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Semigroups of operators on a locally convex space

In this paper we prove some results in the theory of semigroups of continuous linear operators on a locally convex Hausdorff space E. (A family $\{T(s)\}_{s>0}$ of linear operators on a vector space is called a *semigroup* if $T(s)\circ T(t) = T(s+t)$, $s, t > 0$.) In Section 1, we study properties of the infinitesimal operator of a semigroup which satisfies good conditions at the origin such as for instance $T(s)$ converges to a limit operator as s converges to zero or integrability of $T(s)x$ near $s = 0$ ($x \in E$). In Section 2, we prove a perturbation theorem which is a generalization of a result due to Phillips [7] whereas the results of Section 1 are generalizations of results in [5] chapter 9.

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1. E always denotes a locally convex Hausdorff space and a semigroup of operators on E means a semigroup of continuous linear operators on E. $\mathscr{L}(E, E)$ denotes the space of all continuous linear operators on E; $\mathscr{L}_s(E, E)$ is the space $\mathscr{L}(E, E)$ furnished with the topology of simple convergence and $\mathscr{L}_{\mathfrak{G}}(E, E)$ is the space $\mathscr{L}(E, E)$ furnished with the topology of uniform convergence on the system $\mathfrak{G}$ of all bounded subsets of E (see [3]).

DEFINITION 1. A semigroup $\{T(s)\}_{s>0}$ of operators on a locally convex space E is said to be *measurable* if for $x \in E$, the function $s \to T(s)x$ is measurable from $(0, \infty)$ to E.

For the definition of measurability of a vector valued function and properties of measurable semigroups refer to [9].

DEFINITION 2. A semigroup $\{T(s)\}_{s>0}$ of operators on E is said to be

continuous if for $x \in E$, the function $s \to T(s)x$ is continuous from $(0, \infty)$ to E (or the map $s \to T(s)$ from $(0, \infty)$ to $\mathscr{L}_s(E, E)$ is continuous).

DEFINITION 3. A semigroup $\{T(s)\}_{s>0}$ of operators on E is said to be *locally equicontinuous* if for every $[\alpha, \beta] \subset (0, \infty)$ the family $\{T(s)\}_{\alpha \leqslant s \leqslant \beta}$ is equicontinuous.

DEFINITION 4. The operator A_0 on E defined by

$$A_0 x = \operatorname*{Lt}_{t \to 0+} \frac{T(t)x - x}{t} = \operatorname*{Lt}_{t \to 0+} A(t)x \qquad \left(A(t) \equiv \frac{T(t) - I}{t}\right)$$

for each $x \in E$ at which the limit exists is called the *infinitesimal operator of the semigroup* $\{T(s)\}_{s>0}$. Clearly the domain $\mathscr{D}(A_0)$ of A_0 is a subspace of E and A_0 is a linear operator from $\mathscr{D}(A_0)$ to E; it can also be verified that the range of A_0 is contained in the closure of $\bigcup_{s>0} T(s)E = E_0$.

Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on E. For $[\alpha, \beta] \subset (0, \infty)$ and $x \in E$, $\int_\alpha^\beta T(t)x\,dt \in E$ if for instance E is quasi-complete [2].

THEOREM 1. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on E such that for $[\alpha, \beta] \subset (0, \infty)$ and $x \in E$, $\int_\alpha^\beta T(t)x\,dt \in E$. Then* (i) *$\mathscr{D}(A_0)$ is dense in E_0,* (ii) *$\mathscr{D}(A_0)$ and E_0 have the same closure.*

Proof. For $[\alpha, \beta] \subset (0, \infty)$ and $y \in E$ let $y_{\alpha\beta} = \int_\alpha^\beta T(t)y\,dt$,

$$T\left(\frac{\alpha}{2}\right) y_{\frac{\alpha}{2}, \beta - \frac{\alpha}{2}} = \int_{\alpha/2}^{\beta - \frac{\alpha}{2}} T(\alpha/2)T(t)y\,dt$$

$$= \int_{\alpha/2}^{\beta - \frac{\alpha}{2}} T(t + \alpha/2)y\,dt = \int_\alpha^\beta T(s)y\,ds = y_{\alpha\beta}.$$

Thus $y_{\alpha\beta} \in E_0$ and

$$A(t)y_{\alpha\beta} = \frac{T(t) - I}{t} \int_\alpha^\beta T(s)y\,ds = \int_\alpha^\beta \frac{T(t) - I}{t} T(s)y\,ds$$

$$= \frac{1}{t} \int_\alpha^\beta [T(s + t) - T(s)]y\,ds = \frac{1}{t} \int_\beta^{\beta + t} T(s)y\,ds - \frac{1}{t} \int_\alpha^{\alpha + t} T(s)y\,ds.$$

Now $\frac{1}{t} \int_\beta^{\beta + t} T(s)y\,ds$ belongs to the closed convex envelope of $\{T(s)y\}_{\beta \leqslant s \leqslant \beta + t}$ so that

$$\operatorname*{Lt}_{t \to 0} \frac{1}{t} \int_\beta^{\beta + t} T(s)y\,ds = T(\beta)y.$$

Similarly $\operatorname{Lt}_{t\to 0}\frac{1}{t}\int_a^{a+t} T(s)y\,ds = T(a)y$. Hence $y_{a\beta}\in\mathscr{D}(A_0)$. Also

$$\operatorname{Lt}_{\beta\to a}\frac{y_{a\beta}}{\beta-a} = \operatorname{Lt}_{\beta\to a}\frac{1}{\beta-a}\int_a^{\beta} T(s)y\,ds = T(a)y,$$

so that for $x\in E_0$, say $x = T(a)y$, $x = \operatorname{Lt}_{\beta\to a}\frac{1}{\beta-a}y_{a\beta}$. Thus $\mathscr{D}(A_0)\cap E_0$ is dense in E_0.

(ii) For any $x\in\mathscr{D}(A_0)$, $x = \operatorname{Lt}_{t\to 0} T(t)x$ so that $\mathscr{D}(A_0)\subset\bar{E}_0$. Also by (i) $E_0\subset\overline{\mathscr{D}(A_0)}$ so that $\overline{\mathscr{D}(A_0)} = \bar{E}_0$.

THEOREM 2. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on a Fréchet space E. If $\mathscr{D}(A_0)$ is of the second category in E, then* (i) $\mathscr{D}(A_0) = E$, (ii) $A_0\in\mathscr{L}(E,E)$, (iii) $T(s)\to I$ *in* $\mathscr{L}_{\mathfrak{G}}(E,E)$ *as* $s\to 0+$.

Proof. Let $\{t_n\}_{n=1,2,\ldots}$ be an arbitrarily fixed sequence of real numbers converging to $0+$ and let

$$X = \{x\in E/\ \operatorname{Lt}_{n\to\infty} A(t_n)x \text{ exists in } E\},$$

X contains $\mathscr{D}(A_0)$ and is therefore of the second category. Further it is a subspace since each $A(t_n)$ is linear and it is B-measurable since each $A(t_n)$ is continuous (Theorem 2, p. 18, [1]). Hence by Theorem 1, p. 36, [1], X is equal to E. Thus for any $x\in E$, $\operatorname{Lt}_{n\to\infty} A(t_n)x$ exists for any sequence $\{t_n\}_{n=1,2,\ldots}$ converging to $0+$. It can be verified immediately that this limit is independent of the sequence $\{t_n\}_{n=1,2,\ldots}$ converging to $0+$ so that $\mathscr{D}(A_0) = E$. Then A_0 is the limit in $\mathscr{L}_s(E,E)$ of a sequence in $\mathscr{L}(E,E)$, say of $\{A(t_n)\}_{n=1,2,\ldots}$ for any $\{t_n\}_{n=1,2,\ldots}$ converging to zero and is therefore continuous (since E is Fréchet). Also the set $\{A(t)x/t\in(0,1]\}$ is bounded in E for any $x\in E$. For otherwise there exists an $x_0\in E$ and a continuous seminorm p on E and a sequence $s_n\in(0,1]$ such that $p[A(s_n)x_0] > n$ $(n = 1,2,\ldots)$. $\{s_n\}$ has a subsequence $\{s_{n_k}\}_{k=1,2,\ldots}$ converging to some $s_0\in[0,1]$. Then $A(s_{n_k})x_0\to A(s_0)x_0$ (where $A(s_0) = A_0$ if $s_0 = 0$) so that $p[A(s_{n_k})x_0]_{k=1,2,\ldots}$ is bounded which is a contradiction. Thus $\{A(t)/0 < t\leqslant 1\}$ is bounded in $\mathscr{L}_s(E,E)$ and therefore equicontinuous (as E is Fréchet). We now show that $T(s)\to I$ in $\mathscr{L}_{\mathfrak{G}}(E,E)$ as $s\to 0$. Let B be a bounded set in E and V a convex circled neighbourhood of 0 in E. We show that there exists $t_0 > 0$ such that $T(t)x - x\in V$ for $t\in(0,t_0)$ and $x\in B$. As $\{A(t)/0 < t\leqslant 1\}$ is equicontinuous, there exists a neighbourhood U of 0 in E such that $A(t)U\subset V$, $0 < t\leqslant 1$ or $T(t)x - x\in tV$ for $x\in U$ and $t\in(0,1]$. Let $\lambda > 0$ such that $\lambda B\subset U$. Then

$$T(t)x - x\in\frac{t}{\lambda}V \quad \text{for } x\in B \text{ and } t\in(0,1].$$

It suffices to choose t_0 to be any real number in $(0, 1)$ such that $\frac{t_0}{\lambda} < 1$.

THEOREM 3. *If $\{T(s)\}_{s>0}$ is a measurable and locally equicontinuous semigroup of operators on E, then for $x \in \mathcal{D}(A_0)$,*

$$\frac{d}{ds}T(s)x = A_0T(s)x = T(s)A_0x, \quad s > 0.$$

Proof. By Proposition 2 [9], $\{T(s)\}_{s>0}$ is a continuous semigroup,

$$\frac{[T(s+t)-T(s)]x}{t} = A(t)T(s)x = T(s)A(t)x, \quad s, t > 0.$$

For $x \in \mathcal{D}(A_0)$, $\underset{t\to 0+}{\text{Lt}}\, A(t)x = A_0x$ so that $T(s)A(t)x \to T(s)A_0x$. Thus $\underset{t\to 0+}{\text{Lt}} \frac{[T(s+t)-T(s)]x}{t}$ exists and equals $T(s)A_0x = A_0T(s)x$. It remains to prove that

$$\underset{t\to 0+}{\text{Lt}} -\frac{1}{t}[T(s-t)-T(s)]x = \underset{t\to 0+}{\text{Lt}}\, T(s-t)A(t)x = T(s)A_0x.$$

For $s > 0$ fixed let t_0 be fixed such that $0 < t_0 < s$. $T(s-t)A(t)x = T(s-t)[A(t)x - A_0x] + T(s-t)A_0x$. Since $\{T(s-t)/0 \leqslant t \leqslant t_0\}$ is equicontinuous and $\underset{t\to 0+}{\text{Lt}}\, A(t)x = A_0x$, we have

$$\underset{t\to 0+}{\text{Lt}}\, T(s-t)[A(t)x - A_0x] = 0$$

and since the semigroup is continuous we have

$$\underset{t\to 0+}{\text{Lt}}\, T(s-t)A_0x = T(s)A_0x$$

which completes the proof.

The proofs of the two following theorems are almost the same as in the case when E is a Banach space. We include them for the sake of completeness.

THEOREM 4. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on a quasi-complete space. If $x \in \mathcal{D}(A_0)$ is such that*

$$\underset{\delta\to 0+}{\text{Lt}} \int_{\delta}^{1} T(t)A_0x\,dt \quad \textit{exists in } E.$$

Then

$$T(s)x - x = \int_0^s T(t)A_0x\,dt \quad (s > 0).$$

Proof. By Theorem 3, for $x \in \mathscr{D}(A_0)$,

$$\frac{d}{dt}T(t)x = T(t)A_0x$$

so that for every $x' \in E'$, the topological dual of E,

$$\frac{d}{dt}\langle T(t)x, x'\rangle = \langle T(t)A_0x, x'\rangle.$$

Integrating this continuous function we have for $s > \delta > 0$,

$$\int_\delta^s \frac{d}{dt}\langle T(t)x, x'\rangle dt = \int_\delta^s \langle T(t)A_0x, x'\rangle dt \quad (x' \in E'),$$

i.e.

$$\langle T(s)x, x'\rangle - \langle T(\delta)x, x'\rangle = \Big\langle \int_\delta^s T(t)A_0x\,dt,\ x'\Big\rangle.$$

Hence

$$T(s)x - T(\delta)x = \int_\delta^s T(s)A_0x\,dt.$$

By assumption on x,

$$\operatorname*{Lt}_{\delta\to 0+} \int_\delta^s T(t)A_0x\,dt \quad \text{exists for any } s > 0$$

and as $x \in \mathscr{D}(A_0)$,

$$\operatorname*{Lt}_{\delta\to 0+} T(\delta)x = x.$$

We therefore have

$$T(s)x - x = \int_0^s T(t)A_0x\,dt.$$

THEOREM 5. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup and E be quasi-complete. Then $\bigcap_{n=1}^{\infty} \mathscr{D}(A_0^n)$ is dense in E_0.*

Proof. Let $\mathscr{D}(0, \infty)$ denote the space of functions indefinitely differentiable with compact support contained in $(0, \infty)$. For $F(t) \in \mathscr{D}(0, \infty)$ with support contained in $[\alpha, \beta]$, $F(t)T(t)x$ is a continuous E-valued function on $[\alpha, \beta]$. Hence $\int_\alpha^\beta F(t)T(t)x\,dt$ exists in E. Let it be denoted by $\int_0^\infty F(t)T(t)x\,dt = y$. For $\delta > 0$ sufficiently small, $F(t+\delta) \in \mathscr{D}(0, \infty)$ and

$$y = \int_0^\infty F(t+\delta)T(t+\delta)x\,dt = T(\delta)\int_0^\infty F(t+\delta)T(t)x\,dt$$

so that $y \in E_0$,

$$A(t)y = \frac{1}{t}\int_0^\infty F(s)\,[T(s+t)-T(s)]x\,ds$$

$$= \int_0^\infty \frac{1}{t}[F(s-t)-F(s)]T(s)x\,ds$$

$$= \int_a^b \frac{1}{t}[F(s-t)-F(s)]T(s)x\,ds \quad \text{for some } [a,b] \subset (0,\infty)$$

$$= \int_a^b \left\{\frac{1}{t}[F(s-t)-F(s)]+F'(s)\right\} T(s)x\,ds - \int_a^b F'(s)T(s)x\,ds.$$

Let $t \to 0$. Then $\frac{1}{t}[F(s-t)-F(s)]+F'(s) \to 0$ uniformly for $s \in [a,b]$ since $F'(s)$ is uniformly continuous on $(0,\infty)$.

Hence

$$\operatorname*{Lt}_{t\to 0} A(t)y = -\int_a^b F'(s)T(s)x\,ds = -\int_0^\infty F'(s)T(s)x\,ds,$$

i.e. $y \in \mathscr{D}(A_0)$ and $A_0 y = -\int_0^\infty F'(s)T(s)x\,ds$.

Repeating the argument we have for any $n > 0$,

$$A_0^n y = (-1)^n \int_0^\infty F^n(s)T(s)x\,ds.$$

Thus $y \in \mathscr{D}(A_0^n)$. Let $E_1 = \{y/y = \int_0^\infty F(s)T(s)x\,ds, F(s) \in \mathscr{D}(0,\infty)$ and $x \in E\}$. E_1 is a subspace of E_0 and is contained in $\bigcap_{n=1}^\infty \mathscr{D}(A_0^n)$. If we prove that E_1 is dense in E_0, the proposition will be proved. If E_1 is not dense in E_0, there exists $x' \in E'$ (the topological dual of E) such that $\langle y, x'\rangle = 0$ for every $y \in E_1$ and x' does not vanish identically on E_0. Then $x'[\int_0^\infty F(s)\,T(s)x\,ds] = \int_0^\infty F(s)x'[T(s)x]ds = 0$ for every $F \in \mathscr{D}(0,\infty)$ and $x \in E$. It follows that $x'[T(s)x] = 0$ for every $s > 0$ and $x \in E$ i.e. x' vanishes on E_0 which is contradictory to the choice of x'.

THEOREM 6. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on a barreled and quasi-complete space E such that for some $s_0 > 0$, $T(s_0)E \subset \mathscr{D}(A_0)$. Then*

(i) $$T^{(1)}(s) = \frac{d}{ds}T(s) = A_0 T(s)$$

exists as a continuous linear operator for $s \geqslant s_0$ *and* $s \to T(s)$ *from* (s_0, ∞) *to* $\mathscr{L}_{\mathfrak{G}}(E, E)$ *is continuous*;

(ii) $$T^n(s) = A_0^n(Ts)$$

exists as a continuous linear operator for $s \geqslant ns_0$ *and* $s \to T^{(n)}(s)$ *from* (ns_0, ∞) *to* $\mathscr{L}_{\mathfrak{G}}(E, E)$ *is continuous.*

Proof. For $s > 0$, $t \to A(t)T(s)$ is continuous from $(0, \infty)$ to $\mathscr{L}_s(E, E)$ and for $s \geqslant s_0$,

$$\operatorname*{Lt}_{t \to 0} A(t)T(s) \text{ exists in } \mathscr{L}_s(E, E).$$

Hence for $s \geqslant s_0$, $\{A(t)T(s)\}_{0<t\leqslant 1}$ is bounded in $\mathscr{L}_s(E, E)$ and therefore equicontinuous (E being barreled). Thus for $s \geqslant s_0$, $A_0T(s) \in \mathscr{L}(E, E)$. We now show that $s \to A_0T(s)$ from (s_0, ∞) to $\mathscr{L}_{\mathfrak{G}}(E, E)$ is continuous at any $s > s_0$. Let $\delta > 0$ be such that $s > s_0 + 2\delta$ and let $|t| < \delta$. We have to show that for $B \in \mathfrak{G}$

(*) $$T(s+t)x - T(s)x \to 0 \text{ as } t \to 0 \text{ uniformly for } x \in B.$$

For $x \in E$, $T(\tau - s_0)x$ is a continuous function of τ from $[s-\delta, s+\delta]$ to E. Since $A_0T(\tau) = A_0T(s_0)T(\tau - s_0)$ and $A_0T(s_0) \in \mathscr{L}(E, E)$, it follows that for $x \in E$, $\tau \to A_0T(\tau)x$ is continuous from $[s-\delta, s+\delta]$ to E. Thus for $|t| < \delta$,

$$T(s+t)x - T(s)x = \int_s^{s+t} A_0T(\tau)x\,d\tau.$$

For any continuous seminorm p on E we therefore have

$$p[T(s+t)x - T(s)x] \leqslant |t|\,p[A_0T(\tau_0)x]$$
$$\text{for some } \tau_0 \text{ with } |s - \tau_0| \leqslant |t| < \delta.$$

Also since E is barreled, the continuous semigroup $\{T(t)\}_{t>0}$ is locally equicontinuous (by Remark 2 after Proposition 2 [9]) and for $\tau \in [s-\delta, s+\delta]$, $\tau - s_0 \in [s - s_0 - \delta, s - s_0 + \delta] \subset [\delta, s - s_0 + \delta]$. It follows that $\{T(\tau - s_0)\}_{s-\delta<\tau\leqslant s+\delta}$ is equicontinuous and therefore so is $\{A_0T(\tau) / s - \delta \leqslant \tau \leqslant s + \delta\}$ (as $A_0T(\tau) = A_0T(s_0)T(\tau - s_0)$ and $A_0T(s_0) \in \mathscr{L}(E, E)$). Thus $\{A_0T(\tau) / s - \delta \leqslant \tau \leqslant s + \delta\}$ is bounded in $\mathscr{L}_{\mathfrak{G}}(E, E)$ (cf. [3]) and therefore $\{p[A_0T(\tau)x] / x \in B,\ s = \delta \leqslant \tau \leqslant s + \delta\}$ is a bounded set of real numbers. It follows that $p[T(s+t)x - T(s)x] = O(|t|)$ uniformly for $x \in B$ and (*) is proved.

If $s \geqslant 2s_0$ and $x \in E$,

$$T^{(1)}(s)x = A_0T(s)x = T(s/2)A_0T(s/2)x \in \mathscr{D}(A_0)$$

and

$$T^{(2)}(s) = \operatorname*{Lt}_{t \to +0} A(t)\{T(s/2)\,[A_0T(s/2)x]\} = [A_0T(s/2)]^2x = A_0^2T(s)x.$$

Similarly for any n for $s \geqslant ns_0$ and $x \in E$,

$$T^{(n)}(s)x = \left[A_0 T\left(\frac{s}{n}\right)\right]^n x,$$

i.e.

$$T^{(n)}(s)x = A_0^n T(s)x.$$

As $A_0 T(s/n) = T^{(1)}(s/n) \in \mathscr{L}(E, E)$,

$$T^{(n)}(s) \in \mathscr{L}(E, E) \quad \text{for } s \geqslant ns_0$$

and

$$s \to T^{(n)}(s) = T(s - ns_0)\,[A_0^n T(ns_0)]$$

is continuous from $\big((n+1)s_0, \infty\big)$ to $\mathscr{L}_{\mathfrak{S}}(E, E)$.

THEOREM 7. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on a quasi-complete space E such that* (i) *for some $\alpha > 0$, $\{T(s)\}_{0<s\leqslant\alpha}$ is equicontinuous,* (ii) *for every $x \in E$,*

$$\operatorname*{Lt}_{\varepsilon\to 0} \int_\varepsilon^\alpha T(s)x\,ds = \int_0^\alpha T(s)x\,ds$$

and

$$\operatorname*{Lt}_{t\to 0} \frac{1}{t}\int_0^t T(s)x\,ds = Jx$$

exist in E.

Then J is a continuous projection, $J^2 = J$, which maps E onto $\bar{E}_0$ and

$$T(s)J = T(s) = JT(s) \qquad (s > 0).$$

Proof. Let $0 < \varepsilon < t$. $\dfrac{1}{t}\displaystyle\int_\varepsilon^t T(s)x\,ds \in E$ and the map $\dfrac{1}{t}\displaystyle\int_\varepsilon^t T(s)\,ds$ defined by

$$\left[\frac{1}{t}\int_\varepsilon^t T(s)\,ds\right]x = \frac{1}{t}\int_\varepsilon^t T(s)x\,ds$$

from E into E is linear. For $t \leqslant a$, this map belongs to the closed convex circled envelope C of $\{T(s)\}_{0<s\leqslant a}$ in $\mathscr{L}_s(E, E)$ and C is equicontinuous. Thus $\dfrac{1}{t}\displaystyle\int_\varepsilon^t T(s)\,ds \in C \subset \mathscr{L}(E, E)$. The operator $\dfrac{1}{t}\displaystyle\int_0^t T(s)x\,ds$ defined by

$$\left[\frac{1}{t}\int_0^t T(s)\,ds\right]x = \frac{1}{t}\int_0^t T(s)x\,ds = \operatorname*{Lt}_{\varepsilon\to 0}\frac{1}{t}\int_\varepsilon^t T(s)x\,ds$$

belongs to C and therefore also the operator J. Thus J is continuous.

We have

$$T(s)\left[\frac{1}{t}\int_0^t T(\tau)x\,d\tau\right] = T(s)\left[\operatorname*{Lt}_{\varepsilon\to 0}\frac{1}{t}\int_\varepsilon^t T(\tau)x\,d\tau\right]$$

$$= \operatorname*{Lt}_{\varepsilon\to 0}\frac{1}{t}\int_0^t T(s)\,[T(\tau)x]dx = \operatorname*{Lt}_{\varepsilon\to 0}\frac{1}{t}\int_\varepsilon^t T(\tau)\,[T(s)x]d\tau.$$

Hence

$$T(s)\left[\frac{1}{t}\int_0^t T(\tau)x\,d\tau\right] = \frac{1}{t}\int_0^t T(s+\tau)x\,d\tau = \frac{1}{t}\int_0^t T(\tau)\,[T(s)x]d\tau.$$

Taking limit as $t\to 0$,

$$T(s)Jx = T(s) = JT(s)x.$$

Using continuity of J, we have

$$J\left[\frac{1}{t}\int_0^t T(\tau)x\,d\tau\right] = \frac{1}{t}\int_0^t JT(\tau)x\,d\tau$$

so that $J^2x = Jx$ for every $x\in E$. Clearly $Jx\in \bar{E}_0$. For $x\in E_0$, i.e. $x = T(s)y$ for some $s>0$, we have,

$$Jx = \operatorname*{Lt}_{t\to 0}\frac{1}{t}\int_0^t T(\tau)T(s)y\,d\tau = \operatorname*{Lt}_{t\to 0}\frac{1}{t}\int_0^t T(s+\tau)y\,d\tau = T(s)y = x.$$

As J is continuous, $Jx = x$ for every $x\in \bar{E}_0$. Thus J is a continuous projection of E onto $\bar{E}_0$.

THEOREM 8. *Let E be a quasi-complete space which is either barreled or strong dual of a metrizable locally convex space and $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on E such that for some $a>0$ and every $x\in E$*

$$\operatorname*{Lt}_{\varepsilon\to 0}\int_\varepsilon^a T(s)x\,ds \equiv \int_0^a T(s)x\,ds$$

and

$$\operatorname*{Lt}_{t\to 0}\frac{1}{t}\int_0^t T(s)x\,ds \equiv Jx$$

exist in E. Then J is a continuous projection mapping E onto $\bar{E}_0$ and

$$T(s)J = T(s) = JT(s) \qquad (s>0).$$

Proof. The hypothesis on E viz. E is barreled or the strong dual of a metrizable space implies that

(1) *if* F *is a locally convex Hausdorff space and* $u_n \in \mathscr{L}(E, F)$, $n = 1, 2, \ldots$, *such that* $\operatorname{Lt}_{n\to\infty} u_n x \equiv ux$ *exists in* F *for every* $x \in E$, *then* $u \in \mathscr{L}(E, F)$ (refer to [3] and [4]);

(2) $\{T(s)\}_{s>0}$ *is locally equicontinuous* (refer to [9]).

From (2) and the fact that E is quasi-complete it follows that the operators $u_n = \frac{1}{t}\int_{1/n}^{t} T(s)\,ds$ defined by $u_n x = \frac{1}{t}\int_{1/n}^{t} T(s)x\,ds$ $(1/n < t)$ belong to $\mathscr{L}(E, E)$ and by hypothesis for any $t \leqslant a$,

$$u_n x \to \left[\frac{1}{t}\int_0^t T(s)\,ds\right]x \equiv \frac{1}{t}\int_0^t T(s)x\,ds \quad \text{as } n \to \infty \text{ for every } x \in E.$$

Hence by (1), $\frac{1}{t}\int_0^t T(s)\,ds \in \mathscr{L}(E, E)$ for $0 < t < a$. Also by hypothesis for any sequence $\{t_n\}_{n=1,2,\ldots}$ such that $t_n \to 0+$ we have

$$\operatorname{Lt}_{n\to\infty}\left[\frac{1}{t_n}\int_0^{t_n} T(s)\,ds\right]x = \operatorname{Lt}_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds = Jx \quad \text{for } x \in E.$$

Hence by (1), $J \in \mathscr{L}(E, E)$. That J is a projection satisfying the given properties can be proved as in Theorem 7.

THEOREM 9. *Let* $\{T(s)\}_{s>0}$ *be a semigroup of operators on a Fréchet space* E *such that*

$$(*) \qquad \operatorname{Lt}_{s\to 0+} T(s)x = Jx \text{ exists for every } x \in E.$$

Then (1) J *is a continuous projection,* $J^2 = J$, *mapping* E *onto* $\bar{E}_0$ *satisfying*

$$JT(s) = T(s)J = T(s) \qquad (s > 0)$$

and with $T(0) = J$, *the map* $s \to T(s)$ *is continuous from* $[0, \infty)$ *to* $\mathscr{L}_s(E, E)$.

(2) *In order that* (*) *should be satisfied with* $J = I$ *it is necessary and sufficient that*

(i) $\{T(s)\}_{s>0}$ *is a measurable semigroup,*

(ii) *For any* $a > 0$, $\{T(s)\}_{0<s\leqslant a}$ *is equicontinuous,*

(iii) $\bar{E}_0 = E$.

Proof. J can be considered as limit in $\mathscr{L}_s(E, E)$ of a sequence $\{T(t_n)\}_{n=1,2,\ldots}$, where $\{t_n\}_{n=1,2,\ldots}$ is any sequence such that $t_n \to 0+$. As E is Fréchet, $J \in \mathscr{L}(E, E)$. Taking limit as $t \to 0+$ of the two extreme sides of

$$(**) \qquad T(s)T(t)x = T(s+t)x = T(t)T(s)x$$

we have

$$T(s)J = JT(s).$$

Let $x \in E$ and p be a continuous seminorm on E. From (**), $\operatorname{Lt}_{t\to 0+} p[T(s+t)x] = p[T(s)Jx]$ exists. Hence for each $\varepsilon > 0$ and any $s_0 > 0$ at which the oscillation of $p[T(s)x]$ is greater than or equal to ε, there is an open interval $(s_0, s_0+\delta)$ such that the oscillation of $p[T(s)x]$ in that interval is less than ε. It follows that the oscillation of $p[T(s)x]$ is greater than or equal to ε at most at countable set of points so that $p[T(s)x]$ is discontinuous only at countably many points. Let $\{p_n\}_{n=1,2,\ldots}$ be a sequence of seminorms defining the topology of E. There exists a countable subset of $(0, \infty)$ outside which all the functions $p_n[T(s)x]$ are continuous. It follows that for $x \in E$, the E-valued function $T(s)x$ is continuous except on a countable set and is thus measurable. E being Fréchet, $\{T(s)\}_{s>0}$ is a continuous semigroup (cf. [9]). Then (**) gives us in the limit as $t \to 0+$,

$$T(s)Jx = T(s)x = JT(s)x \qquad (x \in E),$$

i.e. we have proved the equation in (1).

We now assert that for every $\alpha > 0$, $\{T(s)\}_{0<s\leqslant\alpha}$ is equicontinuous. If this is not true there exists an $\alpha > 0$ such that $\{T(s)\}_{0<s\leqslant\alpha}$ is not equicontinuous and therefore not bounded in $\mathscr{L}_s(E, E)$ (E being barreled); i.e. there exists $x \in E$ and a continuous seminorm p on E such that $\{p[T(s)x]\}_{0<s\leqslant\alpha}$ is not bounded. One can then find a sequence $\{s_n\}_{n=1,2,\ldots}$, $s_n \in (0, \alpha]$ such that $s_n \to s_0 \in [0, \alpha]$ and $p[T(s_n)x] \to \infty$ as $n \to \infty$. But $p[T(s_n)x] \to p[T(s_0)x]$ as $n \to \infty$ (where $T(s_0) = J$ if $s_0 = 0$). The contradiction proves the assertion.

Let $s_n, t_n \to 0+$ as $n \to \infty$ and let $x \in E$;

$$\begin{aligned} Jx &= \operatorname*{Lt}_{n\to\infty} T(s_n+t_n)x \\ &= \operatorname*{Lt}_{n\to\infty} T(s_n)Jx + T(s_n)\,[T(t_n)x - Jx] \\ &= \operatorname*{Lt}_{n\to\infty} T(s_n)Jx + \operatorname*{Lt}_{n\to\infty} T(s_n)\,[T(t_n)x - Jx]. \end{aligned}$$

Since $\{T(s_n)\}_{n=1,2,\ldots}$ is equicontinuous and $\operatorname*{Lt}_{n\to\infty} T(t_n)x = Jx$, we have

$$\operatorname*{Lt}_{n\to\infty} T(s_n)\,[T(t_n)x - Jx] = 0.$$

Hence

$$Jx = \operatorname*{Lt}_{n\to\infty} T(s_n)\,[T(t_n)x] = \operatorname*{Lt}_{n\to\infty} T(s_n)Jx = J^2x.$$

J is thus a continuous projection. Clearly $Jx \in \bar{E}_0$ for every $x \in E$ and $Jx = x$ for $x \in E_0$. As J is continuous, $Jx = x$ for $x \in \bar{E}_0$. So far we have proved (1) and the necessity part of (2). For proving the sufficiency part of (2) we observe that (i) implies that $\{T(s)\}_{s>0}$ is a continuous semigroup since E is Fréchet [9] and then for $x \in E_0$, we have $\operatorname*{Lt}_{t\to 0} T(t)x = x$.

(ii) i.e. equicontinuity of $\{T(t)\}_{0<t\leqslant 1}$ and (iii) then imply that $\operatorname{Lt}_{t\to 0} T(t)x = x$ for every $x \in E$.

THEOREM 10. *Let $\{T(s)\}_{s>0}$ be a semigroup of operators on E which satisfies the conditions in the hypothesis of Theorem 7. Then A_0 is closed.*

Proof. Let G denote the graph of A_0 in $E\times E$ and (x, y) be adherent to G in $E\times E$. We have to show that $x \in \mathscr{D}(A_0)$ and $A_0x = y$. The set $C =$ closed convex circled envelope of $\{T(s)\}_{0<s\leqslant\alpha}$ in $\mathscr{L}_s(E, E)$ is equicontinuous and for $s \leqslant \operatorname{Min}(\alpha, 1)$, $\int_0^s T(t)\,dt \in C$, where for every $x \in E$, $(\int_0^s T(t)\,dt)x = \int_0^s T(t)x\,dt = \operatorname{Lt}_{\varepsilon\to 0} \int_\varepsilon^s T(t)x\,dt$. Let V be any neighbourhood of zero and let V_1 be another neighbourhood of zero such that $V_1+V_1+V_1 \subset V$. Let U be a closed convex, circled neighbourhood of zero such that $U \subset V_1$ and $CU \subset V_1$. As (x, y) is adherent to G, there exists $z \in \mathscr{D}(A_0)$ such that $x-z \in U$ and $y-A_0z \in U$. Let $s \leqslant \operatorname{Min}(\alpha, 1)$;

$$\int_0^s T(t)y\,dt - [T(s)x-x]$$

$$= \Big(\int_0^s T(t)\,dt\Big)(y-A_0z) + \Big[\int_0^s T(t)A_0z\,dt - \big(T(s)z-z\big)\Big] + T(s)\,(z-x)+x-z.$$

By Theorem 4 we have $\int_0^s T(t)A_0z\,dt = T(s)z-z$. Hence

$$\int_0^s T(t)y\,dt - [T(s)x-x] = \Big(\int_0^s T(t)\,dt\Big)(y-A_0z) + T(s)\,(z-x)+x-z.$$

As $y-A_0z \in U$ and $\pm(z-x) \in U$ and $\int_0^s T(t)\,dt \in C$, the right-hand side of this equation belongs to $V_1+V_1+U \subset V$, where V is an arbitrary neighbourhood of zero in the Hausdorff space E. Thus

$$T(s)x-x = \int_0^s T(t)y\,dy \qquad \text{for } 0 < s \leqslant \operatorname{Min}(\alpha, 1)$$

and

$$\operatorname*{Lt}_{s\to 0+} A(s)x = \operatorname*{Lt}_{s\to 0+} \frac{1}{s}\int_0^s T(t)y\,dt = Jy,$$

i.e. $x \in \mathscr{D}(A_0)$ and $A_0x = Jy$. As y is adherent to the range of A_0 which is contained in $\bar{E}_0$, we have $Jy = y$ and the proof is completed.

THEOREM 11. *Let $\{T(s)\}_{s>0}$ be a continuous semigroup of operators on E, which satisfies the conditions in hypotheses of Theorem 7. If $y_n = A(\delta_n)x_0$ converges weakly to a limit y_0 for some sequence $\{\delta_n\}_{n=1,2,\ldots}$ such that $\delta_n \to 0+$, then $x_0 \in \mathscr{D}(A_0)$ and $A_0x_0 = y_0$.*

Proof. $\{A(\delta_n)x_0\}_{n=1,2,\ldots}$ being weakly convergent is bounded in E, i.e. $T(\delta_n)x_0 - x_0 \in \delta_n B$, $n = 1, 2, \ldots$, for some bounded set B of E. It follows that $T(\delta_n)x_0 \to x_0$ in E so that $x_0 \in \bar{E}_0$ and $Jx_0 = x_0$ (Theorem 7). Further $A(\delta_n)x_0 \in \bar{E}_0$ and the same is true of the weak limit y_0 since the closed subspace $\bar{E}_0$ is also weakly closed. Thus $Jy_0 = y_0$. Setting

$$x_n = \frac{1}{\delta_n}\int_0^{\delta_n} T(t)x_0\,dt \quad \text{and} \quad z_n = \frac{1}{\delta_n}\int_s^{s+\delta_n} T(t)x_0\,dt,$$

where $s + \delta_n \leqslant a$, we have $x_n \to Jx_0$ and $z_n \to T(s)x_0$ in E. Also

$$\begin{aligned} z_n - x_n &= \frac{1}{\delta_n}\int_0^{s} [T(t+\delta_n)x_0 - T(t)x_0]\,dt \\ &= \int_0^{s} T(t)A(\delta_n)x_0\,dt = \int_0^{s} T(t)y_n\,dt \\ &= \Big(\int_0^{s} T(t)\,dt\Big)(y_n). \end{aligned}$$

For $s < a$, the continuous linear operator $\int_0^s T(t)\,dt$ on E is also weakly continuous. As $y_n \to y_0$, we have $\big(\int_0^s T(t)\,dt\big)(y_n)$ converges to $\big(\int_0^s T(t)\,dt\big)(y_0)$ weakly i.e. $z_n - x_n$ converges weakly to $\big(\int_0^s T(t)\,dt\big)(y_0) = \int_0^s T(t)y_0\,dt$. But $z_n - x_n$ converges to $T(s)x_0 - x_0$ in the topology of E and therefore also in the weak topology. Hence

$$T(s)x_0 - x_0 = \int_0^{s} T(t)y_0 \quad \text{for } s < a$$

and $\operatorname{Lt}_{s\to 0} A(s)x_0 = Jy_0 = y_0$, i.e. $x_0 \in \mathcal{D}(A_0)$ and $A_0x_0 = y_0$.

THEOREM 12. *Let $\{T(s)\}_{s\geqslant 0}$ with $T(0) = I$ be a locally equicontinuous semigroup of operators on E such that $T(t)x$ converges weakly to x as $t \to 0+$ for every $x \in E$. Then $\{T(s)\}_{s\geqslant 0}$ is a continuous semigroup.*

Proof. For $x \in E$, $T(s)x$ is weakly right continuous from $(0, \infty)$ to E and hence weakly measurable. Further it is almost separably valued (i.e. for almost all s, $T(s)x$ belongs to a separable subspace of E) since if $\{r_n\}_{n=1,2,3,\ldots}$ is the sequence of rationals in $(0, \infty)$, then the closed subspace M of E spanned by $\{T(r_n)x\}_{n=1,2,\ldots}$ is separable and also weakly closed and therefore by weak right continuity of $T(s)x$ and the density of $\{r_n\}_{n=1,2,\ldots}$ in $[0, \infty)$ we have $T(s)x \in M$ for $s \geqslant 0$. Then by Proposition 1, [9], for any continuous seminorm q on E there exists a sequence $\{f_n(s)\}_{n=1,2,\ldots}$ of countably valued functions such that $q[T(s)x - f_n(s)] \to 0$ as $n \to \infty$, uniformly for s outside a set of measure zero of $(0, \infty)$. Further

since $\{T(s)\}_{s\geqslant 0}$ is locally equicontinuous, it follows from Remark 1 after Proposition 2 [9] that $\{T(s)\}_{s>0}$ is a continuous semigroup.

It remains to prove that $\operatorname{Lt}_{s\to 0} T(s)x = x$. Let N denote the linear space spanned by $\{T(r_n)x\}_{n=1,2,\ldots}$. For any r_n, we have $T(s)T(r_n)x = T(s+r_n)x$ and as $\{T(s)\}_{s>0}$ is a continuous semigroup we have $\operatorname{Lt}_{s\to 0} T(s+r_n)x = T(r_n)x$ so that $\operatorname{Lt}_{s\to 0} T(s)T(r_n)x = T(r_n)x$. Since each $y\in N$ is a finite linear combination of elements of the form $T(r_n)x$ we have $\operatorname{Lt}_{s\to 0} T(s)y = y$. For $t\in[0,1]$ and p a continuous seminorm on E and $y\in N$, we have $p[T(t)x-x]\leqslant p[T(t)x-T(t)y]+p[T(t)y-y]+p(y-x)$. The semigroup $\{T(s)\}_{s\geqslant 0}$ being locally equicontinuous $\{T(t)\}_{0\leqslant t\leqslant 1}$ is equicontinuous and so given p there exists a continuous seminorm q and a constant a such that

$$p[T(t)z]\leqslant aq(z) \text{ whatever } z\in E \text{ and } t\in[0,1].$$

Thus

$$p[T(t)x-x]\leqslant aq(x-y)+p[T(t)y-y]+p(y-x) \quad \text{for } t\in[0,1].$$

Now as $T(s)x$ converges weakly to x as $s\to 0$, x belongs to the weak closure of N. But the weak closure of N is the same as its closure, i.e. it is M. Thus for any $\varepsilon>0$ we can choose $y\in N$ such that $p(y-x)<\varepsilon/3$ and $aq(x-y)<\varepsilon/3$. Then we can choose $s_0\in(0,1)$ such that $p[T(t)y-y]<\varepsilon/3$ for $0\leqslant t\leqslant s_0$. Then $p[T(t)x-x]<\varepsilon$ for $0\leqslant t\leqslant s_0$. As p is any continuous seminorm on E this proves that $\operatorname{Lt}_{s\to 0} T(s)x = x$.

2. The semigroup $\{T(s)\}_{s\geqslant 0}$ is continuous means that $T(0)=I$ and the map $s\to T(s)$ from $[0,\infty)$ to $\mathscr{L}_s(E,E)$ is continuous.

DEFINITION. A semigroup $\{T(s)\}_{s>0}$ of operators on E is said to be *equicontinuous* if $\{T(s)\}_{s>0}$ is an equicontinuous family of operators on E.

If A_0 is the infinitesimal operator of a continuous semigroup $\{T(s)\}_{s>0}$, then the closed linear extension of the infinitesimal operator if it exists is called the *infinitesimal generator of the semigroup* $\{T(s)\}_{s>0}$. It is known that the infinitesimal operator of a continuous and equicontinuous semigroups $\{T(s)\}_{s\geqslant 0}$ is closed so that it is the infinitesimal generator of the semigroup. In Theorem 1 below we recall the Hille–Yosida Theorem ([8] Appendix or [10]) which gives necessary and sufficient condition in order that a closed linear densely defined operator be the infinitesimal operator of a continuous and equicontinuous semigroup. Theorem 2 is deduced immediately from Theorem 1 and is used to prove the perturbation Theorem 4.

THEOREM 1. *Let A be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space E. Then in order that A be the infinitesimal generator of a continuous and equicontinuous semigroup*

$\{T(s)\}_{s\geqslant 0}$ *it is necessary and sufficient that the family* $\{\lambda^n[R(\lambda, A)]^n/\lambda > 0, n = 1, 2, \ldots\}$ *is equicontinuous, where* $R(\lambda, A)$ *denotes the resolvant operator* $(\lambda I - A)^{-1}$ *of* A.

For the proof see [8], Appendix or [10].

Using Theorem 1 one can prove immediately,

THEOREM 2. *Let* A *be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space* E. *Then in order that* A *be the infinitesimal generator of a continuous semigroup* $\{T(s)\}_{s\geqslant 0}$ *with* $\{e^{-ws}T(s)\}_{s\geqslant 0}$ *equicontinuous for some* $w > 0$ *it is necessary and sufficient that the family*

$$\{(\lambda - w)^n[R(\lambda, A)]^n/\lambda > w,\ n = 1, 2, \ldots\}$$

is equicontinuous.

Proof. (i) *Necessity.* Let A be the infinitesimal operator of a continuous semigroup $\{T(s)\}_{s\geqslant 0}$ with $\{e^{-ws}T(s)\}_{s\geqslant 0}$ equicontinuous and let B be the infinitesimal generator of the continuous and equicontinuous semigroup $\{S(s)\}_{s>0}$, where $S(s) = e^{-ws}T(s)$. For $x \in \mathscr{D}(A)$, we have

$$\frac{1}{t}[S(t)x - x] = \frac{1}{t}[\exp(-wt) - 1]T(t)x + \frac{1}{t}[T(t)x - x].$$

Hence Bx exists and $Bx = Ax - wx$. Thus $\mathscr{D}(B) \supset \mathscr{D}(A)$. On the other hand, $T(s) = \exp(ws)S(s)$ so that by the same argument we have $\mathscr{D}(A) \supset \mathscr{D}(B)$. Therefore $\mathscr{D}(B) = \mathscr{D}(A)$ and $B = A - wI$. Since B is closed it follows that A is closed and is the infinitesimal generator of $\{T(s)\}_{s\geqslant 0}$; also $\mu I - B = (\mu + w)I - A$ so that $R(\mu, B) = R(\mu + w, A)$. Now by Theorem 1, the family

$$\{\mu^n[R(\mu, B)]^n/\mu > 0,\ n = 1, 2, \ldots\}$$

is equicontinuous. Hence setting $\lambda = \mu + w$ we have the equicontinuity of the family

$$\{(\lambda - w)^n[R(\lambda, A)]^n/\lambda > w,\ n = 1, 2, \ldots\}.$$

(ii) *Sufficiency.* Let A be a closed linear densely defined operator on E such that the family

$$\{(\lambda - w)^n[R(\lambda, A)]^n/\lambda > w,\ n = 1, 2, \ldots\}$$

is equicontinuous. Then $B = -wI + A$ is also closed and densely defined and $R(\lambda, A) = R(\lambda - w, B)$. Thus

$$\{\mu^n[R(\mu, B)]^n/\mu > 0,\ n = 1, 2, \ldots\}$$

is equicontinuous and therefore by Theorem 1, B is the infinitesimal generator of a continuous and equicontinuous semigroup $\{S(s)\}_{s\geqslant 0}$. Hence A is the infinitesimal generator of the continuous semigroup $\{T(s)\}_{s\geqslant 0}$ with $T(s) = e^{ws}S(s)$ and $\{e^{-ws}T(s)\}_{s\geqslant 0}$ equicontinuous.

Remark 1. Theorem 2 generalizes completely the Hille–Yosida theorem for Banach space [7] viz: A necessary and sufficient condition that a closed linear densely defined operator A generate a continuous semigroup $\{T(s)\}_{s\geqslant 0}$ of operators on a Banach space E is that there exist real numbers $M>0$ and w such that

$$\|[R(\lambda, A)]^n\| \leqslant M(\lambda-w)^{-n} \quad \text{for } \lambda > w,\ n=1,2,\ldots$$

since if $\{T(s)\}_{s\geqslant 0}$ is a continuous semigroups of operators on a Banach space there exists w such that $\{e^{-ws}T(s)\}_{s\geqslant 0}$ is equicontinuous (e.g. $w = w_0+\varepsilon$, where

$$w_0 = \inf_{s\geqslant 0} \frac{\log\|T(s)\|}{s} = \lim_{s\to\infty} \frac{\log\|T(s)\|}{s}$$

and $\varepsilon > 0$).

Remark 2. If $\{T(s)\}_{s\geqslant 0}$ is a continuous semigroup of operators on a locally convex space there may not exist an w such that $\{e^{-ws}T(s)\}_{s\geqslant 0}$ is equicontinuous, e.g. let E denote the space of functions $x(t)$ continuous on the real line and with compact support. For any compact subset of the real line let E_K denote the subspace of E formed of functions which vanish outside K and let E have the inductive limit topology defined by the subspaces E_K [3]. Let $\{T(s)\}_{s\geqslant 0}$ be the translation semigroup

$$[T(s)x](t) = x(t+s).$$

Then $\{T(s)\}_{s\geqslant 0}$ is a continuous semigroup for which there does not exist any w such that $\{e^{-ws}T(s)\}_{s\geqslant 0}$ is equicontinuous; since for any non-zero x in E, the supports of all the functions of the family

$$\{e^{-ws}T(s)x = e^{-ws}x(s+t)/s \geqslant 0\}$$

are not contained in any fixed compact set K and hence the family of operators

$$\{e^{-ws}T(s)/s \geqslant 0\}$$

is not bounded in $\mathscr{L}_s(E, E)$ and therefore is not equicontinuous.

Definition. A bounded operator on E is a *linear map from E* into itself which maps some neighbourhood of zero of E into a bounded subset of E.

Clearly a bounded operator is continuous.

Theorem 3. *Let E be a sequentially complete locally convex Hausdorff space and A be a densely defined closed linear operator on E such that for some $w>0$, the resolvant $R(\lambda, A)$ of A exists as a continuous linear operator on E for $\lambda > w$ and the family*

$$\{(\lambda-w)^n[R(\lambda, A)]^n/\lambda > w,\ n=1,2,\ldots\}$$

is equicontinuous. If B is a bounded operator on E, then there exists $w_1 > 0$ such that the resolvant $R(\lambda, A+B)$ of $A+B$ exists as a continuous linear operator E and the family

$$\{(\lambda-w_1)^n [R(\lambda, A+B)]^n / \lambda > w_1,\ n = 1, 2, \ldots\}$$

is equicontinuous.

Proof. It is easily seen that $A+B$ is a closed linear operator on E with dense domain $\mathscr{D}(A)$ of A. We first show that for λ sufficiently large, the partial sums of the series $\sum_{n=1}^{\infty} [BR(\lambda, A)]^n$ form an equicontinuous sequence which converges pointwise on E so that $\sum_{n=0}^{\infty} [BR(\lambda, A)]^n$ is a continuous linear operator on E. As B is bounded, there exists a neighbourhood U_0 of zero in E such that

$$B(U_0) = K \text{ is bounded;} \tag{1}$$

since the family $\{(\lambda-w)^n [R(\lambda, A)]^n / \lambda > w,\ n = 1, 2, \ldots\}$ is equicontinuous, there exists a convex circled neighbourhood V_0 of zero such that

$$R(\lambda, A) V_0 \subset \frac{1}{(\lambda-w)^n} U_0, \quad \lambda > w,\ n = 1, 2. \tag{2}$$

By (1), there exists $\alpha > 0$ such that

$$K \subset \alpha V_0. \tag{3}$$

Using (2) with $n = 1$, (1) and (3) we have

$$[BR(\lambda, A)] V_0 \subset \frac{1}{(\lambda-w)} K \subset \frac{\alpha}{\lambda-w} V_0$$

and

$$[BR(\lambda, A)]^2 V_0 \subset \frac{\alpha}{(\lambda-w)^2} K.$$

Suppose that

$$[BR(\lambda, A)]^n V_0 \subset \frac{\alpha^{n-1}}{(\lambda-w)^n} K. \tag{4}$$

Then

$$R(\lambda, A)\,[BR(\lambda, A)]^n V_0$$

$$\subset R(\lambda, A) \left[\frac{\alpha^{n-1} K}{(\lambda-w)^n}\right] \subset R(\lambda, A) \left[\frac{\alpha^n V_0}{(\lambda-w)^n}\right] \subset \frac{\alpha^n}{(\lambda-w)^{n+1}} U_0$$

so that

$$[BR(\lambda, A)]^{n+1} V_0 \subset \frac{\alpha^n}{(\lambda-w)^{n+1}} K.$$

Thus by induction (4) is true for $n = 1, 2, 3, \ldots$ $(\lambda > w)$. Let U be any convex circled neighbourhood of zero. There exists $\gamma > 0$ such that $\gamma K \subset U$. Then

$$[BR(\lambda, A)]^n(\gamma V_0) \subset \frac{\alpha^{n-1}}{(\lambda - w)^n} U.$$

As U is convex

$$\sum_{n=1}^{m} [BR(\lambda, A)]^n(\gamma V_0)$$

$$\subset \left\{ \sum_{n=1}^{m} \frac{\alpha^{n-1}}{(\lambda - w)^n} \right\} U \subset \frac{1}{\lambda - w} \left\{ \sum_{n=0}^{\infty} \frac{1}{2^n} \right\} U \qquad (\text{if } \lambda - w > 2\alpha)$$

$$= \frac{2}{\lambda - w} U \subset 2U \qquad (\text{if } \lambda - w > 1).$$

Thus for any convex circled neighbourhood U of zero there exists a neighbourhood $V = \frac{1}{2}\gamma V_0$ of zero such that

$$\left\{ \sum_{n=1}^{m} [BR(\lambda, A)]^n \right\} V \subset U$$

for $m = 1, 2, \ldots$ and

$$\lambda > w_1 = \operatorname{Max}(w+1, w+2\alpha). \tag{5}$$

This proves the required equicontinuity of the partial sums. E being sequentially complete, for any $x \in E$, the series $\sum_{n=1}^{\infty} [BR(\lambda, A)]^n x$ will be convergent if for any continuous seminorm p on E the series $\sum_{n=1}^{\infty} p\{[BR(\lambda, A)]^n x\}$ is convergent. Let $x \in E$ and $\beta > 0$ such that $x \in \beta V_0$. Then by (4)

$$[BR(\lambda, A)]^n x \in \beta \frac{\alpha^{n-1}}{(\lambda - w)^n} K.$$

If $k = \sup_{x \in K} p(x)$, then

$$p\{[BR(\lambda, A)]^n x\} \leqslant \frac{k\beta\alpha^{n-1}}{(\lambda - w)^n}$$

so that $\sum_{n=1}^{\infty} p\{[BR(\lambda, A)]^n x\}$ converges for $\lambda > \alpha + w$. We have thus proved that for $\lambda > w_1$, $\sum_{n=0}^{\infty} [BR(\lambda, A)]^n$ is a continuous linear operator on E. It is the inverse of $I - BR(\lambda, A)$.

Let

$$R = R(\lambda, A)\,[I - BR(\lambda, A)]^{-1}, \qquad \lambda > w.$$

Then

$$[\lambda I-(A+B)]R = [\lambda I-(A+B)]R(\lambda, A)\,[I-BR(\lambda, A)]^{-1}$$
$$= [I-BR(\lambda, A)]\,[I-BR(\lambda, A)]^{-1} = I.$$

Further the range of R is $\mathscr{D}(A)$ as the range of $[I-BR(\lambda, A)]^{-1}$ is E and the range of $R(\lambda, A)$ is $\mathscr{D}(A)$. Thus given $x \in \mathscr{D}(A)$, there exists a y such $x = Ry$. Therefore

$$R[\lambda I-(A+B)]x = R[\lambda I-(A+B)]Ry = Ry = x$$

so that R is both the left and right inverse of $\lambda I-(A+B)$, i.e. for $\lambda > w_1$, R is the resolvant operator $R(\lambda, A+B)$ of $A+B$. For any continuous seminorm q, the series

$$\sum_{n=1}^{n} q\{R(\lambda, A)\,[BR(\lambda, A)]^n x\}$$

is convergent for $\lambda > w$ since by equicontinuity of

$$\{(\lambda-w)\,[R(\lambda, A)]^n/\lambda > w,\ n = 1, 2, \ldots\},$$

there exists a continuous seminorm p such that

$$q\{(\lambda-w)^n[R(\lambda, A]^n x\} \leqslant p(x) \quad \text{for } \lambda > w,\ n = 1, 2, \ldots$$

which implies that

$$q\{(\lambda-w)R(\lambda, A)\,[BR(\lambda, A)]^n x\} \leqslant p\{[BR(\lambda, A)]^n x\},$$

i.e.

$$q\{R(\lambda, A)\,[BR(\lambda, A)]^n x\} \leqslant \frac{1}{\lambda-w}\, p\{[BR(\lambda, A)]^n x\}$$
$$\leqslant p\{[BR(\lambda, A)]^n x\} \quad \text{for } \lambda > w_1$$

since $w_1 \geqslant w+1$ by (5) and $\sum p\{[BR(\lambda, A)]^n x\}$ is proved to be convergent. We now wish to regroup the terms of $[R(\lambda, A+B)]^n = \{\sum_{j=0}^{\infty} R(\lambda, A)\times$ $\times[BR(\lambda, A)]^j\}^n$ according to the number of factors B. Let $\{s_m\}_{m=0,1,2,\ldots}$ denote the sequence of partial sums of the series $\sum_{j=0}^{\infty} R(\lambda, A)\,[BR(\lambda, A)]^j$. This sequence is equicontinuous and therefore it can be easily verified that

$$\Big\{\sum_{j=0}^{\infty} R(\lambda, A)\,[BR(\lambda, A)]^j\Big\}^n = \operatorname*{Lt}_{m\to\infty} s_m^n$$

the limit being in $\mathscr{L}_s(E, E)$.

s_m being a finite sum,

$$s_m^n = \{R(\lambda, A)+R(\lambda, A)BR(\lambda, A)+R(\lambda, A)\,[BR(\lambda, A)]^2+ \\ + \ldots + R(\lambda, A)\,[BR(\lambda, A)]^m\}^n$$

may be written in the form

$$V(m, 0)+V(m, 1)+\ldots+V(m, mn),$$

where $V(m, k)$ denotes the sum of terms in s_m^n which contain precisely k factors B.

For $m \geqslant k$, $V(m, k) = V(m+1, k) = \ldots = V_k$, say. Hence it follows that

$$\Big\{\sum_{j=0}^{\infty} R(\lambda, A)\,[BR(\lambda, A)]^j\Big\}^n = \operatorname*{Lt}_{m\to\infty} s_m^n = \sum_{k=0}^{\infty} V_k.$$

Any term u_k of V_k will possess $n+k$, $R(\lambda, A)$'s, since each B introduces another $R(\lambda, A)$. Further the $R(\lambda, A)$'s will be in $k+1$ non-empty groups separated from each other by the kB's. In other words u_k will be of the form

$$[R(\lambda, A)]^{r_1}B[R(\lambda, A)]^{r_2}B \ldots [R(\lambda, A)]^{r_k}B[R(\lambda, A)]^{n_{k+1}},$$

where

$$\sum_{i=1}^{k+1} r_i = n+k \quad \text{and} \quad r_i > 0 \tag{6}$$

using (1), (2) and (3) repeatedly one can verify that

$$u_k(V_0) \subset [R(\lambda, A)]^{r_1}[(\lambda-w)^{-r_2}\alpha(\lambda-w)^{-r_3}\alpha \ldots (\lambda-w)^{-r_{k+1}}K], \tag{7}$$

where α occurs $k-1$ times.

The family $\{(\lambda-w)^n[R(\lambda, A)]^n/\lambda > w,\ n = 1, 2, \ldots\}$ being equicontinuous, it is bounded on the bounded subset K. For any continuous seminorm q let

$$M = \sup\{q\{(\lambda-w)^n[R(\lambda, A)]^n x\}/x \in K, \lambda > w,\ n = 1, 2, \ldots\}.$$

Then for $x \in K$,

$$q\{[R(\lambda, A)]^{r_1}x\} \leqslant M(\lambda-w)^{-r_1}.$$

Hence from (6) and (7), for $x \in V_0$ we have

$$q[u_k(x)] \leqslant (\lambda-w)^{-n-k}\alpha^{k-1}$$

so that if p is the Minkowski functional of V_0, then $p(x) \leqslant 1$ implies that

$$q[u_k(x)] \leqslant \frac{M}{\alpha}(\lambda-w)^{-n-k}\alpha^k$$

which proves that

$$q[u_k(x)] \leqslant \frac{M}{a}(\lambda-w)^{-n-k}a^k p(x) \qquad (x \in E).$$

Hence

$$q\big([R(\lambda, A+B)]^n x\big) \leqslant \sum_{k=0}^{\infty} C_k^n (\lambda-w)^{-n}\frac{M}{a}\left(\frac{a}{\lambda-w}\right)^k p(x)$$

for $\lambda > w_1$, where C_k^n is the coefficient of x^k in $(1-x)^{-n}$, i.e.

$$q\big([R(\lambda, A+B)]^n x\big) \leqslant \frac{M}{a}(\lambda-w)^{-n}\left(1-\frac{a}{\lambda-w}\right)^{-n} p(x)$$

$$= \frac{M}{a}(\lambda-w-a)^{-n}p(x)$$

$$\leqslant \frac{M}{a}(\lambda-w_1)^{-n}p(x) \qquad \text{(since by (5), } \lambda-w-a > \lambda-w_1).$$

We have thus proved that for a continuous seminorm q there exists a continuous seminorm p and a constant $a = \dfrac{M}{a} > 0$ such that

$$q\{(\lambda-w_1)^n[R(\lambda, A+B)]^n x\} \leqslant ap(x), \qquad x \in E, \lambda > w_1, n = 1, 2, \ldots,$$

i.e. we have proved the required equicontinuity of

$$\{(\lambda-w_1)^n[R(\lambda, A+B)]^n / \lambda > w_1,\ n = 1, 2, \ldots\}.$$

Combining Theorem 2 and Theorem 3. We have the following perturbation theorem:

THEOREM 4. *Let E be a sequentially complete locally convex Hausdorff space and let A be the infinitesimal generator of a continuous semigroup $\{T(s)\}_{s\geqslant 0}$ such that for some $w > 0$ $\{e^{-ws}T(s)\}_{s\geqslant 0}$ is equicontinuous. If B is a bounded operator on E, then $A+B$ is the infinitesimal generator of a continuous semigroup $\{S(s)\}_{s\geqslant 0}$ such that for some $w_1 > 0$, $\{e^{-w_1 s}S(s)\}_{s\geqslant 0}$ is equicontinuous.*

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