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On a certain boundary problem for the equation $\Delta^2 u - C^4 u = 0$

1. In the paper [3] it has been constructed the fundamental solution for the equation

$$\Delta^2 u(X) - C^4 u(X) = 0, \tag{1}$$

where $X = (x_1, x_2)$ and C is a positive constant. Subsequently it was applied to solve the boundary problems of Lauricelli and of Riquier for equation (1) in the bounded domain D. In this paper we construct the Green function and the solution of the boundary problem for equation (1) for the half-plane $x_2 > 0$ with the conditions:

$$u(x_1, 0) = f_1(x_1), \qquad \Delta u(x_1, 0) = f_2(x_1). \tag{1a}$$

2. We shall construct the Green function by the method of symmetric images for the half-plane $x_2 > 0$. Let the two points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, $X \neq Y$ be given in this half-plane and let us denote by $\bar{X} = (x_1, -x_2)$ the symmetric image of the point X with respect to the axis y_1. Futher let us write:

$$r = XY = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}, \quad r_1 = \bar{X}Y = [(x_1 - y_1)^2 + (x_2 + y_2)^2]^{1/2}$$

and

$$\varrho = [(x_1 - y_1)^2 + x_2^2]^{1/2}.$$

As we know [3] the fundamental solution of equation (1) is the function

$$U(r) = \pi Y_0(Cr) + 2K_0(Cr), \tag{2}$$

where $Y_0(z)$ is the Bessel function of order zero of the second kind and $K_0(z)$ is the Mac Donald function of order zero ([2], p. 113 and 117).

We are going to prove:

THEOREM 1. *The function*

$$G(X, Y) = U(r) - U(r_1), \tag{3}$$

where $U(r)$ is given by (2) *is a Green function with the pole X for equation* (1) *and for the half-plane $x_2 > 0$ with the boundary conditions*:

$$G(X, Y)|_{y_2=0} = 0, \quad \Delta_Y G(X, Y)|_{y_2=0} = 0. \tag{4}$$

Proof. $U(r_1)$ as a function of Y is defined and of class C^4 in the half-plane $x_2 > 0$ and satisfies equation (1) in this set. For $Y = (y_1, 0)$ we have $r = r_1 = \varrho$ and $G(X, Y)|_{y_2=0} = U(\varrho) - U(\varrho) = 0$. Since $\Delta_Y U(r) = U''(r) + r^{-1} U'(r)$ and so

$$\Delta_Y G(X, Y)|_{y_2=0} = [\Delta_Y U(r) - \Delta_Y U(r_1)]|_{y_2=0} = 0.$$

3. Let $u(X)$ be a function of class C^4 in the domain $x_2 > 0$ satisfying equation (1) and the boundary conditions (1a). Applying formally the fundamental formula of [3] to functions $u(X)$ and $U(r_1)$ and applying the definition of the fundamental solution we obtain respectively:

$$0 = -(8\pi C^2)^{-1} \int_{-\infty}^{+\infty} \left[\Delta u(Y) \frac{\partial U(r_1)}{\partial y_2} - \Delta U(r_1) \frac{\partial u(Y)}{\partial y_2} + \right.$$

$$\left. + u(Y) \frac{\partial \Delta U(r_1)}{\partial y_2} - U(r_1) \frac{\partial \Delta u(Y)}{\partial y_2} \right]\Big|_{y_2=0} dy_1$$

and

$$u(X) = -(8\pi C^2)^{-1} \int_{-\infty}^{+\infty} \left[U(r) \frac{\partial \Delta u(Y)}{\partial y_2} - u(Y) \frac{\partial \Delta U(r)}{\partial y_2} + \right.$$

$$\left. + \Delta U(r) \frac{\partial u(Y)}{\partial y_2} - \Delta u(Y) \frac{\partial U(r)}{\partial y_2} \right]\Big|_{y_2=0} dy_1.$$

If we add the above formulas then we obtain in view of (1a), (3) and (4)

$$u(X) = (8\pi C^2)^{-1} \int_{-\infty}^{+\infty} \left[f_1(y_1) \frac{\partial \Delta_Y G(X, Y)}{\partial y_2} + \right. \tag{5}$$

$$\left. + f_2(y_1) \frac{\partial G(X, Y)}{\partial y_2} \right]\Big|_{y_2=0} dy_1.$$

From the recurrence formulas ([2], p. 111 and 117)

$$\frac{d}{dz}[z^{-n} K_n(z)] = -z^{-n} K_{n+1}(z), \quad \frac{d}{dz}[z^{-n} Y_n(z)] = -z^{-n} Y_{n+1}(z)$$

and from the identities

$$\Delta_Y Y_0(Cr) + C^2 Y_0(Cr) = 0, \quad \Delta_Y K_0(Cr) - C^2 K_0(Cr) = 0$$

we have

$$(6)\qquad \left.\frac{\partial G(X,Y)}{\partial y_2}\right|_{y_2=0} = 2Cx_2N_2(\varrho), \qquad \left.\frac{\partial \Delta_Y G(X,Y)}{\partial y_2}\right|_{y_2=0} = 2C^3x_2N_1(\varrho),$$

where

$$N_2(\varrho) = \varrho^{-1}[\pi Y_1(C\varrho)+2K_1(C\varrho), \qquad N_1(\varrho) = \varrho^{-1}[-\pi Y_1(C\varrho)+2K_1(C\varrho)].$$

Substituting expressions (6) into (5) we obtain:

$$(7)\qquad u(X) = (4\pi C)^{-1}x_2 \int_{-\infty}^{+\infty} [f_1(y_1)C^2N_1(\varrho)+f_2(y_1)N_2(\varrho)]dy_1.$$

LEMMA 1. *Let* $\int_{-\infty}^{+\infty} |f_i(y_1)|\,dy_1$, $i = 1, 2$ *be convergent. Then the function* $u(X)$ *given by formula* (7) *is of class* C^4 *in the half-plane* $x_2 > 0$ *and satisfies equation* (1) *in this set.*

Proof. We shall prove that the integral on the right-hand side of formula (7) as well as the integrals we get by differentiating n-times ($n = 1, 2, 3, 4$) with respect to x_1 and x_2 the functions under the sign of integral (7) exist and are uniformly convergent for X belonging the rectangle $P(a, A, B)$: $\{|x_1| \leqslant B,\ a \leqslant x_2 \leqslant A\}$, where a, A, B are arbitrary positive constants. The functions we obtain by n-times differentiation of the kernels $N_1(\varrho)$ and $N_2(\varrho)$ with respect to x_1 and x_2 are linear combinations of the functions:

$$\Phi_1 = x_2^{\gamma}(x_1-y_1)^{\alpha}\varrho^{-n}Y_n(C\varrho), \qquad \Phi_2 = x_2^{\gamma}(x_1-y_1)^{\alpha}\varrho^{-n}K_n(C\varrho),$$

where $n = 1, 2, 3, 4, 5$, $\gamma = 0, 1, 2, 3, 4$, $\alpha = 0, 1, 2, 3, 4$, $\gamma+\alpha \leqslant n-1$. It is enough to know that the integrals

$$(8)\qquad \int_{-\infty}^{+\infty} f_i(y_1)\Phi_1 dy_1, \qquad \int_{-\infty}^{+\infty} f_i(y_1)\Phi_2 dy_1, \qquad i = 1, 2$$

exist and are uniformly convergent for $X \in P(a, A, B)$. For $X \in P(a, A, B)$ we have $a \leqslant \varrho$. From that and from ([2], p. 145) we see that the functions $Y_n(C\varrho)$ and $K_n(C\varrho)$ are bounded for $a \leqslant \varrho$.

From these results we can obtain the inequalities:

$$|\Phi_1| \leqslant \varrho^{-n+\gamma+\alpha}|Y_n(C\varrho)| \leqslant A_{n,\gamma,\alpha}, \qquad |\Phi_2| \leqslant \varrho^{-n+\gamma+\alpha}|K_n(C\varrho)| \leqslant B_{n,\gamma,\alpha}$$

for $|x_1| \leqslant B$, $a \leqslant x_2 \leqslant A$, $-\infty < y_1 < +\infty$, where $A_{n,\gamma,\alpha}$ and $B_{n,\gamma,\alpha}$ are positive constants.

It follows from this that

$$\left|\int_{-\infty}^{+\infty} f_i(y_1)\Phi_1 dy_1\right| \leqslant A_{n,\gamma,\alpha}\int_{-\infty}^{+\infty} |f_i(y_1)|\,dy_1, \qquad i = 1, 2$$

and

$$\left| \int_{-\infty}^{+\infty} f_i(y_1) \Phi_2 dy_1 \right| \leqslant B_{n,\gamma,a} \int_{-\infty}^{+\infty} |f_i(y_1)| dy_1, \quad i = 1, 2.$$

Hence and by assumption of Lemma 1 integrals (8) exist and are uniformly convergent for $X \in P(a, A, B)$.

The function $u(X)$ defined by formula (5) or (7) is of class C^4 in the domain $x_2 > 0$ and its derivatives up to the order four may be found by differentiation under the sign of the integral.

We shall prove now that the function defined by formula (5) satisfies equation (1). Taking into consideration the above properties and the fact that the function $G(X, Y)$ as a function of the point of X, $X \neq Y$ satisfies equation (1) we have

$$\Delta^2 u(X) - C^4 u(X) = (8\pi C^2)^{-1} \int_{-\infty}^{+\infty} \left\{ f_1(y_1) \frac{\partial}{\partial y_2} \left[\Delta_Y \left(\Delta_X^2 G(X, Y) - \right. \right. \right.$$

$$\left. \left. \left. - C^4 G(X, Y) \right) \right] + f_2(y_1) \frac{\partial}{\partial y_2} \left[\Delta_X^2 G(X, Y) - C^4 G(X, Y) \right] \right\}_{y_2 = 0} dy_1 = 0.$$

LEMMA 2. *Let $\varphi(y_1)$ be a function defined on the real axis and continuous at the point $\bar{x}_1$. Let the integral $\int_{-\infty}^{+\infty} |\varphi(y_1)| dy_1$ be convergent. Then the function defined by the formula*

$$L(x_1, x_2) = (\pi)^{-1} x_2 \int_{-\infty}^{+\infty} \varphi(y_1) \varrho^{-2} dy_1$$

is defined for the half-plane $x_2 > 0$ and convergent to $\varphi(\bar{x}_1)$ when $(x_1, x_2) \to (\bar{x}_1, 0)$.

The above lemma is a certain modification of the theorem from the monograph [1] (p. 268) and its proof is similar to that of the theorem.

LEMMA 3. *Let $f_i(y_1)$, $i = 1, 2$, be two functions defined on the real axis and continuous at the point $\bar{x}_1$. Let the integrals*

$$\int_{-\infty}^{+\infty} |f_i(y_1)| dy_1, \quad i = 1, 2,$$

be convergent. Then the function defined by (7) satisfies the following boundary conditions:

$$\text{(9)} \qquad \operatorname{Lim} u(x_1, x_2) \Delta = f_1(\bar{x}_1), \quad \text{as } x_1 \to \bar{x}_1, \ x_2 \to 0^+,$$

$$\text{(10)} \qquad \operatorname{Lim} \Delta u(x_1, x_2) = f_1(\bar{x}_1), \quad \text{as } x_1 \to \bar{x}_1, \ x_2 \to 0^+.$$

Proof. We shall now prove that the function $u(X)$ satisfies the condition of (9). For that purpose we present the function $u(X)$ given by formula of (7) in the form:

$$u(X) = (\pi)^{-1}x_2 \int_{-\infty}^{+\infty} f_1(y_1)\varrho^{-2}dy_1 + x_2 \int_{-\infty}^{+\infty} f_1(y_1)W_1(\varrho)dy_1 + \\ + x_2 \int_{-\infty}^{+\infty} f_2(y_1)W_2(\varrho)dy_1,$$

where

$$W_1(\varrho) = (4\pi)^{-1}CN_1(\varrho) - (\pi)^{-1}\varrho^{-2}, \qquad W_2(\varrho) = (4\pi C)^{-1}N_2(\varrho).$$

In view of Lemma 2, we have

$$\operatorname{Lim}\left[(\pi)^{-1}x_2 \int_{-\infty}^{+\infty} f_1(y_1)\varrho^{-2}dy_1\right] = f_1(\bar{x}_1) \qquad \text{as } x_1 \to \bar{x}_1,\ x_2 \to 0^+.$$

It is enough to show that integrals:

$$x_2 \int_{-\infty}^{+\infty} |f_i(y_1)W_i(\varrho)|dy_1, \qquad i = 1, 2, \tag{11}$$

are convergent to zero if $x_1 \to \bar{x}_1$, $x_2 \to 0^+$.

By using the expansion of $Y_1(C\varrho)$ and $K_1(C\varrho)$ into the series ([2], p. 113 and 117), the function $W_1(\varrho)$ may be written in the form

$$W_1(\varrho) = C(2\pi\varrho)^{-1} \sum_{k=0}^{\infty} [(2k+1)!(2k+2)!]^{-1} \left(\frac{C\varrho}{2}\right)^{2(2k+1)+1} \left\{2\ln\frac{C\varrho}{2} - \right. \\ \left. - \psi(2k+2) - \psi(2k+3)\right\} = o(\varrho) \qquad \text{for } \varrho \to 0^+.$$

Hence we find that the function $W_1(\varrho)$ is bounded for $\varrho > 0$. Thus there exists a number $B > 0$ such that $|W_1(\varrho)| \leqslant B$ for $0 < \varrho < +\infty$. So we have

$$x_2 \int_{-\infty}^{+\infty} |f_1(y_1)W_1(\varrho)|dy_1 \leqslant Bx_2 \int_{-\infty}^{+\infty} |f_1(y_1)|dy_1.$$

From this result and by assumption of Lemma 3 it follows that

$$\lim x_2 \int_{-\infty}^{+\infty} |f_1(y_1)W_1(\varrho)|dy_1 = 0 \qquad \text{as } x_1 \to \bar{x}_1,\ x_2 \to 0^+.$$

We shall now prove that integral (11) for $i = 2$ is convergent to

zero when $x_1 \to \bar{x}_1$, $x_2 \to 0^+$. Applying the expansions of $Y_1(C\varrho)$ and $K_1(C\varrho)$ into series we obtain:

$$W_2(\varrho) = (2\pi C\varrho)^{-1} \sum_{k=0}^{\infty} [(2k)!(2k+1)!]^{-1} \left(\frac{C\varrho}{2}\right)^{2(2k)+1} \left\{ 2\ln\frac{C\varrho}{2} - \right.$$

$$\left. - \psi(2k+1) - \psi(2k+2) \right\} = o(\varrho^{-1}) \quad \text{for } \varrho \to 0^+.$$

Thus there exist a number $\varrho_0 > 0$ such that

$$|W_2(\varrho)| \leqslant \varrho^{-1} \quad \text{for } 0 < \varrho < \varrho_0. \tag{12}$$

It follows by the continuity of the function $f_2(x_1)$ at the point $\bar{x}_1$ that there exist numbers $\delta > 0$ and $M_\delta > 0$ such that

$$|f_2(x_1)| \leqslant M_\delta \quad \text{for } |x_1 - \bar{x}_1| < \delta. \tag{13}$$

Let

$$\delta_1 = \min\left(\frac{\varrho_0}{2}, \frac{\delta}{2}\right). \tag{14}$$

The integral given by formula (11) for $i = 2$ may be written in the form

$$x_2 \int_{-\infty}^{+\infty} |f_2(y_1) W_2(\varrho)|\, dy_1 = x_2 \int_{|y_1 - \bar{x}_1| < \delta_1} |f_2(y_1) W_2(\varrho)|\, dy_1 + \tag{15}$$

$$+ x_2 \int_{|y_1 - \bar{x}_1| \geqslant \delta_1} |f_2(y_1) W_2(\varrho)|\, dy_1.$$

If $|x_1 - \bar{x}_1| < \delta_1/2$, then

$$x_2 \int_{|y_1 - \bar{x}_1| \geqslant \delta_1} |f_2(y_1) W_2(\varrho)|\, dy_1 \leqslant x_2 \int_{|y_1 - x_1| \geqslant \frac{\delta_1}{2}} |f_2(y_1) W_2(\varrho)|\, dy_1.$$

If $|y_1 - x_1| \geqslant \delta_1/2$, then $\varrho \geqslant \delta_1/2$. From the estimates concerning functions $Y_n(C\varrho)$ and $K_n(C\varrho)$ for $\varrho \geqslant \delta_1/2$ we have

$$|W_2(\varrho)| \leqslant D_{\delta_1} \quad \text{for } |y_1 - x_1| \geqslant \frac{\delta_1}{2},\ x_2 > 0,$$

where D_{δ_1} is a positive constant. Thus

$$x_2 \int_{|y_1 - x_1| \geqslant \frac{\delta_1}{2}} |f_2(y_1) W_2(\varrho)|\, dy_1 \leqslant x_2 D_{\delta_1} \int_{-\infty}^{+\infty} |f_2(y_1)|\, dy_1.$$

Let $\varepsilon > 0$ be given; then there exists a number $\eta > 0$ such that the condition $0 < x_2 < \eta$ implies

$$x_2 D_{\delta_1} \int_{-\infty}^{+\infty} |f_2(y_1)| \cdot dy_1 < \frac{\varepsilon}{2}.$$

Therefore

$$x_2 \int_{|y_1 - \bar{x}_1| \geqslant \delta_1} |f_2(y_1) W_2(\varrho)| \, dy_1 \leqslant \frac{\varepsilon}{2} \tag{16}$$

for $0 < x_2 < \eta$ and $|x_1 - \bar{x}_1| < \delta_1/2$.

Let us now take into consideration the integral $x_2 \int_{|y_1 - \bar{x}_1| < \delta_1} |f_2(y_1) \times$ $\times W_2(\varrho)| dy_1$. Let $\eta_1 = \min\left(\eta, \frac{\sqrt{7}}{2} \delta_1\right)$. If $|x_1 - \bar{x}_1| < \frac{\delta_1}{2}$ and $0 < x_2 > \eta_1$, then

$$|y_1 - x_1| \leqslant |y_1 - \bar{x}_1| + |x_1 - \bar{x}_1| \leqslant \frac{3\delta_1}{2} \quad \text{and} \quad 0 < \varrho < 2\delta_1.$$

Therefore from formulas (12), (13) and (14) we have

$$x_2 \int_{|y_1 - \bar{x}_1| < \delta_1} |f_2(y_1) W_2(\varrho)| \, dy_1 \leqslant x_2 \int_{|y_1 - x_1| < \frac{3\delta_1}{2}} |f_2(y_1) W_2(\varrho)| \, dy_1$$

$$\leqslant x_2 M_\delta \int_{|y_1 - x_1| < \frac{3\delta_1}{2}} \varrho^{-1} dy_1.$$

Since the integral

$$x_2 \int_{|y_1 - x_1| < \frac{3}{2}\delta_1} \varrho^{-1} dy_1 = 2x_2 \left\{ \ln\left(\frac{3\delta_1}{2} + \sqrt{\frac{9\delta_1^2}{4} + x_2^2} \right) - \ln x_2 \right\}$$

is convergent to zero for $x_2 \to 0^+$, there exists a number $\eta_2 > 0$ such that the conditions: $0 < x_2 < \min(\eta_1, \eta_2)$ and $|x_1 - \bar{x}_1| < \delta_1/2$ imply

$$x_2 \int_{|y_1 - \bar{x}_1| < \delta_1} |f_2(y_1) W_2(\varrho)| \, dy_1 < \varepsilon/2. \tag{17}$$

It follows from (15), (16) and (17) that

$$\int_{-\infty}^{+\infty} |f_2(y_1) W_2(\varrho)| \, dy_1 < \varepsilon$$

for $|x_1 - \bar{x}_1| < \delta_1/2$ and $0 < x_2 < \eta_3$, where $\eta_3 = \min(\eta_1, \eta_2)$. So we have proved that the function defined by (7) satisfies the boundary condition (9).

We shall now prove that the function $u(X)$ satisfies condition (10). In view of formulas (5) and (6) we have:

$$\frac{\partial}{\partial y_2}[\Delta_X G(X, Y)]|_{y_2=0} = \frac{\partial}{\partial y_2}[\Delta_Y G(X, Y)]|_{y_2=0} = 2C^3 x_2 N_1(\varrho),$$

and

$$\frac{\partial}{\partial y_2}\left[\Delta_X\left(\Delta_Y G(X, Y)\right)\right]|_{y_2=0} = \frac{\partial}{\partial y_2}[\Delta_X^2 G(X, Y)]|_{y_2=0}$$

$$= \frac{\partial}{\partial y_2}[C^4 G(X, Y)]|_{y_2=0} = C^4 \frac{\partial}{\partial y_2} G(X, Y)|_{y_2=0} = 2C^5 x_2 N_2(\varrho).$$

We have then

$$\Delta u(X) = (4\pi C)^{-1} x_2 \int_{-\infty}^{+\infty} [f_2(y_1) C^2 N_1(\varrho) + f_1(y_1) C^4 N_2(\varrho)] dy_1.$$

Now the proof that condition (10) is satisfied if similar to that of the proof for condition (9).

From Lemmas 1 and 3 we get

THEOREM 2. *By the assumptions of Lemma 3 the function $u(X)$ defined by formula* (7) *is the solution of equation* (1) *in the half-plane $x_2 > 0$ with the boundary conditions* (9) *and* (10).

Rcferences

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