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Intrinsic topologies on a poset

Given a poset $(X; \leqslant)$ with at least two elements, the interval topology $\mathscr{I}$ on X is defined by using as a subbase for the closed sets, the sets

$$Pa = \{x\colon x \in X \text{ and } x \leqslant a\} \quad \text{and} \quad bP = \{x\colon x \in X \text{ and } b \leqslant x\},$$

where $a, b \in X$ and where $P = \{(x, y) \in X \times X\colon x \leqslant y\} \subset X \times X$ is the partial order on X. The purpose of this note is to define another intrinsic topology $\mathscr{U}$ on X in terms of the partial order P and to study the relationship between $\mathscr{I}$ and $\mathscr{U}$, and to show, inter alia, that for a conditionally complete, order-dense semilattice, the two topologies coincide if and only if the order is linear (Proposition 6 and its Corollary). This also helps to characterise some semilattices for which the associated partial order is strongly continuous (Proposition 10). The relevant definitions appear below.

1. Preliminery results. Unless otherwise stated, we follow Birkhoff [1] in notation and the terminology. For the definition of the interval topology $\mathscr{I}$ and some of its properties, one may refer to [1] or [3] among others. For $a \in X$, let

$$P_0a = \{x \in X\colon x < a\} \quad \text{and} \quad aP_0 = \{x \in X\colon a < x\},$$

where $x < a$ means $x \leqslant a$ but $x \neq a$. Let $\mathscr{U}$ denote the topology on X obtained by using as a subbase for the topology (i.e. for the open sets) the sets of the type P_0a and bP_0, where $a, b \in X$. In case X is linear, it is clear that $\mathscr{I}$ and $\mathscr{U}$ coincide. Simple examples show that this is not the case in general. However, since $\mathscr{I}$ is always T_1 [3], we have that $\mathscr{I} \supseteq \mathscr{U}$ when X is finite.

DEFINITION. A poset X is *order-dense* (or simply *dense*) if, whenever $a < b$, there is $c \in X$ such that $a < c < b$. A lattice is order-dense if it is order-dense in the associated partial order.

PROPOSITION 1. *If* $(X; \wedge, \vee)$ *is an order-dense lattice, then* $\mathscr{U} \supseteq \mathscr{I}$.

Proof. If suffices to show that a subbasic open set $X \setminus Pa$ or $X \setminus bP$ in $\mathscr{I}$ is also open in $\mathscr{U}$. Let $x \in X \setminus Pa$. Then either $x > a$ or x is incomparable with a. If $x > a$, then $x \in aP_0 \subset X \setminus Pa$. In the latter case, $x \neq a \wedge x \neq a$ and there is $z \in X$ such that $a \wedge x < z < x$. Then $x \in zP_0 \subset X \setminus Pa$. Thus $X \setminus Pa$ is $\mathscr{U}$-open. One argues similarly to show that $X \setminus aP$ is also $\mathscr{U}$-open.

Simple examples can be constructed to show that the condition that X be an order-dense lattice is necessary.

As remarked earlier, if X is linearly ordered, $\mathscr{I}$ and $\mathscr{U}$ coincide without any additional restriction on X. The properties of this common topology are discussed, for example, in [4] or [5]. To prove the converse, some preliminary results are needed.

PROPOSITION 2. *Every maximal loset* L *in a poset* $(X, \leqslant)$ *is closed in the interval topology* $\mathscr{I}$ *on* X.

Proof. Writing $N(a) = \{x \in X: x \text{ is incomparable with } a\}$, we note that $N(a) = X \setminus (Pa \coprod aP)$ is open in $\mathscr{I}$ for every $a \in X$. If L is a loset in X, then $L \subset X \setminus \coprod \{N(a): a \in L\}$ and every $p \in X \setminus \coprod \{N(a): a \in L\}$ is comparable with every $a \in L$. Hence, if L is maximal, $L = X \setminus \coprod \{N(a): a \in L\}$ and is thus $\mathscr{I}$-closed.

DEFINITION. A poset $(X, \leqslant)$ is *order-complete* if every non-empty subset of X has both the supremum and the infimum. It is said to be *conditionally order-complete* if every non-empty subset of X bounded above has the supremum and dually.

PROPOSITION 3. *A maximal loset* M *in an order-complete* [*resp. conditionally order-complete*] *poset* X *is order-complete* [*resp. conditionally order-complete*] *in the inherited order.*

Proof. Straightforward.

COROLLARY. *With* M *and* X *as above, for any* $A \subset M$,

$$\inf_X A = \inf_M A \quad \text{and} \quad \sup_X A = \sup_M A$$

whenever they exist.

PROPOSITION 4. *Let* M *be a maximal loset in a conditionally order-complete poset* X. *Then the interval topology of* M *is the same as its subspace topology induced from the interval topology* $\mathscr{I}$ *on* X.

Proof. If $Q = P \coprod (M \times M)$, then Q is linear on M and the interval topology $\mathscr{I}_1$ on M is obtained by using as a subbase for the closed sets, the family $\{aQ, Qb: a, b \in M\}$. Since for any $a \in M$, $Qa = Pa \coprod M$ and $aQ = aP \coprod M$ and since by Proposition 2, M is $\mathscr{I}$-closed, it follows that Qa, aQ are $\mathscr{I}$-closed, so that $\mathscr{I}_1 \subseteq \mathscr{I}_M$, the subspace topology on M.

Conversely, let F be any subset of M closed in $\mathscr{I}_M$. Then $F = F_1 \coprod M$, where $F_1 = \coprod_\lambda (\coprod_i Pa_{i,\lambda} \coprod \coprod_j b_{j,\lambda} P)$ for some $a_{i,\lambda}, b_{j,\lambda} \in X$ and where i, j

take values in some finite index sets and λ in some index set. Hence,

$$F = F_1 \prod M = \coprod_\lambda \left(\coprod_i Pa_{i,\lambda} \coprod \coprod_j b_{j,\lambda} P\right) \prod M$$
$$= \coprod_\lambda \left(\coprod_i \left(Pa_{i,\lambda} \prod M\right) \coprod \coprod_j \left(b_{j,\lambda} P \prod M\right)\right)$$

and by Proposition 3 and its Corollary, this yields,

$$F = \coprod_\lambda \left(\coprod_i Qu_{i,\lambda} \coprod \coprod_j v_{j,\lambda} Q\right)$$

for some $u_{i,\lambda}, v_{j,\lambda} \in M$. Hence F is $\mathscr{I}_1$-closed and $\mathscr{I}_M \subseteq \mathscr{I}_1$.

2. Intrinsic topologies on a poset. We start with a preliminary result needed to establish the main result.

PROPOSITION 5. *Let X be an order-dense, conditionally order-complete poset such that for every $a, b \in X$, there is $t \in X$ such that both a and b are comparable with t. Then X is connected in its interval topology.*

Proof. Let $a, b \in X$ be arbitrary. Then by hypothesis, there is $t \in X$ such that $\{a, t\}$ and $\{b, t\}$ are both losets. Applying Hausdorff's maximality principle, we obtain maximal losets M and N such that $\{a, t\} \subset M$ and $\{b, t\} \subset N$. By Proposition 3, both M and N are order-dense and conditionally order-complete.

It is well known (see [4], for example) that M and N are connected in their respective interval topologies. Hence by Proposition 4; M, N are connected subsets of X. Since $t \in M \prod N$, this implies that $M \coprod N$ is a connected subset of X. Thus every pair of elements in X lies in the same $\mathscr{I}$-component. Hence X is connected in $\mathscr{I}$.

PROPOSITION 6. *Let X be an order-dense, conditionally order-complete poset such that for every $a, b \in X$, there is $t \in X$ comparable with both of them. Then X is a loset if and only if $\mathscr{I} = \mathscr{U}$.*

Proof. If X is a loset, then $\mathscr{I} = \mathscr{U}$ without any more restriction on X. Conversely, let X be as above and suppose that $\mathscr{I} = \mathscr{U}$. Suppose there exist incomparable elements $a, b \in X$. By hypothesis, there is $t \in X$ comparable with both a and b. We cannot have $a \leqslant t$ and $t \leqslant b$ for then $a \leqslant b$ contradicting the assumption that a and b are incomparable. For the same reason, we cannot have $t \leqslant a$ and $b \leqslant t$. Thus either $a \leqslant t$ and $b \leqslant t$ or $t \leqslant a$ and $t \leqslant b$.

Suppose $t \leqslant a$ and $t \leqslant b$. The other case is dealt with similarly. Then $t \in P_0 a \prod P_0 b$. Without loss of generality, we may assume $t = \sup(P_0 a \prod P_0 b)$, so that $Pt = P_0 a \prod P_0 b$. Now, since $P_0 a$ and $P_0 b$ are $\mathscr{U}$-open, they are open in the common topology and Pt is open. But Pt is $\mathscr{I}$-closed and hence closed in the common topology. We may thus write $X =$

$= Pt \amalg tP_0 \amalg N(t)$, where it may be recalled that $N(t)$ is the set of elements incomparable with t and is open in $\mathscr{I}$ and hence in the common topology.

Thus X is a disjoint union of three open sets at least two of which, namely Pt and tP_0 are non-empty. Hence X is disconnected in the common topology, i.e. in $\mathscr{I}$. But this contradicts Proposition 5. Hence X must be a loset.

COROLLARY. *An order-dense, conditionally complete semilattice is linearly ordered if and only if* $\mathscr{I} = \mathscr{U}$.

3. Intrinsic topologies and the strong continuity of the partial order. A partial order P on a topological space X is customarily called *continuous* if $P = P^*$ (where * always denotes the closure of a set). Topological spaces with a continuous partial order have been studied, among others, by Ward [6]. A partial order P is said to be *strongly continuous* (see [2]) (i) if whenever $a < b$, there exist (open) neighbourhoods G_a and G_b of a and b respectively such that $x \in G_a$ and $y \in G_b$ implies $x < y$ and (ii) if whenever a is incomparable with b, there exist neighbourhoods G_a and G_b of a and b respectively such that every $x \in G_a$ is incomparable with every $y \in G_b$. It was shown in [2] that this is equivalent to saying that $\partial P \subset \Delta$, where ∂P is boundary of P and $\Delta = \{(x, x): x \in X\}$ is the *diagonal* of X.

It is well known (see [6]) that if P is a continuous partial order on a topological space $(X, \mathscr{T})$, then $\mathscr{T} \supseteq \mathscr{I}$, the interval topology on X. A similar relation holds between the strong continuity of P and the topology $\mathscr{U}$ defined earlier.

PROPOSITION 7. *If P is a strongly continuous partial order on a topological space $(X, \mathscr{T})$, then $\mathscr{T} \supseteq \mathscr{U}$.*

Proof. It suffices to prove that for any $a \in X$, the sets P_0a and aP_0 are $\mathscr{T}$-open. Consider

$$P_0a \times aP_0 = \{(x, y): x < a < y\} \subset X \times X.$$

Let $(p, q) \in P_0a \times aP_0$ be arbitrary. Then $p < a < q$ and applying the strong continuity of P, we find $\mathscr{T}$-open neighbourhoods G_p, G_a and G_q of p, a and q respectively such that every $x \in G_p$ and $y \in G_q$ satisfy $x < a < y$. Thus

$$(p, q) \in G_p \times G_q \subset P_0a \times aP_0.$$

Hence $P_0a \times aP_0$ is open in $X \times X$. Since the projection maps π_1 and π_2 are open, it follows that the sets $\pi_1(P_0a \times aP_0) = P_0a$ and $\pi_2(P_0a \times \times aP_0) = aP_0$ are both-open. The proof is valid even if either aP_0 or P_0a is empty, with obvious modifications.

DEFINITION. In a poset $(X; \leqslant)$, (a, b) is said to be a *gap* if $a < b$ and if there is no $x \in X$ such that $a < x < b$.

DEFINITION. A semilattice $(X; \wedge)$ with a topology on X is a *topo-*

logical semilattice if X is a Hausdorff space and if the semilattice $\wedge: X \times X \to X$ is a continuous map.

It is well known that a semilattice with a topology on it is Hausdorff if the associated partial order $\leqslant$ is continuous in the usual sense. Thus if $\leqslant$ is strongly continuous on X, X is Hausdorff.

PROPOSITION 8. *Let $(X, \mathscr{T})$ be a topological space with a semilattice $\wedge$ on X such that every component of X is a subsemilattice. If the partial order P (or $\leqslant$) derived from $\wedge$ is strongly continuous, then every component of X is linearly ordered. If, in addition, X is a compact space, then it is a topological semilattice.*

Proof. Since P is strongly continuous, by Proposition 7, $\mathscr{T} \supseteq \mathscr{U}$ and $\mathscr{T} \supseteq \mathscr{I}$. Suppose there are incomparable elements a, b in a component C of X. Then $z = a \wedge b \in C$, by hypothesis and $Pz = P_0 a \prod P_0 b$ is an open set in $\mathscr{U}$ and hence in $\mathscr{T}$. Similarly Pz is a closed set in $\mathscr{I}$ and therefore in $\mathscr{T}$. Thus we may write

$$C = (C \prod Pz) \coprod (C \prod (X \setminus Pz))$$

a disjoint union of two non-empty open sets. This contradiction proves that C must be linearly ordered.

By the remark above, X is a Hausdorff space. Now, to show that X is a topological semilattice, let $a, b \in X$ be arbitrary. If $a < b$, then $a = a \wedge b$ and for a neighbourhood G of $a = a \wedge b$ take $U = P_0 b \prod C_a \prod G$ and $V = a P_0 \prod C_b$ if (a, b) is a gap. Then U and V are neighbourhoods of a and b respectively and if $x \in U$ and $y \in V$, then $x < a < b$ and $a < b < y$, so that $x < y$ or $x \wedge y = x \in G$. If there is $z \in (a, b)$, then take $U = P_0 z \prod G$ and $V = z P_0$. Again, U and V are neighbourhoods of a and b respectively and if $x \in U$ and $y \in V$, then $x < z < y$, so that $x \wedge y = x \in G$.

If $a = b$, then $a = a \wedge b$ and for any neighbourhood G of a, take $U = V = G \prod C_a$; then $x \in U$ and $y \in V$ implies that x and y are comparable and in G so that $x \wedge y$ is either x or y, but in either case, $x \wedge y \in G$. Note that we did not need the hypothesis of compactness so far.

Suppose a and b are incomparable. Let $\{(x_n, y_n); n \in D\}$ be a net in $X \times X$ converging to (a, b). We must show that the net $\{x_n \wedge y_n: n \in D\}$ in X converges to $a \wedge b = z$. Since X is compact, the net $\{x_n \wedge y_n: n \in D\}$ has a cluster point in X, say z_1. Also, $(x_n \wedge y_n, x_n) \in P$ for every $n \in D$ and since $\leqslant$ is strongly continuous, P is closed. Hence, by considering the subnet of $\{x_n \wedge y_n: n \in D\}$ converging to z_1 and the corresponding subnet of $\{x_n: n \in D\}$ converging to a, it follows that $z_1 \leqslant a$. Similarly, $z_1 \leqslant b$, so that $z_1 \leqslant z = a \wedge b$.

If $z_1 < z$, the nets $\{x_n: n \in D\}$ and $\{y_n: n \in D\}$ will eventually be in the neighbourhood $z P_0$ of a and of b. Thus, $z < x_n$ and $z < y_n$ so that, $z \leqslant x_n \wedge y_n$ eventually. In other words, the net $\{x_n \wedge y_n: n \in D\}$ cannot be

frequently in the neighbourhood $P_0 z$ of z_1. This contradiction shows that z_1 must equal z. In other words, $\{x_n \wedge y_n : n \in D\}$ has a unique cluster point z. Hence it converges to z.

In the reverse direction, we have the following

PROPOSITION 9. *Let $(X, \mathscr{T}; \wedge)$ be a topological semilattice such that every component of X is a complete, linear subsemilattice. Then the partial order $\leqslant$ derived from $\wedge$ is strongly continuous.*

Proof. It may be noted first that since X is a topological semilattice, $\leqslant$ is continuous in the usual sense [6]. A comparison between the definitions of the usual continuity and the strong continuity of $\leqslant$ shows that we need to prove only the following: "If $a < b$, then there exist neighbourhoods U and V of a and b respectively, such that if $x \in U$ and $y \in V$, then $x < y$."

Since $\leqslant$ is continuous in the usual sense, $\mathscr{T} \supseteq \mathscr{S}$ and for any $x \in X$; Px and xP are closed sets. Suppose $a < b$ and that a and b belong to the same component C. If (a, b) is a gap in C, take $U = (X \setminus bP) \prod C$ and $V = (X \setminus Pa) \prod C$, then $x \in U$ and $y \in V$ imply $x < a < b < y$ so that $x < y$. If there is $z \in (a, b) \prod C$, then it is easily seen that the neighbourhoods $U = (X \setminus zP) \prod C$ and $V = (X \setminus Pz) \prod C$ of a and b respectively will do.

Now suppose a and b belong to different components, say C_a and C_b respectively. Let $r = \wedge\{x \in C_b : a < x \leqslant b\}$, $s = \wedge\{x \in C_a : x \text{ non } \leqslant r\}$. Take $s = a$ if there is no such $x \in C_a$ such that x non $\leqslant r$. Then $r \in C_b$, $s \in C_a$; take $z = r \wedge s$. Then we have $a \leqslant z \leqslant r \leqslant b$. Several cases arise depending upon whether any of the above is a strict inequality or an equality.

Case (i) Suppose $a < z < r < b$. Take $U = (X \setminus zP) \prod C_a$ and $V = (X \setminus Pr) \prod C_b$, as the neighbourhoods of a and b respectively. Then $x \in U$ and $y \in V$ imply that $x \leqslant s$ so that $x \leqslant r < y$ or $x < y$.

Case (ii) Suppose $a = z < r < b$. Then take $U = (X \setminus rP) \prod C_a$ and $V = (X \setminus Pr) \prod C_b$ as the neighbourhoods of a and b respectively. Clearly, if $x \in U$ and $y \in V$, then $x < r < y$, so that $x < y$.

Case (iii) If $a < z = r < b$, take $U = (X \setminus zP) \prod C_a$ and $V = (X \setminus Pz) \prod C_b$.

The choice of U and V in other cases is obvious. Thus the proof is completed.

Finally, one gets a characterisation of the strong continuity of a partial order by combining the hypothesis of the last two propositions. This yields the following

PROPOSITION 10. *Let $(X, \mathscr{T})$ be a compact space with a semilattice $\wedge$ on X such that every component of X is a complete subsemilattice. Then the partial order $\leqslant$ on X derived from $\wedge$ is strongly continuous if and only if X is a topological semilattice and every component of X is linear.*

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