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Some remarks on Toeplitz methods and continuity*

1. DEFINITION. Let f be a real function, $f\colon R \to R$, and let A be a Toeplitz method of summability. We shall say that f is *A-continuous* if, for every A-summable sequence (t_n), the sequence $(f(t_n))$ is also A-summable.

The following facts are simple consequences of the above definition:

1.1. Given a Toeplitz method A, the set of all A-continuous functions contains all linear functions and is closed with respect to superposition of functions.

1.2. A function f is continuous iff it is continuous in the sense of identity method (I-continuous).

Moreover, for a wide class of Toeplitz methods A-continuity implies continuity:

THEOREM 1. *Let A be a permanent Toeplitz method. If f is A-continuous, then f is continuous in ordinary sense.*

Proof. Suppose the contrary. Let x_0 be a point of discontinuity of f. Suppose, first, that there exist two sequences (x_n) and (y_n), each converging to x_0, and such that $f(x_n) \to a$ and $f(y_n) \to b$, $a \neq b$. In view of 1.1 we can assume without loss of generality that $a = 0$ and $b = 1$. As it is well known, [1], for every permanent Toeplitz method there exists a sequence (ε_n) which is not A-summable and such that

$$\varepsilon_n = \begin{cases} 0 & \text{for } n = n_k, \\ 1 & \text{for } n = n'_k, \end{cases} \qquad k = 0, 1, \ldots,$$

the sequences (n_k), (n'_k) exhausting all non-negative integers. Let (t_n) be defined as follows:

$$t_n = \begin{cases} x_k & \text{for } n = n_k, \\ y_k & \text{for } n = n'_k, \end{cases} \qquad k = 0, 1, \ldots$$

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Since the sequence (t_n) is convergent (with x_0 as its limit), it is also A-summable. The sequence $(f(t_n))$ can be written in the form

$$f(t_n) = \varepsilon_n + c_n,$$

where $c_n \to 0$. Therefore $(f(t_n))$ is not A-summable, what contradicts the hypothesis. To complete the proof, we suppose now that there exists a sequence (x_n) such that $x_n \to x_0$ and $f(x_n) \to \infty$. The method A being permanent, it is easy to find a sequence (t_n) of the form

$$x_0, \ldots, x_0, x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k, \ldots$$

(obviously convergent), such that $(f(t_n))$ is not A-summable (it is A-summable to infinity). Therefore f must be continuous at x_0.

THEOREM 2. *If a function f is A-continuous for every permanent Toeplitz method A, then f is a linear function.*

Proof. The theorem is implied by the following

EXAMPLE 1. There exists a permanent Toeplitz method A such that the only A-continuous functions are linear functions. Let A be defined as follows:

$$\begin{aligned} A_{3k}((t_n)) &= -rt_{3k} + rt_{3k+1} + t_{3k+2}, \\ A_{3k+1}((t_n)) &= A_{3k+2}((t_n)) = t_{3k+1}, \end{aligned} \qquad k = 0, 1, \ldots; 0 \neq r \neq 1.$$

It is easy to verify that a sequence (t_n) is A-summable iff it can be written in the form

$$\begin{aligned} t_{3k} &= s_k, \quad t_{3k+1} = b_k, \\ t_{3k+2} &= (1-r)b_k + rs_k + d_k, \end{aligned} \qquad k = 0, 1, \ldots,$$

where (s_k) is arbitrary sequence, (b_k) — convergent sequence, and (d_k) is a sequence convergent to 0. Since f is A-continuous, we can find (for every (s_k), (b_k), (d_k)) a sequence (d'_k), $d'_k \to 0$, such that

$$f((1-r)b_k + rs_k + d_k) = (1-r)f(b_k) + rf(s_k) + d'_k.$$

In particular, let $b_k = b$, $d_k = 0$ $(k = 0, 1, \ldots)$ and let (s_k) be dense in R. For any real s we can find a sequence of indices (n_k) such that $s_{n_k} \to s$. Taking into account that f is continuous (Theorem 1) and that $d'_{n_k} \to 0$, we obtain

$$f((1-r)b + rs) = (1-r)f(b) + rf(s).$$

Without loss of generality we can assume that $f(0) = 0$. Simple reasoning leads now to the conclusion, that the identity

$$f(x+y) = f(x) + f(y)$$

holds for arbitrary reals x, y. Since the function f is continuous, it must be linear.

The question arises if there exist a permanent Toeplitz method $A \neq I$ and a non-linear function f, such that f is A-continuous. An answer to this is given by the following

EXAMPLE 2. Let $A = A_{(2^n)}$ be so called *single-sequence method*, i.e. a permanent Toeplitz method such that the only A-summable sequences (d_n) are

$$d_n = a2^n + c_n,$$

where a denotes arbitrary constant, (c_n) — arbitrary convergent sequence (cf. [2], p. 48). Let g_0 be an arbitrary continuous function defined on the interval $\langle -1, 1\rangle$, vanishing at each end of the interval. Let

$$g(t) = \begin{cases} g_0(t-2^k) & \text{for } t \in \langle 2^k-1, 2^k+1\rangle,\ k = 1, 2, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

It is easy to verify that the sequence (y_n), $y_n = g(a2^n + c_n)$, is convergent for any a and (c_n). Therefore g is A-continuous.

The function constructed in the above example is A-continuous because of its very special behaviour at infinity. Now we shall state some necessary conditions a function must satisfy in order to be continuous in the sense of some non-trivial Toeplitz method.

2. A Toeplitz method A is said *to have a strict rate of growth* if there exists a sequence (ϑ_n) such that the following conditions are fulfilled:

— $(\vartheta_n t_n)$ is bounded for every A-summable sequence (t_n);

— if for some sequence (σ_n) the sequence $(\sigma_n t_n)$ is bounded for every A-summable (t_n), then $\sigma_n = O(\vartheta_n)$.

LEMMA 1. *The following conditions can not be satisfied simultaneously*:

(a) *A Toeplitz method A has a strict rate of growth*;

(b) *a function f is A-continuous*;

(c) *there exists an A-summable sequence* (t_n) *such that for some sequence of indices* (n_k), $t_{n_k} \to +\infty$, *and the quotient* $f(t_{n_k})/t_{n_k}$ *tends to infinity.*

Proof. Let (ϑ_n) be a strict rate of growth for A. Since the function f is A-continuous, there exist such constants M_1 and M_2 that for every n we have $|\vartheta_n t_n| \leqslant M_1$, $|f(t_n)\vartheta_n| \leqslant M_2$. Let $\theta_k = f(t_{n_k})/t_{n_k}$, and

$$\sigma_n = \begin{cases} \vartheta_n & \text{for } n \neq n_k \\ \vartheta_{n_k}|\theta_k| & \text{for } n = n_k \end{cases} \qquad (k = 0, 1, 2, \ldots).$$

Since $|t_n \sigma_n| \leqslant \max(M_1, M_2)$ and (ϑ_n) is a strict rate of growth, there must exist a constant K such that $\sigma_n \leqslant K\vartheta_n$. This, however, contradicts condition (c).

We shall say that two Toeplitz methods are equivalent if their fields are equal. A method A is said to be equivalent to a method extracted

from I, if there exists an increasing sequence of indices (n_k) such that a sequence (t_n) is A-summable iff its subsequence (t_{n_k}) is convergent.

The following are simple consequences of Lemma 1:

COROLLARY 2.1. *Let A be a Toeplitz method such that the method $\overline{A}$ obtained from A by omitting all zero-columns is convergence preserving Toeplitz method with a strict rate of growth. Let f be A-continuous real function such that $|f(t)/t| \to \infty$ when $t \to +\infty$ $(t \to -\infty)$. Then A is equivalent to a method extracted from I.*

Proof. It is sufficient to observe that $\overline{A}$ is equivalent to I. Suppose the contrary. Then there exists an unbounded from above $\overline{A}$-summable sequence (t_n) and a sequence of indices (n_k) such that $t_{n_k} \to +\infty$. Now we can apply Lemma 1.

COROLLARY 2.2. *Let A have a strict rate of growth. If there exists an A-summable sequence (t_n) dense in a half-line, then every A-continuous function f must satisfy the condition $\overline{\lim}|f(t)/t| < +\infty$ when $t \to +\infty$.*

Proof. The hypotheses of Lemma 1 are satisfied.

The following weaker results are valid for permanent methods which not necesarilly have a strict rate of growth.

LEMMA 2. *If $A = (a_{kn})$ is permanent for null-sequences (i.e. sequences convergent to zero) and if f is A-continuous function, then for every A-summable sequence (t_n) and every convergent to zero sequence τ_n the following equality holds:*

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big) = 0.$$

Proof. Let (ε_n) be an arbitrary sequence with terms equal to 0 or 1, let (t_n) and (τ_n) satisfy the hypotheses of the lemma. Since $f(t_n+\varepsilon_n\tau_n)-f(t_n) = \varepsilon_n\big(f(t_n+\tau_n)-f(t_n)\big)$ and the sequence $\big(f(t_n+\varepsilon_n\tau_n-f(t_n)\big)$ is A-summable, the limit

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big)\varepsilon_n$$

does exist. Applying Schur lemma (cf. [1], p. 133) to the matrix (b_{kn}), $b_{kn} = a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big)$ we conclude that $\sum_n |b_{kn}| \underset{k}{\to} 0$. The proof is completed.

Now we can prove what follows:

COROLLARY 2.3. *Let $A = (a_{kn})$ be a permanent Toeplitz method such that there exists an A-summable sequence (t_n), $t_n \to +\infty$. If a function f satisfies the condition: $f'(t) \to +\infty$ when $t \to +\infty$, and if f' is increasing for $t > T$, then f is not A-continuous function.*

Proof. Let (t_n), $t_n \to +\infty$, be an A-summable sequence. Assuming $\tau_n = \big(f'(t_n)\big)^{-1}$ and taking into account that $f(t_n+\tau_n)-f(t_n) = f'(t_n+$

$+\theta_n\tau_n)\tau_n \geqslant f'(t_n)\tau_n = 1$, we conclude in view of Lemma 2 that $\sum_n |a_{kn}| \to 0$. This contradicts the permanency of the method A.

In all we have proved above some additionary assumptions about the structure of the field of the method under consideration were made. Now we shall prove

THEOREM 3. *Let a differentiable function f satisfy the following condition: there exist $\varepsilon > 0$ and $T > 0$ such that $|f'(t)/t| \geqslant \varepsilon$ for $t \geqslant T$. If f is A-continuous for a convergence preserving Toeplitz method $A = (a_{kn})$, then A is equivalent to a method extracted from I.*

Proof. We can assume that the matrix (a_{kn}) contains no zero-columns, and under this assumption we shall prove that A is equivalent to I.

(1) We shall prove first, that for every A-summable sequence (t_n) there exists a constant $M = M((t_n))$ such that $\sum_n |a_{kn}t_n| \leqslant M$ for $k = 0, 1, 2, \ldots$ Let f satisfy the assumptions of the theorem, let ϱ be arbitrary real number such that $|\varrho| \leqslant 1/N$, where $N = (T+1)/\varepsilon T$, and let $|t| \geqslant T+1$. Since the function f is increasing (or decreasing) on the interval $\langle t-1, t+1\rangle$ ($f'(t)$ exists and does not vanish), we conclude that one of the numbers

$$\frac{f(t+N\varrho)-f(t)}{t}, \quad \frac{f(t-N\varrho)+f(t)}{t}$$

is positive, and the other is negative. It is easy to verify that absolute value of each of them is not smaller than $|\varrho|$. It means that the equation

$$\frac{f(t+\tau)-f(t)}{t} = \varrho$$

(with unknown τ, and fixed ϱ and t) has at least one solution in the interval $|\tau| \leqslant N|\varrho|$.

Let (t_n) be an arbitrary A-summable sequence, let (t_{n_i}) be a subsequence consisting of all t_n such that $t_n \geqslant T+1$. If (ϱ_i) is an arbitrary sequence convergent to zero and such that $|\varrho_i| \leqslant N^{-1}$ for every i, then for every equation

$$\frac{f(t_{n_i}+\tau)-f(t_{n_i})}{t_{n_i}} = \varrho_i$$

we can find a solution τ_{n_i}, $|\tau_{n_i}| \leqslant N\varrho_i$. Obviously $\tau_{n_i} \underset{i}{\to} 0$. Take now $\tau_n = 0$ for $n \neq n_i$ ($i = 0, 1, 2, \ldots$). Since f is A-continuous, the sequence (s_n),

$$s_n = f(t_n+\tau_n)-f(t_n),$$

is A-summable and the limit

$$\lim_k \sum_n a_{kn}\big(f(t_n+\tau_n)-f(t_n)\big) = \lim_k \sum_i a_{kn_i}t_{n_i}\varrho_i$$

does exist. But the sequence (ϱ_i) was arbitrarily chosen, and the existence of the right-hand limit means that every null-sequence (ϱ_i) (with $|\varrho_i| \leqslant N^{-1}$) is summable by the method (b_{ki}), $b_{ki} = a_{kn_i} t_{n_i}$. Therefore every null-sequence is (b_{ki})-summable and there exists a positive number $K = K\big((t_{n_i})\big)$ such that $\sum_i |a_{kn_i} t_{n_i}| \leqslant K$ for every k. Since there exists a constant L such that $\sum_n |a_{kn}| \leqslant L$ for every k, we obtain

$$\sum_n |a_{kn} t_n| \leqslant K + L(T+1) = M,$$

what was to be proved.

(2) Now we shall prove that A is equivalent to I (this part of the proof is a modification of the proof of Theorem 3 of [1]). Let $\alpha_n = \sup_k |a_{kn}|$, $\gamma = \inf_n \alpha_n$. It is sufficient to prove that γ is bigger than 0. Indeed, if $\gamma > 0$, then by the first part of the proof we have $|t_n| \leqslant M/\alpha_n \leqslant \gamma^{-1} M$. That means that every A-summable sequence is bounded, and A must be equivalent to I (cf. Theorem 7 of [1]). It remains to prove that $\gamma > 0$. Suppose, then, that $\gamma = 0$. There exists an increasing sequence of indices (n_m) such that $\sum_m \alpha_{n_m}^{1/3} < +\infty$. From what we know of f it follows that there exists such a number $T' > 0$ that for every $t \geqslant T'$ we have: $f(t) \geqslant t^2 \varepsilon/2 + C$ for some constant C. We can assume that $\alpha_{n_m}^{-2/3} \geqslant T'$, $m = 0, 1, 2, \ldots$ Let

$$t_n = \begin{cases} \alpha_{n_m}^{-2/3} & \text{for } n = n_m,\ m = 0, 1, \ldots, \\ 0 & \text{elsewhere.} \end{cases}$$

The sequence (t_n) is A-summable, for the series $\sum_m |a_{kn_m} \alpha_{n_m}^{-2/3}|$ converges uniformly with respect to k. The sequence (v_n), $v_n = f(t_n) - C$ is also A-summable then. In view of (1) we have: $\sum_n |a_{kn}(f(t_n) - C)| \leqslant M\big((v_n)\big) = M_1$. The last inequality leads to $\alpha_{n_m} \geqslant (\varepsilon/2M_1)^3$, what contradicts the hypothesis that $\alpha_{n_m} \underset{m}{\to} 0$, and that completes the proof.

Some minor generalisations of the theorem are possible, e.g.:

Let for a differentiable function f there exist $\alpha > 0$, $\varepsilon > 0$ and $T > 0$ such that $\big|f'(t)/|t|^{\alpha}\big| \geqslant \varepsilon$ for $|t| \geqslant T$. If f is A-continuous for a convergence preserving Toeplitz method A, then A is equivalent to a method extracted from I.

Proof. The function $f^m = f \circ f \circ \ldots \circ f$, for sufficiently large positive integer m, satisfies the assumptions of the preceding theorem.

4. A fairly good characterisations of some classes of A-continuous functions related to Toeplitz methods of some types can be given (cf. Example 1). However, in all such examples known to the author, the

existence of non-linear A-continuous functions is connected with the fact that the method A does not sum any bounded divergent sequence.

It is open question whether there exists a Toeplitz method A which sums some divergent bounded sequences and which yields a non-linear A-continuous function. Author believes that the answer is negative. In any case, the notion of A-continuity does not seem to be interesting.

References

[1] S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Math. 14 (1954).

[2] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin 1958.