ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVI (1972)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVI (1972)

cm ptm 1955

Ministerstwo Nauki i Szkolnictwa Wyższego

HELENA MUSIELAK (Poznań)

Inequalities for the norms of some functions defined by Stieltjes integrals

In this paper a theorem of Civin [2] concerning the derivatives of fractional orders is extended correspondingly.

1. Let a complex-valued function $\omega(t)$ of bounded variation in the interval $\langle -R, R\rangle$ be given, and let

$$f_0(x) = \int_{-R}^{R} e^{ixt}\,d\omega(t) \quad \text{for } x\in(-\infty, \infty).$$

The derivative of order α $(\alpha > 0)$ of the function $f_0(x)$ is defined as the integral

$$f_0^{(\alpha)}(x) = \int_{-R}^{R} i^\alpha t^\alpha e^{ixt}\,d\omega(t).$$

In place of these two integrals one can consider more general functions

$$f(x) = \int_{-R}^{R} \gamma(t) e^{ixt}\,d\omega(t)$$

and

$$g(x) = \int_{-R}^{R} \mu(t)\gamma(t) e^{ixt}\,d\omega(t),$$

where $\mu(t)$ and $\gamma(t)$ are continuous functions in $\langle -R, R\rangle$. Writing $p(\nu) = \nu\pi/R$ $(\nu = 0, \pm 1, \pm 2, \ldots)$, let us suppose, that for the suitable numbers a, s $(a \geqslant 0,\ 0 < s \leqslant 1)$,

$$\mu(t)e^{iat} = \sum_{\nu=-\infty}^{\infty} c_\nu e^{ip(\nu)t} \quad \text{for } t\in\langle -R, R\rangle,$$

and

$$\sum_{\nu=-\infty}^{\infty} |c_\nu|^s < \infty.$$

Then

$$g(x) = \sum_{\nu=-\infty}^{\infty} c_\nu f[p(\nu) - a + x]$$

for every real x (see [2], and [5], p. 229).

In the next Section we shall show the following fundamental inequality

$$\|g\| \leqslant \|f\| \sum_{\nu=-\infty}^{\infty} |c_\nu|^s, \tag{1}$$

where $\|\ \|$ means the norm in any of spaces $L^\varphi(-\infty, \infty)$ and M^φ, φ being an s-convex function. In case of the space of continuous functions in $(-\infty, \infty)$ inequality (1), with $s = 1$, was given by P. Civin ([2], see also [5], p. 229). In case of the Marcinkiewicz–Orlicz space and functions $\mu(t) = it$, $\gamma(t) = 1$, an analogous estimate was obtained by Albrycht [1].

2. Let $\varphi(u)$ be an s-convex φ-function, i.e. $\varphi(u)$ is non-negative and continuous in the interval $\langle 0, \infty)$, $\varphi(u) = 0$ if and only if $u = 0$, satisfying the condition: $\varphi(\alpha u + \beta v) \leqslant \alpha^s \varphi(u) + \beta^s \varphi(v)$, for $\alpha, \beta > 0$, $\alpha^s + \beta^s = 1$ $(0 < s \leqslant 1)$ (cf. [4]). Write

$$\|h\|_{L^\varphi} = \|h(x)\|_{L^\varphi} = \inf\left\{\lambda > 0 : \int_{-\infty}^{\infty} \varphi\left(\frac{|h(x)|}{\lambda^{1/s}}\right) dx \leqslant 1\right\};$$

$\|h\|_{L^\varphi}$ is the norm of the measurable function h in space $L^\varphi(-\infty, \infty)$

THEOREM 1. *If* $f \in L^\varphi(-\infty, \infty)$, *then*

$$\|g\|_{L^\varphi} \leqslant \|f\|_{L^\varphi} \sum_{\nu=-\infty}^{\infty} |c_\nu|^s. \tag{2}$$

Proof. We write

$$g_n(x) = \sum_{\nu=-n}^{n} c_\nu f[p(\nu) - a + x]. \tag{3}$$

By Fatou lemma,

$$\int_{-\infty}^{\infty} \varphi\left(\frac{|g(x)|}{\lambda^{1/s}}\right) dx \leqslant \varliminf_{n\to\infty} \int_{-\infty}^{\infty} \varphi\left(\frac{|g_n(x)|}{\lambda^{1/s}}\right) dx \quad \text{for } \lambda > 0. \tag{4}$$

If $\varepsilon > \sup_n \|g_n\|_{L^\varphi}$, then

$$\int_{-\infty}^{\infty} \varphi\left(\frac{|g_n(x)|}{\varepsilon^{1/s}}\right) dx \leqslant 1 \quad \text{for } n = 0, 1, 2, \ldots$$

Hence, by (4),

$$\int_{-\infty}^{\infty} \varphi\left(\frac{|g(x)|}{\varepsilon^{1/s}}\right) dx \leqslant 1.$$

Therefore $\varepsilon \geqslant \|g\|_{L^\varphi}$. Consequently,

$$\|g\|_{L^\varphi} \leqslant \sup_n \|g_n\|_{L^\varphi}.$$

But

$$\sup_n \|g_n\|_{L^\varphi} \leqslant \sup_n \sum_{\nu=-n}^{n} |c_\nu|^s \|f[p(\nu)-a+x]\|_{L^\varphi} = \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \|f[p(\nu)-a+x]\|_{L^\varphi}$$

and $\|f[p(\nu)-a+x]\|_{L^\varphi} = \|f(x)\|_{L^\varphi}$. Thus we obtain inequality (2).

Now, let M^φ denote the Marcinkiewicz–Orlicz space, i.e. the set of all complex-valued measurable functions h such that $\varrho(\eta h) < \infty$ for an $\eta > 0$ depending on h, where

$$\varrho(h) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \varphi(|h(x)|)\,dx.$$

Moreover, let

$$\|h\|_{M^\varphi} = \|h(x)\|_{M^\varphi} = \inf\left\{\lambda > 0:\ \varrho\left(\frac{h}{\lambda^{1/s}}\right) \leqslant 1\right\}$$

(cf. [1]).

THEOREM 2. *For any f (defined in Section 1)*

$$\|g\|_{M^\varphi} \leqslant \|f\|_{M^\varphi} \sum_{\nu=-\infty}^{\infty} |c_\nu|^s. \tag{5}$$

Proof. Since $f(x)$ is continuous and bounded in $(-\infty, \infty)$, the relation $f \in M^\varphi$ is obvious. Proceeding as in the proof of Theorem 1, we obtain

$$\varrho\left(\frac{g}{\lambda^{1/s}}\right) \leqslant \overline{\lim_{T\to\infty}} \lim_{n\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi\left(\frac{|g_n(x)|}{\lambda^{1/s}}\right) dx \quad \text{for } \lambda > 0,$$

where g_n are defined by (3). Let us introduce in the Orlicz space $L^\varphi\langle -T, T\rangle$ the norm

$$\|h\|_{L^\varphi\langle -T,T\rangle} = \|h(x)\|_{L^\varphi\langle -T,T\rangle} = \inf\left\{\lambda > 0: \frac{1}{2T} \int_{-T}^{T} \varphi\left(\frac{|h(x)|}{\lambda^{1/s}}\right) dx \leqslant 1\right\}.$$

Let $\varepsilon > \sup_n \|g_n\|_{L^\varphi\langle -T,T\rangle}$. Then

$$\frac{1}{2T} \int_{-T}^{T} \varphi\left(\frac{|g_n(x)|}{\varepsilon^{1/s}}\right) dx \leqslant 1 \quad \text{for } n = 0, 1, 2, \ldots$$

Hence $\varrho(g/\varepsilon^{1/s}) \leqslant 1$, i.e. $\varepsilon \geqslant \|g\|_{M^\varphi}$. Thus $\|g\|_{M^\varphi} \leqslant \overline{\lim_{T\to\infty}} \sup_n \|g_n\|_{L^\varphi\langle -T,T\rangle}$.

But

$$\sup_n \|g_n\|_{L^\varphi\langle -T,T\rangle} \leqslant \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \cdot \|f[p(\nu)-a+x]\|_{L^\varphi\langle -T,T\rangle}.$$

Consequently,

$$\|g\|_{M^\varphi} \leqslant \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \overline{\lim_{T\to\infty}} \|f[p(\nu)-a+x]\|_{L^\varphi\langle -T,T\rangle}. \tag{6}$$

Now, we show that for every $h \in M^\varphi$ there holds the inequality

$$\overline{\lim_{T\to\infty}} \|h\|_{L^\varphi\langle -T,T\rangle} \leqslant \|h\|_{M^\varphi}. \tag{7}$$

Let $\varepsilon > \|h\|_{M^\varphi}$. We choose η such that $\varepsilon > \eta > \|h\|_{M^\varphi}$. Then $\varrho\left(\frac{h}{\varepsilon^{1/s}}\right) \leqslant \frac{\eta}{\varepsilon}\varrho\left(\frac{h}{\eta^{1/s}}\right) < 1$, since the function φ is s-convex. Hence there exists a $T_0 > 0$, such that

$$\frac{1}{2T}\int_{-T}^{T} \varphi\left(\frac{|h(x)|}{\varepsilon^{1/s}}\right) dx < 1 \quad \text{for } T > T_0.$$

Therefore

$$\varepsilon \geqslant \|h\|_{L^\varphi\langle -T,T\rangle} \quad \text{for } T > T_0.$$

Thus

$$\|h\|_{L^\varphi\langle -T,T\rangle} \leqslant \|h\|_{M^\varphi} \quad \text{for } T > T_0,$$

and this implies inequality (6).

From inequalities (6), (7) follows that

$$\|g\|_{M^\varphi} \leqslant \sum_{\nu=-\infty}^{\infty} |c_\nu|^s \|f[p(\nu)-a+x]\|_{M^\varphi}.$$

To prove (5) it is sufficient to show that, for an arbitrary real u,

$$\|f(x+u)\|_{M^\varphi} = \|f(x)\|_{M^\varphi}. \tag{8}$$

We have

$$|\varrho(f(x+u)) - \varrho(f(x))| \leqslant \overline{\lim_{T\to\infty}} \frac{1}{2T}\left|\int_{-T}^{-T+u} \varphi(|f(x)|)\,dx\right| + \overline{\lim_{T\to\infty}} \frac{1}{2T}\left|\int_{T}^{T+u} \varphi(|f(x)|)\,dx\right| = 0,$$

because $f(x)$ is bounded. Hence $\varrho(f(x+u)) = \varrho(f(x))$ for every real u. This implies formula (8), and Theorem 2 is proved.

3. In this Section $\|\ \|$ will mean the norm in any of the spaces $C(-\infty, \infty)$, $L^{\varphi}(-\infty, \infty)$, M^{φ}, where φ is s-convex $(1/2 < s \leqslant 1)$. It is known (cf. [2]) that if $\mu(t)$ is absolutely continuous, $\mu(-R) = e^{i\theta}\mu(R)$ $(0 \leqslant \theta \leqslant 2\pi)$, $a \in \theta/2R$, and $\mu'(t)$ is equal almost everywhere to a function $k(t)$ of bounded variation $Vk(t)$ in $\langle -R, R\rangle$; then

$$|c_\nu| \leqslant \frac{R}{2}\left\{|k(R)| + |k(-R)| + Vk(t)\right\}\frac{1}{(\frac{1}{2}\theta - \nu\pi)^2}.$$

Thus, in the special case when $\mu(t) = i^a t^a$ $(a \geqslant 1)$, we choose

$$\theta = \begin{cases} a\pi & \text{if } 1 \leqslant a \leqslant 2, \\ (a - 2l) & \text{if } 2l < a \leqslant 2(l+1),\ l = 1, 2, \ldots \end{cases}$$

Then

$$|c_\nu| \leqslant 2aR^a \frac{1}{(\frac{1}{2}a\pi - \nu\pi)^2} \quad \text{for } \nu = 0, \pm 1, \pm 2, \ldots \tag{9}$$

THEOREM 3. *Under the assumption* $\|f_0\| < \infty$ *and* $s = 1$,

$$\|f_0^{(a)}\| \leqslant R^a \|f_0\| \quad \text{if } a = 0, 1, 2, \ldots, \tag{i}$$

$$\|f_0^{(a+q)}\| \leqslant \left\{\left(\frac{4}{3} - \frac{8}{\pi^2}\right)a + \frac{2}{a+1}\right\}R^{a+q}\|f_0\| \quad \text{if } 1 < a < 2, \tag{ii}$$

$$q = 0, 1, 2, \ldots$$

In particular,

$$\begin{aligned} &\|f_0^{(a)}\| \leqslant 1.8R^a\|f_0\| && \text{for any } a > 1, \\ &\|f_0^{(a)}\| \leqslant 7a^{-1}R^a\|f_0\| && \text{if } 0 < a < 1. \end{aligned} \tag{iii}$$

Proof. The inequality in (i) will be proved by induction. For $a = 0$ the inequality is obvious. Let us suppose it is satisfied for an integer $a \geqslant 0$. Taking $\gamma(t) = i^a t^a$, $\mu(t) = it$, we set

$$f(x) = \int_{-R}^{R} \gamma(t) e^{ixt} d\omega(t), \qquad g(x) = \int_{-R}^{R} \mu(t)\gamma(t) e^{ixt} d\omega(t).$$

Thus, applying Lemma 1 of [2] or inequalities (2) or (5), we obtain

$$\|f_0^{(a+1)}\| = \|g\| \leqslant \|f\| \sum_{\nu=-\infty}^{\infty} |c_\nu| = \|f_0^{(a)}\| R \leqslant R^{a+1}\|f_0\|,$$

by the induction hypothesis (see [2]).

Now, we shall deduce inequality (ii). Let $\sum''_{\nu=-\infty}^{\infty} a_\nu$ be the sum of all terms of the series $\sum_{\nu=-\infty}^{\infty} a_\nu$ with exception of a_0 and a_1. Then, by estimate (9),

$$\sum_{\nu=-\infty}^{\infty}{}'' |c_\nu| \leqslant 2aR^a \sum_{\nu=-\infty}^{\infty}{}'' \frac{1}{(\frac{1}{2}a\pi - \nu\pi)^2}.$$

But

$$\sum_{\nu=-\infty}^{\infty}{}'' \frac{1}{(\frac{1}{2}a\pi-\nu\pi)^2} = \sum_{\nu=1}^{\infty} \frac{1}{(\frac{1}{2}a\pi+\nu\pi)^2} + \sum_{\nu=2}^{\infty} \frac{1}{(\frac{1}{2}a\pi-\nu\pi)^2}$$

$$\leqslant \frac{4}{\pi^2} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^2} + \frac{1}{\pi^2} \sum_{\nu=2}^{\infty} \frac{1}{(\nu-1)^2} = \frac{4}{\pi^2}\left(\frac{\pi^2}{8}-1\right) + \frac{1}{6} = \frac{2}{3} - \frac{4}{\pi^2}.$$

Hence

$$\sum_{\nu=-\infty}^{\infty}{}'' |c_\nu| \leqslant 4\left(\frac{1}{3} - \frac{2}{\pi^2}\right) aR^a.$$

Since

$$|c_0| \leqslant \frac{1}{a+1} R^a, \qquad |c_1| \leqslant \frac{1}{a+1} R^a, \tag{10}$$

we have

$$\sum_{\nu=-\infty}^{\infty} |c_\nu| \leqslant \left\{\left(\frac{4}{3} - \frac{8}{\pi^2}\right) a + \frac{2}{a+1}\right\} R^a < \frac{9}{5} R^a.$$

Applying Lemma 1 of [2] or inequalities (2) or (5) and reasoning as before we obtain (ii).

Inequality given in (iii) follows at once from the suitable Civin results (see [2]) and our Theorems 1, 2.

THEOREM 4. *Suppose that* $\|f_0\| < \infty$, $1/2 < s < 1$, $1 \leqslant a \leqslant 2$, $q = 0, 1, 2, \ldots$ *Then*

$$\|f_0^{(a+q)}\| \leqslant A(s, a)\,\{A(s, 1)\}^q R^{(a+q)s} \|f_0\|,$$

where

$$A(s, a) = \frac{2^{s+1} a^s}{\pi^{2s}}\, \zeta(2s) + \frac{2}{(a+1)^s}, \qquad \zeta(z) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^z}.$$

Proof. In view of (9) and (10),

$$\sum_{\nu=-\infty}^{\infty} |c_\nu|^s \leqslant A(s, a) R^{as}.$$

Hence

$$\|f_0^{(a+1)}\| = \|g\| \leqslant \|f\|\, A(s, 1) R^s \leqslant A(s, a) A(s, 1) R^{(a+1)s} \|f_0\|.$$

Analogously,

$$\|f_0^{(a+2)}\| = \|g\|\, A(s, 1) R^s \leqslant A(s, a)\,\{A(s, 1)\}^2 R^{(a+2)s} \|f_0\|, \qquad \text{etc.}$$

Considering the cases $1 \leqslant \alpha \leqslant 2$, $2l < \alpha \leqslant 2(l+1)$ $(l = 1, 2, \ldots)$ and applying estimates (9) and (10), we get easily the following

THEOREM 5. *Let* $\|f_0\| < \infty$, $1/2 < s < 1$, $\alpha \geqslant 1$.

Then

$$\|f_0^{(\alpha)}\| \leqslant A(s, \alpha) R^{\alpha s} \|f_0\|,$$

where $A(s, \alpha)$ *has the same meaning as above.*

References

[1] J. Albrycht, *A generalization of a Zygmund–Bernstein theorem*, Ann. Polon. Math. 2 (1955), p. 64-66.

[2] P. Civin, *Inequalities for trigonometric integrals*, Duke Math. 8 (1941), p. 658-665.

[3] М. А. Красносельский и Я. Б. Рутицкий, *Выпуклые функции и пространства Орлича*, Москва 1958.

[4] W. Matuszewska, *Przestrzenie funkcji φ-całkowalnych*, Część II, *Uogólnione przestrzenie Orlicza*, Prace Mat. 6 (1961), p. 149-164.

[5] А. Ф. Тиман, *Теория приближения функций действительного переменного*, Москва 1960.