n-1}} \dots \int f_i(Y_i) \varrho^{-1-n} dY_i; \qquad x_i \int_{E_{n-1}} \dots \int f_i(Y_i)(y_j - x_j) \varrho^{-n-1} dY_i;$$

$$x_i \int_{E_{n-1}} \dots \int f_i(Y_i) \varrho^{-n-1} dY_i; \qquad x_i^3 \int_{E_{n-1}} \dots \int f_i(Y_i) \varrho^{-1-n} dY_i;$$

$$x_i \int_{E_{n-1}} \dots \int f_i(Y_i)(y_j - x_j)(y_k - x_k) \varrho^{-n-2} dY_i, \qquad i \neq j, \ i \neq k$$

are uniformly convergent as $X \in W(\alpha, A, B)$.

Proof. The functions $\varrho^{-1}|x_i|$; $\varrho^{-2}|x_i(y_j - x_j)|$; $\varrho^{-3}|x_i^3|$ and $\varrho^{-3}|x_i(y_j - x_j)(y_k - x_k)|$ being not greater than 1, the integral $\int_{E_{n-1}} \dots \int f_i(Y_i) \varrho^{-n} dY_i$ is a common majorant for all the integrals we deal in Lemma 9. This together with Lemma 8 implies the assertion of Lemma 9.

Proof. By (4.1), $F_1 = I_1+I_2+I_3+I_4$, where

$$I_1 = 2\int_{E_{n-1}}\dots\int f_n^1(Y_n)[\sigma^2+(h-x_n)^2]^{\frac{1}{2}(2-n)}dY_n,$$

$$I_2 = 2\int_{E_{n-1}}\dots\int f_n^1(Y_n)[\sigma^2+(3h+x_n)^2]^{\frac{1}{2}(2-n)}dY_n,$$

$$I_3 = 2\int_{E_{n-1}}\dots\int f_n^1(Y_n)[\sigma^2+(5h-x_n)^2]^{\frac{1}{2}(2-n)}dY_n,$$

$$I_4 = 2\int_{E_{n-1}}\dots\int f_n^1(Y_n)\sum_{k=3}^{\infty}\{\sigma^2+[(2k+1)h(-1)^{2k+1}x_n]^2\}^{\frac{1}{2}(2-n)}dY_n.$$

By Lemma 12

$$|I_4| \leqslant \int_{E_{n-1}}\dots\int |f_n^1(Y_n)|\,h^{1-n}\sum_{i=3}^{\infty}(i)^{2-n}dY_n \leqslant \int_{E_{n-1}}\dots\int |f_n^1(Y_n)|\,dY_n \leqslant \int_{E_{n-1}}\dots\int \omega(r)\,dY_n.$$

Introducing the spherical coordinates (2.11) in the last integral we have

$$\int_{E_{n-1}}\dots\int \omega(r)\,dY_n = M\int_0^{\infty}\omega(r)r^{n-1}dr,$$

M being a constant. Since the last integral converges by our assumption it follows that the integral I_4 is uniformly convergent in the cylinder $W(\eta, A)$. By the triangle inequality

$$|\overline{XY}| \geqslant |\overline{OY}|-|\overline{OX}| \geqslant |\overline{OY}|-(\eta^2+A^2)^{\frac{1}{2}} \geqslant (Y_n^2)^{\frac{1}{2}}-(\eta^2+A^2)^{\frac{1}{2}}$$

whence

$$|\overline{XY}| \geqslant (Y_n^2)^{\frac{1}{2}}-\tfrac{1}{2}(Y_n^2)^{\frac{1}{2}} = \tfrac{1}{2}(Y_n^2)^{\frac{1}{2}} \quad \text{for } |Y_n| > R.$$

Thus

$$|I_1| \leqslant N\int_{E_{n-1}}\dots\int |f_n^1(Y_n)|\,[\tfrac{1}{2}(Y_n^2)^{\frac{1}{2}}]^{\frac{1}{2}(2-n)}dY_n,$$

N being a constant. Applying the spherical coordinates (2.11) we obtain

$$|I_1| \leqslant \int_0^{\infty}\omega(r)r^{2-n}r^{n-2}dr = N\int_0^{\infty}\omega(r)dr$$

which implies uniform convergence in $W(\eta, A)$ of the integral I_1 in $W(\eta, A)$, since it admits an uniformly convergent majorant. The integrals I_2 and I_3 may be majorized similary, since $|h-x_n| \leqslant |3h+x_n| \leqslant |5h-x_n|$, for $x_n \in (-h, +h)$ which leads to uniform convergence of these integrals

and consequently of the integral F_1. The proof concerning the remaining integrals is similar.

LEMMA 2c. *Let* $\beta_n = \int_{E_{n-1}} \dots \int (1+T_n^2)^{-\frac{1}{2}n} dT_n$, $x_n > h$; *then*

$$\frac{x_n - h}{\beta_n} \int_{E_{n-1}} \dots \int [\sigma^2 + (x_n - h)^2]^{-\frac{1}{2}n} dY_n = 1. \tag{4.5}$$

Proof. Applying the transformation

$$y_1 = x_1 + t_1(x_n - h), \dots, y_{n-1} = x_{n-1} + t_{n-1}(x_n - h) \tag{4.6}$$

with the Jacobi determinant $|J| = |x_n - h|^{n-1}$, we obtain

$$\frac{1}{\beta_n} \int_{E_{n-1}} \dots \int (x_n - h)^{1-n} (1+T_n^2)^{-\frac{1}{2}n} (x_n - h)^{n-1} dT_n,$$

whence results the Lemma 2c.

Remark. Multiplication of both sides of (4.5) by $f_n^1(X_n)$ gives

$$\frac{1}{\beta_n} \int_{E_{n-1}} \dots \int f_n^1(X_n)\,(x_n - h)\,[\sigma^2 + (x_n - h)^2]^{-\frac{1}{2}n} dY_n = f_n^1(X_n). \tag{4.7}$$

THEOREM 9. *Let* $f_n^1(Y_n)$ *and* $f_n^2(Y_n)$ *satisfy in* E_{n-1} $(n \geqslant 4)$ *the Hölder's condition with the exponent* $0 < \alpha < 1$ *and belong to* H_3; *then the solution of the Neumann problem for* Σ *with the boundary conditions*

$$\lim \frac{\partial u}{\partial x_n} = f_n^1(X_n) \quad \text{as } X \to (x_1, \dots, x_{n-1}, h), \tag{c}$$

$$\lim \frac{\partial u}{\partial x_n} = f_n^2(X_n) \quad \text{as } X \to (x_1, \dots, x_{n-1}, -h) \tag{d}$$

is of form

$$u(X) = \int_{E_{n-1}} \dots \int f_n^1(Y_n) G_2(X, Y)\Big|_{y_n = h} dY_n + \int_{E_{n-1}} \dots \int f_n^2(Y_n) G_2(X, Y)\Big|_{y_n = -h} dY_n, \tag{4.8}$$

where $G_2(X, Y)|_{y_n=h}$ *and* $G_2(X, Y)|_{y_n=-h}$ *are defined by* (4.1) *and* (4.2) *respectively.*

Proof. By Lemma 13 the function $u(X)$ is of class C^2 and is harmonic in Σ so the boundary conditions are only to be checked. Let

$$G_2(X, Y)|_{y_n=h} = G_3(X, Y), \quad G_2(X, Y)|_{y_n=-h} = G_4(X, Y).$$

By (4.1)

$$\frac{\partial G_3}{\partial x_n} = 2(2-n)\sum_{k=0}^{\infty}[(2k+1)h+(-1)^{k+1}x_n]\{\sigma^2+[(2k+1)h+ \\ +(-1)^{k+1}x_n]^2\}^{-n/2}. \tag{4.9}$$

By (4.2)

$$\frac{\partial G_4}{\partial x_n} = 2(2-n)\sum_{k=0}^{\infty}[(-1)^k(2h+1)h+x_n]\{\sigma^2+[(-1)^k(2k+ \\ +1)h+x_n]^2\}^{-\frac{1}{2}n}. \tag{4.10}$$

By (4.7)

$$\frac{\partial u}{\partial x_n} = \int_{E_{n-1}}\dots\int f_n^1(Y_n)\frac{\partial G_3}{\partial x_n}\,dY_n + \int_{E_{n-1}}\dots\int f_n^2(Y_n)\frac{\partial G_4}{\partial x_n}\,dY_n, \tag{4.11}$$

where $\frac{\partial G_3}{\partial x_n}$ and $\frac{\partial G_4}{\partial x_n}$ are defined by formulae (4.9) and (4.10), respectively. Let us prove the boundary condition (c). Taking into account (4.9), (4.10) and (4.11) we see that our proof reduces to formula

$$\lim_{x_n\to h}\frac{\partial u}{\partial x_n} = 2(2-n)\lim_{x_n\to h}\int_{E_{n-1}}\dots\int f_n^1(Y_n)\,(x_n-h)\,[\sigma^2+(x_n-h)^2]^{-\frac{1}{2}n}dY_n.$$

Since all the remaining integrals in (4.11) tend to zero as $x_n\to h$. Let

$$R_3(X) = \frac{1}{\beta_n}\int_{E_{n-1}}\dots\int f_n^1(Y_n)\,(x_n-h)\,[\sigma^2+(x_n-h)^2]^{-\frac{1}{2}n}dY_n - f_n^1(X_n).$$

By (4.7) implies

$$|R_3(X)| \leqslant \frac{1}{\beta_n}\int_{E_{n-1}}\dots\int |f_n^1(X_n)-f_n^1(Y_n)|\,|x_n-h|\,[\sigma^2+(x_n-h)^2]^{-\frac{1}{2}n}dY_n.$$

Applying the transformation (4.6) to the integral at the right-hand side we obtain

$$|R_3(X)| \leqslant \frac{1}{\beta_n}\int_{E_{n-1}}\dots\int |f_n^1[x_1+t_1(x_n-h),\dots,x_{n-1}+t_{n-1}(x_n-h)]- \\ -f_n^1(x_1,\dots,x_{n-1})|\,|x_n-h|\,(1+T_n^2)^{-\frac{1}{2}n}(x_n-h)^{-n}(x_n-h)^{n-1}dT_n.$$

By the Hölder condition

$$|R_3(X)| \leqslant \frac{|x_n - h|}{\beta_n} \int_{E_{n-1}} \dots \int (T_n^2)^{\alpha} (1 + T_n^2)^{-\frac{1}{2}n} dT_n$$

and, by Lemma 5, $R_3 \to 0$ as $x_n \to h$.

Similary we check the boundary condition (b). In this case we apply formulae (4.9), (4.10) and (4.11) and reduce the proof to the formula

$$\lim_{x_n \to -h} \frac{\partial u}{\partial x_n} = 2(2-n) \lim_{x_n \to -h} \int_{E_{n-1}} \dots \int f_n^1(Y_n)\,(x_n + h)\,[\sigma^2 + (x_n + h)^2]^{-\frac{1}{2}n} dY_n.$$

5. In this paragraph we construct the Geen function and solve the mixed Neumann–Dirichlet problem for the strip Σ in the case $n \geqslant 3$.

THEOREM 10. *The Green function for the unbounded strip Σ with the pole at X with Neumann boundary condition an the hyperplane $y_n = h$ and Dirichlet condition at the hyperplane $y_n = -h$ is of the form*

$$G_5(X, Y) = r^{2-n} - (r_1^{(2)})^{2-n} + (r_1^{(1)})^{2-n} - (r_2^{(1)})^{2-n} + H_5(X, Y), \tag{5.1}$$

where

$$H_5(X, Y) = \sum_{\nu=1}^{\infty} (-1)^{\nu} \{[(r_{2\nu}^{(2)})^{2-n} - (r_{2\nu+1}^{(2)})^{2-n} + (r_{2\nu+1}^{(1)})^{2-n} - (r_{2\nu+2}^{(1)})^{2-n}]\}, \tag{5.2}$$

i.e.

$$G_5(X, Y) = r^{2-n} + (r_1^{(1)})^{2-n} + H_6(X, Y), \tag{5.3}$$

where

$$H_6(X, Y) = \sum_{\nu=1}^{\infty} (-1)^{\nu} \{[(r_{\nu}^{(2)})^{2-n} + (r_{\nu+1}^{(2)})^{2-n}] + [(r_{\nu+1}^{(1)})^{2-n} + (r_{\nu+2}^{(1)})^{2-n}]\}.$$

Proof. We have for Y at the hyperplane π_1

$$r^{n-1} = (r_1^{(1)})^{n-1}, \quad (r_{\nu}^{(2)})^{n-2} = (r_{\nu+1}^{(2)})^{n-2}, \quad \nu = 1, 2, \dots$$

Hence by Lemma 10 $\left.\frac{\partial G_5}{\partial y_n}\right|_{y_n = h} = 0$. If Y belongs to the hyperplane π_2, then

$$r^{n-2} = (r_1^{(2)})^{n-2}, \quad (r_{\nu}^{(1)})^{n-2} = (r_{\nu+1}^{(1)})^{n-2}, \quad \nu = 1, 2, \dots$$

whence $G_5(X, Y) = 0$.

Let us prove now that series (5.2) and its derivative series $\frac{\partial H_5}{\partial y_i}$, $\frac{\partial^2 H_5}{\partial y_i^2}$, $i = 1, \dots, n$, are uniformly convergent. We shall first prove this in the case $n \geqslant 4$, the case $n = 3$ being treated separately.

Series (5.2) admits the majorant

$$\sum_{\nu=1}^{\infty}(-1)^{\nu}\{[(r_{2\nu}^{(2)})^{2-n}+(r_{2\nu+1}^{(2)})^{2-n}]+[(r_{2\nu+1}^{(1)})^{2-n}+(r_{2\nu+2}^{(1)})^{2-n}]\},$$

which is majorized by the series $h^{2-n}\sum i^{2-n}<\infty$, which implies uniform convergence of the series representing $H_5(X, Y)$. The proof of the uniform convergence of the derivative series: $\dfrac{\partial H_5}{\partial y_i}$ and $\dfrac{\partial^2 H_5}{\partial y_i^2}$, $i=1,\ldots,n$ is similar.

Let us now pass to the case $n=3$. By (5.1)

$$G_6(X, Y)=\frac{1}{r}-\frac{1}{r_1^{(1)}}-\frac{1}{r_1^{(2)}}+H_7(X, Y),$$

where

$$H_7(X, Y)=\sum_{\nu=3}^{\infty}(\gamma_\nu-\delta_\nu)-\frac{1}{r_2^{(1)}}$$

and

$$\gamma_\nu=\frac{1}{r_\nu^{(2)}}-\frac{1}{r_\nu^{(1)}},\qquad \delta_\nu=\frac{1}{r_{\nu+1}^{(1)}}-\frac{1}{r_{\nu-1}^{(2)}}.$$

This series converges uniformly as $|y_3|\leqslant h$, $|x_3|\leqslant h$. Indeed

$$|\varrho_2-\varrho_1|>|r_2-r_1|;\qquad |r_1|>\varrho_1,\ |r_2|>\varrho_2$$

whence

$$|\gamma_\nu|<\frac{|r_\nu^{(1)}-r_\nu^{(2)}|}{r_\nu^{(2)}r_\nu^{(1)}}\leqslant\frac{\big||\xi_\nu^{(1)}|-|\xi_\nu^{(2)}|\big|}{|\xi_\nu^{(2)}||\xi_\nu^{(1)}|}=\frac{2|x_n|}{|4h^2\nu^2-x_n^2|}\leqslant\frac{4h}{4\nu^2h^2-h^2}$$

$$\leqslant\frac{1}{h(\nu^2-1)}\leqslant\frac{1}{h(\nu^2-1)^2}\qquad \text{for } \nu>1.$$

Similarly

$$|\delta_\nu|\leqslant\frac{|r_{\nu-1}^{(2)}-r_{\nu+1}^{(1)}|}{r_{\nu+1}^{(1)}r_{\nu-1}^{(2)}}\leqslant\frac{\big||\xi_{\nu-1}^{(2)}|-|\xi_{\nu+1}^{(1)}|\big|}{|\xi_{\nu-1}^{(2)}||\xi_{\nu+1}^{(1)}|}=\frac{|2x_n-4h|}{(2\nu h-h)(2\nu h-3h)}$$

$$\leqslant\frac{6h}{(2\nu h-4h)(2\nu h-4h)}\leqslant\frac{3}{h(\nu-2)^2}\qquad \text{as } \nu>2.$$

Thus

$$|\gamma_\nu+\delta_\nu|\leqslant|\gamma_\nu|+|\delta_\nu|\leqslant[h(\nu-1)^2]^{-1}+3[h(\nu-2)^2]^{-1}\qquad \text{for } \nu>3.$$

Hence the series $H_7(X, Y)$ converges uniformly and the series $G_6(X, Y)$ almost uniformly in Σ.

The derivative series $\dfrac{\partial(H_7+r_2^{(1)})}{\partial y_i}$ and $\dfrac{\partial^2(H_7+r_2^{(1)})}{\partial y_i^2}$ are uniformly convergent in Σ. Indeed,

$$\frac{\partial(H_7+r_2^{(1)})}{\partial y_i} = \sum_{\nu=2}^{\infty}\{[r_\nu^{(1)}(r_\nu^{(1)})^2]^{-1}(y_i-x_i)-[r_{\nu+1}^{(2)}(r_{\nu+1}^{(2)})^2]^{-1}(y_i-x_i)\}+R_i,$$

where $R_i = \dfrac{\partial(r_2^{(1)})}{\partial y_i}$, and since $|(y_i-x_i)(r_k^{(j)})^{-1}|<1$ and $|R_i|<(4h^2)^{-1}$ as $i=1,2,3$, $k=2,3,\ldots$, $j=1,2$, we have

$$\left|\frac{\partial(H_7+r_2^{(1)})}{\partial y_i}\right| \leqslant \sum_{\nu=2}^{\infty}[(r_\nu^{(1)})^{-2}+(r_{\nu+1}^{(2)})^{-2}]+(4h^2)^{-1} \leqslant C_1\sum_{\nu=1}^{\infty}(\nu^2)^{-1}+(4h^2)^{-1}, \tag{5.4}$$

C_1 being a constant. Analogous estimations hold for the second derivatives.

THEOREM 11. *Let the functions $f_n^1(Y_n)$, $f_n^2(Y_n)$ of class H_3 in E_{n-1} $(n\geqslant 3)$ satisfy the Hölder condition with the positive exponent $\alpha<1$; then the solution of the mixed Neumann-Dirichlet problem for Σ with the boundary conditions*

$$\lim_{x_n\to h}\frac{\partial u}{\partial x_n} = f_n^1(X_n), \tag{e}$$

$$\lim_{x_n\to -h} u(X) = f_n^2(X_n) \tag{f}$$

is of the form

$$u(X) = \alpha_1\int_{E_{n-1}}\ldots\int f_n^1(Y_n)G_5(X,Y)\Big|_{y_n=h}dY_n+\alpha_2\int_{E_{n-1}}\ldots\int f_n^2(Y_n)\frac{\partial G_5}{\partial y_n}\Big|_{y_n=-h}dY_n, \tag{5.5}$$

α_1, α_2 being certain constants.

Proof. In the case $n\geqslant 4$, by (5.1) implies

$$G_5(X,Y)|_{y_n=h} = 2[\sigma^2+(h-x_n)^2]^{(2-n)/2}+ + \sum_{i=1}^{\infty}(-1)^{C_i}\{\sigma^2+[(2i+1)h+(-1)^{i+1}x_n]^2\}^{(2-n)/2}, \tag{5.6}$$

where $C_i=\frac{1}{2}i(i+1)$, and

$$\frac{\partial G_5(X,Y)}{\partial y_n}\Big|_{y_n=-h} = 2(2-n)\sum_{i=1}^{\infty}(-1)^{C_i}\{\sigma^2+[x_n+(-1)^{i+1}(2i+1)h]^2\}^{(2-n)/2}. \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that the solution of the mixed problem takes on the form

$$u(X) = S_1(X) + S_2(X), \tag{5.8}$$

where

$$S_1(X) = \beta_1 \int_{E_{n-1}} \dots \int f_n^1(Y_n) \Big\{ [\sigma^2 + (h - x_n)^2]^{\frac{1}{2}(2-n)} + \sum_{k=1}^{\infty} (-1)^k \Big[\big(\sigma^2 + \big((4k-1)h + x_n\big)^2\big)^{(2-n)/2} + \big(\sigma^2 + (4k+1)h - x_n\big)^2\big)^{(2-n)/2} \Big] \Big\} dY_n$$

and

$$S_2(X) = \beta_2 \int_{E_{n-1}} \dots \int f_n^2(Y_h) \sum_{k=0}^{\infty} (-1)^{k+1} \{ [(4k+1)h + x_n] [\sigma^2 + ((4k+1)h + x_n)^2]^{-n/2} + \big((4k+3)h - x_n\big) \big[\sigma^2 + \big((4k+3)h - x_n\big)^2\big]^{-n/2} \} dY_n$$

β_1, β_2 being certain constants.

As you can see the obtained integrals $S_i(X)$, $i = 1, 2$ as well as their derivatives of the first and the second order with respect to x_i are integrals of $F_i(X)$ type, $i = 1, \dots, 6$, from Lemma 13, hence they are uniformly convergent in every set $W(\alpha, A)$. The first integral in the formula representing $S_1(X)$ is, by Lemma 9, uniformly convergent together with the derivatives of the first and the second order whence the function $u(X)$ defined by formula (5.5) is of the class C^2 in Σ. Taking into consideration the above properties and the fact that the function $G_5(X, Y)$ is harmonic in Σ we get from (5.5)

$$\Delta u(X) = \alpha_1 \int_{E_{n-1}} \dots \int f_n^1(Y_n) [\Delta_X G(X, Y)] \Big|_{y_n = h} dY_n + \alpha_2 \int_{E_{n-1}} \dots \int f_n^2(Y_n) \left\{ \frac{\partial}{\partial y_n} [\Delta_X G_5(X, Y)] \right\} \Big|_{y_n = -h} dY_n = 0$$

for $X \in \Sigma$.

In the tree dimensional case ($n = 3$) by (5.8) we get

$$u(X) = \beta_1 \iint_{E_2} f_3^1(Y_3) [\sigma^2 + (h - x_3)^2]^{-1/2} dY_3 + \beta_2 \iint_{E_2} f_3^2(Y_3) [\sigma^2 + (h + x_3)]^{-3/2} dY_3 + V_1(X) + V_2(X), \tag{5.9}$$

where

$$V_1(X) = \beta_1 \iint_{E_2} f_3^1(Y_3) \Gamma_1(X, Y) dY_3, \qquad V_2(X) = \beta_2 \iint_{E_2} f_3^2(Y_3) \Gamma_2(X, Y) dY_3$$

and

$$\Gamma_1(X, Y) = \sum_{k=1}^{\infty} (-1)^k \{ [\sigma^2 + ((4k-1)h + x_3)^2]^{-1/2} + [\sigma^2 + ((4k+1)h - x_3)^2]^{-1/2} \}$$

and

$$\Gamma_2(X, Y) = \sum_{k=1}^{\infty} \{(-1)^k [(4k-1)h - x_3] \left[\sigma^2 + \left((4k-1)k - x_3\right)^2\right]^{-3/2} +$$

$$+ (-1)^{k+1} [(4k+1)h + x_3] \left[\sigma^2 + \left((4k+1)h + x_3\right)^2\right]^{-3/2}\}.$$

We can obtain similar majorants for the series $\Gamma_1(X, Y)$ and $\Gamma_2(X, Y)$ as for the series $H_7(X, Y)$. Thus both are uniformly convergent. Let

$$\frac{\partial V_1}{\partial x_i} = \iint_{E_2} f_3^1(Y_3) \frac{\partial \Gamma_1}{\partial x_i} dY_3, \qquad \frac{\partial V_2}{\partial x_i} = \iint_{E_2} f_3^2(Y_3) \frac{\partial \Gamma_2}{\partial x_i} dY_3,$$

$$\frac{\partial^2 V_2}{\partial x_i^2} = \iint_{E_2} f_3^1(Y_3) \frac{\partial^2 \Gamma_1}{\partial x_i^2} dY_3, \qquad \frac{\partial^2 V_2}{\partial x_i^2} = \iint_{E_2} f_3^2(Y_3) \frac{\partial^2 \Gamma_2}{\partial x_i^2} dY_3,$$

where $i = 1, 2, 3$. The functions $\dfrac{\partial \Gamma_j}{\partial x_i}$, $\dfrac{\partial^2 \Gamma_j}{\partial x_i^2}$, $j = 1, 2$, $i = 1, 2, 3$, admit majorants of type (5.4). This easily implies uniform convergence in Σ of the series representing $V_1(X)$ and $V_2(X)$. By Lemma 9 two first integrals in (5.9) together with their first and second order derivatives are uniformly convergent, whence the function defined by (5.9) is of class C^2 in Σ. We shall prove now that the function defined by formula (5.9) is harmonic in Σ. Since functions $[\sigma^2 + (h - x_3)^2]^{-1/2}$, $[\sigma^2 + (h + x_3)^2]^{-3/2}$, $\Gamma_1(X, Y)$ and $\Gamma_2(X, Y)$ are harmonic in Σ we get from (6.9): $\Delta u(X) = 0$ for $X \in \Sigma$. It remains to check the boundary conditions it is case to see from (5.8) for $n \geqslant 3$ that

$$\lim_{x_n \to -h} S_1(X) = 0 \quad \text{and} \quad \lim_{x_n \to -h} S_2(X) = f_n^2(X_n)$$

whence condition (f) is satisfied. By (5.8)

$$\frac{\partial u}{\partial x_n} = S_3(X) + S_4(X), \tag{5.10}$$

where

$$S_3(X) = \frac{\partial S_1}{\partial x_n}$$

$$= \beta_3 \int_{E_{n-1}} \dots \int f_n^1(Y_n) \Big\{ \sum_{k=0}^{\infty} (-1)^k [x_n - (4k+1)h] \left[\sigma^2 + \left(x_n - (4k+1)h\right)^2\right]^{-n/2} +$$

$$+ (-1)^{k+1} [x_n - (4k+3)h] \left[\sigma^2 + \left(x_n - (4k+3)h\right)^2\right]^{-n/2} \Big\} dY_n,$$

and

$$S_4 = \frac{\partial S_2}{\partial x_n}$$

$$= \int_{E_{n-1}} \dots \int f_n^2(Y_n) \Big\{ n \sum_{k=0}^{\infty} \Big[(-1)^{k+1} (x_n + (4k+1)h)^2 \big[\sigma^2 + (x_n + (4k+1)h)^2\big]^{-\frac{1}{2}(n+1)} +$$

$$+ (-1)^{k+2} (x_n - (4k+3)h)^2 \big[\sigma^2 + (x_n - (4k+3)h)^2\big]^{-\frac{1}{2}(n+2)} \Big] + \sum_{k=0}^{\infty} \Big[(-1)^k \big[\sigma^2 +$$

$$+ (x_n + (4k+1)h)^2\big]^{-\frac{1}{2}n} + (-1)^{k+1} \big[\sigma^2 + (x_n + (4k+3)h)^2\big]^{-\frac{1}{2}n} \Big] \Big\} dY_n.$$

It follows easily from (5.10) that

$$\lim_{x_n \to h} S_3(X) = f_n^1(X_n) \quad \text{and} \quad \lim_{x_n \to h} S_4 = 0,$$

which implies the boundary condition (e).

IV. THE SOLUTION OF THE DIRICHLET PROBLEM FOR THE POISSON'S EQUATION IN E_3^+

6. We shall solve the Poisson's equation

$$\Delta u(X) = -F(X) \tag{6.1}$$

with the boundary condition being

$$u(X) = f(X), \tag{6.1a}$$

where $f(0, y_2, y_3) = f_1(y_2, y_3) = f_1(Y_1)$, $f(y_1, 0, y_3) = f_2(y_1, y_3) = f_2(Y_2)$, $f(y_1, y_2, 0) = f_3(y_1, y_2) = f_3(Y_3)$, where function $F(X)$ is of class C^1 in E_3^+ and the function $f(X)$ is continuous on the set $y_i = 0$, $i = 1, 2, 3$. It is known ([2], p. 376, vol. I) that the solution of the problem under consideration is the function of form

$$u(X) = u_1(X) + u_2(X), \tag{6.2}$$

where

$$u_1(X) = \frac{1}{4\pi} \iint_{\partial E_3^+} f(Y) \frac{\partial G(X, Y)}{\partial n_Y} d\sigma_Y, \tag{6.3}$$

and

$$u_2(X) = \frac{1}{4\pi} \iiint_{E_3^+} G(X, Y) F(Y) dV_Y, \tag{6.4}$$

where $X \in E_3^+$ and $G(X, Y)$ denotes the Green function defined by (3.1b), in case of three dimensions ($n = 3$).

The function $u_1(X)$ is harmonic and satisfies the boundary condition (6.1a). We dealt with it in chapter II, 3.

We shall prove that the function $u_2(X)$ satisfies equation (6.1) and the zero boundary condition. Let us examine this function. In the case $n = 3$ we have

$$(6.4a)\quad u_2(X) = \frac{1}{4\pi}\iiint\limits_{E_3^+} \{[(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2]^{-\frac{1}{2}} -$$

$$-[(y_1-x_1)^2+(y_2+x_2)^2+(y_3-x_3)^2]^{-\frac{1}{2}}+[(y_1+x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2]^{-\frac{1}{2}}+$$

$$+[(y_1+x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2]^{-\frac{1}{2}}+[(y_1-x_1)^2+(y_2+x_2)^2+(y_3+x_3)^2]^{-\frac{1}{2}}-$$

$$-[(y_1-x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2]^{-\frac{1}{2}}+[(y_1+x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2]^{-\frac{1}{2}}-$$

$$-[(y_1+x_1)^2+(y_2+x_2)^2+(y_3+x_3)^2]^{-\frac{1}{2}}\}F(Y)dV_Y.$$

Let us consider the integral

$$(6.5)\qquad Q(X) = \frac{1}{4\pi}\iiint\limits_{E_3^+} K_1(X, Y)F(Y)dV_Y,$$

where $K_1(X, Y) = [(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2]^{-\frac{1}{2}}$.

The function $F(Y)$ belong to the class H_4 it satisfies the Hölder condition in E_3^+ with a positive exponent $\alpha < 1$ and the integral $\int_0^\infty \omega_1(r)rdr$, where $\omega_1(r) = \max\limits_{Y^2=r^2}|F(Y)|$ is convergent.

LEMMA 14. *Let the function $F(Y)$ belong to the class H_4; then function* (6.4a) *is of class C^2 and satisfies equation* (6.1) *in E_3^+.*

Proof. Let D_1 denote domain belongs in $D_2 \subset E_3^+$ and let $X \in D_2$. Let $Q(X) = Q_1(X)+Q_2(X)$, where

$$Q_1(X) = \iiint\limits_{D_2} K_1(X, Y)F(Y)dY \text{ and } Q_2(X) = \iiint\limits_{E_3^+-D_2} K_1(X, Y)F(Y)dY.$$

The integral $Q_2(X)$ is uniformly convergent together with first and second order derivatives with respect to $X \in D_1$. Indeed, by the triangle inequality: $r-\varrho_1 < \varrho$; $\varrho_1 < (2A^2+B^2)^{1/2} = M$, $r-M < \varrho$, $\frac{1}{2}r < \varrho$, $r > r_0$. Thus

$$\iiint\limits_{C(\Omega_\delta)} |f(Y)|K_1(X, Y)dY < \iiint\limits_{C(\Omega_\delta)} \omega_1(r)2r^{-1}dY,$$

where $D_2 \subset \Omega_s \subset E_3^+$, and $C(\Omega_s)$ is the set of the points satisfaing the inequality $y_1^2+y_2^2+y_3^2 \geqslant s^2$, $y_1 \geqslant 0$, $y_2 \geqslant 0$, $y_3 \geqslant 0$. Applying in the last integral transformation (2.10) we give

$$C_1 \int_s^{+\infty} \omega_1(r) r\, dr, \tag{6.6}$$

where C_1 is some constant. The functions $\dfrac{\partial Q_2}{\partial x_i}$ and $\dfrac{\partial^2 Q_2}{\partial x_i^2}$ are uniformly convergent for $X \in D_2$, since they have majorants of type (6.6). This implies that the integral $Q_2(X)$ is of class C^2 in E_3^+. By [4], p. 120, vol. II for $X \in D_2$, $\Delta Q_1(X) = -4\pi F(X)$. Since $\Delta_X K_1(X, Y) = 0$ for $X \neq Y$, we get

$$\Delta Q(X) = \Delta Q_1(X)+\Delta Q_2(X) = -4\pi F(X)+ \iiint\limits_{E_3^+-D_2} F(Y)\Delta_X K_1(X, Y)\,dY$$
$$= -4\pi F(X).$$

The remained integrals

$$\iiint\limits_{E_3^+} K_i(X, Y)F(Y)\,dV_Y, \quad i = 2, \ldots, 7,$$

in the form of (6.4a) are also the harmonic functions of points X belonging to $E_3^+ - D_2$, where can be treated analogically as in the integral $Q_2(X)$, because $K_i(X, Y)$ are the harmonic functions of points X outside the space E_3^+. Hence $\Delta u_2(X) = -F(X)$.

LEMMA 15. *Let the function $F(X)$ be integrable and bounded in E_3^+. Then $u_2(X) \to 0$ as X tends to a point $X_0 \in \partial E_3^+$ through the set E_3^+.*

Proof. We have

$$\left|\iiint\limits_{E_3^+} G(X, Y)F(Y)\,dV_Y\right| \leqslant M \iiint\limits_{E_3^+} G(X, Y)\,dV_Y,$$

where $M = \sup\limits_{X \in E_3^+} |F(X)|$. Let K_ϱ denote the part of the ball $y_1^2+y_2^2+y_3^2 \leqslant \varrho^2$ defined by $y_1 \geqslant 0$, $y_2 \geqslant 0$, $y_3 \geqslant 0$, and let $K_{\varrho_1}(X)$ be the ball with centre X and radius $\varrho_1 < \varrho$. Then

$$\iiint\limits_{K_\varrho} G(X, Y)\,dV_Y = Z_1(X)+Z_2(X),$$

where $Z_1(X) = \iiint\limits_{K_{\varrho_1}} G(X, Y)\,dV_Y$ and $Z_2(X) = \iiint\limits_{K_\varrho - K_{\varrho_1}} G(X, Y)\,dV_Y$. By weak singularity of the kernel $G(X, Y)$, $Z_1(X) \to 0$ as $\varrho_1 \to 0$. Applying the mean value theorem to the integral $Z_2(X)$ we obtain

$$Z_2(X) \leqslant \tfrac{4}{3}\pi\varrho^3 G(X, \overline{Y}),$$

where $\overline{Y} \in K_\varrho - K_{\varrho_1}$. Since $G(X, Y)$ is symmetric and since it vanishes on the set $\partial(K_\varrho - K_{\varrho_1})$ we see that $G(X, Y) < \varrho^{-4}$ for X sufficiently close to X_0. Thus

$$\iiint_{K_\varrho - K_{\varrho_1}} G(X, Y)\,dV_Y \leqslant \varrho^{-4}\,\tfrac{4}{3}\pi\varrho^3 = \frac{\pi}{4\varrho} < \varepsilon \quad \text{for } \varrho > \varrho_0.$$

Therefore $\iiint_{E_3} G(X, Y)\,dV_Y \to 0$, as $X \to X_0 \in \partial E_3^+$.

Lemmas 14 and 15 imply

THEOREM 12. *Let $F(X)$ be a bounded function in E_3^+ of class H_4; then the function $u_2(X)$ defined by formula* (6.4) *satisfies* (6.1) *and the zero boundary condition.*

Theorems 6 and 12 imply

THEOREM 13. *Let $u_1(X)$ be a harmonic function satisfying the boundary condition* (6.1a); *let the function $f(Y)$ be in H_4 let it satisfy the Hölder condition with an positive exponent $\alpha < 1$ and let the assumptions of Lemma* 1b *be satisfied. Then the solution of Dirichlet problem in E_3^+ for equation* (6.1) *with the boundary conditions* (6.1a) *is the function defined by* (6.2).

References

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