ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVI (1972)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVI (1972)

Ministerstwo Nauki i Szkolnictwa Wyższego

A. JANKOWSKI and R. RUBINSZTEIN (Warszawa)

Functions with non-degenerate critical points on manifolds with boundary

Let M be a C^∞ manifold with boundary ∂M. Functions $f\colon M \to R$ with the properties

(i) f is constant on each component of ∂M,

(ii) f has no critical points on ∂M,

(iii) f has only non-degenerate critical points were investigated by many authors and were called the *Morse functions on* M.

We are going to consider another class of functions on manifolds with boundary which we will call m-functions.

We will give the definition in Section 1.

The aim of this paper is to prove the Morse inequalities for m-functions. This is done in Section 4. Formula (5) which results from the Morse inequalities was proved by C.T.C. Wall (unpublished) who had used different methods.

We will finish with an example and unsolved problem concerning the existence of m-function which extends given Morse function on ∂M.

1. Definitions and preliminary results. Let M be a compact smooth manifold with boundary ∂M.

DEFINITION 1.1. Let $f\colon M \to R$ be a smooth function on M. We call f to be a *m-function on* M if it has the following properties:

(i) $f|_{\partial M}\colon \partial M \to R$ is a Morse function on ∂M, that means it has only non-degenerate critical points,

(ii) f has no critical points on ∂M,

(iii) f has only non-degenerate critical points.

Let $x \in \partial M$ be a critical point of $f|_{\partial M}$, the index of such a point is an ordered pair (λ, ε), where λ is the index of $f|_{\partial M}$, $\varepsilon = 1$ if $\operatorname{grad} f_x$ points outwards and $\varepsilon = -1$ if $\operatorname{grad} f_x$ points inwards. If $x \in \operatorname{Int} M$ is a critica point of f its index is defined as usual.

With each m-function on M we associate three sequences of integers:

C_λ = number of critical points of index λ,

D_λ^+ = number of critical points of index $(\lambda, +1)$,
D_λ^- = number of critical points of index $(\lambda, -1)$.

We will consider the problem of the existence of the m-functions at first. Let f be the smooth function on ∂M and let $C(M,f)$ be the topological space (in C^2 topology) of all real smooth functions on M, which are equal to f on ∂M.

PROPOSITION 1.2. *If $f \in C^\infty(\partial M)$ is a Morse function, then the set of m-functions on M is open and dense in $C(M,f)$.*

Proof. The first statement is obvious. We are going to prove the second one. Let $g \in C(M,f)$ and let

$$h\colon \partial M \times R^+ \to M$$

be a collar neighborhood of the boundary. Consider the sets

$$A = \left\{ x \in \partial M, \left.\frac{dgh}{dt}\right|_{(x,0)} = 0 \right\},$$

$$P = \{\text{critical points of } f\}.$$

The set of critical points of g on ∂M is the intersection $A \cap P$ and consists of the finite number of points $y_1, \ldots, y_s$ say. For arbitrary real numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $s_1 > s_0 > 0$ let $\mu\colon \partial M \to R$, $\lambda\colon R \to R$ be smooth functions with the properties

(i) $\mu \in V_{\varepsilon_1}(0) \subset C(\partial M)$,
(ii) $\mu(y_i) > 0$, $i = 1, \ldots, s$,
(iii) $\mu^{-1}(0) \subset P \setminus A$,
(iv) $\lambda \in V_{\varepsilon_2}(0)$,
(v) $\lambda(s) = \lambda_0 > 0$ for $s \leqslant s_0$,
(vi) $\lambda(s) = 0$ for $s \geqslant s_1$.

Consider

$$g_1\colon \partial M \times R^+ \to R$$

given by the formula

$$g_1(x,t) = gh(x,t) + \mu(x)\lambda(t)t;$$

we have from (vi) that

$$g_1(x,t) = gh(x,t)$$

for $t \geqslant s_1$ and it results that the function

$$g_1 h^{-1}\colon h(\partial M \times R^+) \to R$$

coincide with g on the set $h(\partial M \times (s_1, \infty))$. Putting

$$g_2(y) = \begin{cases} g(y) & \text{for } y \in M \setminus h(\partial M \times [0, s_1]), \\ g_1 h^{-1}(y) & \text{for } y \in h(\partial M \times R^+), \end{cases}$$

we get a smooth function g_2 such that

(i) for each $\varepsilon > 0$ there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$g_2 \in V_\varepsilon(g),$$

(ii) g_2 has no critical points on ∂M.

The first statement results from Lemma C of [3], p. 12. Now

$$\left.\frac{dg_2 h}{dt}\right|_{(x,0)} = \left.\frac{dgh}{dt}\right|_{(x,0)} + \mu(x)\cdot\lambda_0$$

and thus on $A \cap P$

$$\frac{dg_2 h}{dt} \neq 0.$$

For $x \in \partial M \setminus P$ we have

$$dg_2 h|_{(x,0)}|_{T(\partial M)} = df_x \neq 0$$

and for $x \in P \setminus A$

$$\left.\frac{dg_2 h}{dt}\right|_{(x,0)} = \left.\frac{dgh}{dt}\right|_{(x,0)} \neq 0.$$

It results that g_2 has no critical points on some neighborhood U_0 of the boundary ∂M. To get the rest of the proof we proceed as in [3], p. 16 and 17.

Let $f\colon \partial M \to R$ be a Morse function with the critical points $y_1, \ldots, y_k$ and $\varepsilon_1, \ldots, \varepsilon_k$ a sequence with $\varepsilon_i = \pm 1$, $i = 1, \ldots, k$.

DEFINITION 1.3. We will call a *m-function g on M* to be an *extension of* $(f, \{y_i\}, \{\varepsilon_i\})$ if

(i) $g \in C(M, f)$,

(ii) grad g_{y_i} points inwards if $\varepsilon_i = -1$ and outwards if $\varepsilon_i = +1$.

As a simple consequence of Proposition 1.2 we get the following corollary:

COROLLARY 1.4. *For each* $(f, \{y_i\}, \{\varepsilon_i\})$ *there is an extension on* M.

Proof. Let $\varphi\colon \partial M \to [-1, 1]$ be a C^∞ function such that $\varphi(y_i) = \varepsilon_i$ and let

$$h\colon \partial M \times R^+ \to M$$

be the collar neighborhood. Consider $g_1\colon \partial M \times R^+ \to R$ given by

$$g_1(x, t) = f(x) + \varphi(x)t$$

and let $g\colon M \to R$ be a C^∞ extension of

$$g_2 = g_1 h^{-1}|_{h(\partial M \times [0,1])}$$

on M. It results from Proposition 1.2 that g can be approximated by m-function with desired properties.

2. Homology of morphisms in the category of pairs. For the later use we will briefly recall the definition and some properties of the homology groups of morphisms in the category of pairs.

Let $\mathfrak{C}$ be the category of topological spaces and maps. Let $\mathfrak{C}_p$ be the category of pairs of $\mathfrak{C}$, i.e. objects of $\mathfrak{C}_p$ are morphisms of $\mathfrak{C}$ and a morphism in $\mathfrak{C}_p$ from α to β, where $\alpha\colon A_1 \to A_2$, $\beta\colon B_1 \to B_2$ is a pair (f_1, f_2), $f_1\colon A_1 \to B_1$, $f_2\colon A_2 \to B_2$ such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

is commutative.

As there was shown by Hilton [1] we have an exact homology sequence of a map $\Phi = (f_1, f_2)$:

$$(1) \qquad \ldots \to H_n(\beta) \to H_n(\Phi) \to H_{n-1}(\alpha) \xrightarrow{\Phi_*} H_{n-1}(\beta) \to \ldots$$

which can be constructed as follows:

Let $(C_n(A_1), d'_n)$ be the singular chain complex of A_1, and $(C_n(A_2), d''_n)$ such a complex of A_2. The singular complex of α is $(C_n(\alpha), d^\alpha_n)$, where

$$C_n(\alpha) = C_{n-1}(A_1) \oplus C_n(A_2),$$
$$d^\alpha_n(x, y) = (-d_{n-1}(x),\ d_n(y) + \alpha x)$$

(here and in the sequel we use the same symbol for a map an a chain map that it induces).

A short exact sequence of chain complexes

$$0 \to C_n(A_2) \to C_n(\alpha) \to C_{n-1}(A_1) \to 0$$

give us the exact homology sequence of α:

$$\ldots \to H_n(A_2) \to H_n(\alpha) \to H_{n-1}(A_1) \xrightarrow{\alpha_*} H_{n-1}(A_2) \to \ldots$$

If α is an inclusion, then $H_n(\alpha) = H_n(A_2, A_1)$. To obtain (1) let $(C_n(\alpha), d^\alpha_n)$ (resp. $(C_n(\beta), d^\beta_n)$) be the complex of α (resp. β). The complex of $\Phi = (f_1, f_2)$ is $(C_n(\Phi), d_n)$, where

$$C_n(\Phi) = C_{n-1}(\alpha) \oplus C_n(\beta),$$
$$d_n(x, y) = (-d_{n-1}x,\ d_n y + \Phi x).$$

(1) is the exact homology sequence of a short exact sequence of chain complexes

$$0 \to C_n(\beta) \to C_n(\Phi) \to C_{n-1}(\alpha) \to 0.$$

Let Φ^T be the transposition of Φ, i.e. $\Phi^T = (\alpha, \beta)$. As there was shown in [1] there is a chain isomorphism

$$C(\Phi) \cong C(\Phi^T)$$

which implies that if $\lambda\colon A \to B$, $\mu\colon B \to C$ are maps and $\nu = \mu\lambda\colon A \to C$, then there is the exact sequence

$$\ldots \to H_n(\lambda) \xrightarrow{(1,\mu)_*} H_n(\nu) \xrightarrow{(\lambda,1)_*} H_n(\mu) \to H_{n-1}(\lambda) \to \ldots$$

All this construction can be redone in the category $(\mathfrak{C}_p)_p$. Let Φ_1, $\Phi_2 \in \mathrm{Obj}(\mathfrak{C}_p)_p$, and $\mathfrak{N}\colon \Phi_1 \to \Phi_2$. We have

$$\Phi_1\colon \alpha_1 \to \beta_1, \qquad \Phi_2\colon \alpha_2 \to \beta_2$$

and

$$\mathfrak{N} = \big((g_1, g_2), (g_3, g_4)\big),$$

where $(g_1, g_2)\colon \alpha_1 \to \alpha_2$, $(g_3, g_4)\colon \beta_1 \to \beta_2$. We put

$$C_n(\mathfrak{N}) = C_{n-1}(\Phi_1) \oplus C_n(\Phi_2),$$
$$d_n(u, v) = (-d_{n-1}u,\ d_n v + \mathfrak{N}u)$$

and hence we get an exact homology sequence

$$\ldots \to H_n(\Phi_2) \to H_n(\mathfrak{N}) \to H_{n-1}(\Phi_1) \to H_{n-1}(\Phi_2) \to \ldots$$

Let $\mathfrak{N}^T$ be the transposition of $\mathfrak{N}$, i.e. $\mathfrak{N}^T = (\Phi_1, \Phi_2)$. Just by the same method as for Φ^T it can be shown that there is a natural chain isomorphism

$$C(\mathfrak{N}) \cong C(\mathfrak{N}^T)$$

and hence if $\Phi_3 = \Phi_2 \circ \Phi_1$ there is the exact sequence

$$(2) \qquad \ldots \to H_n(\Phi_1) \xrightarrow{(1,\Phi_2)_*} H_n(\Phi_3) \xrightarrow{(\Phi_1,1)_*} H_n(\Phi_2) \to H_{n-1}(\Phi_1) \to \ldots$$

Remark. Following [1] one can easily show that if

$$\Phi = (\alpha, \beta)\colon f \to g$$

and α, β are inclusions, then

$$H_n(\Phi) = H_n(C_g, C_f),$$

where C_g (resp. C_f) is the mapping cone of g (resp. f).

3. Some lemmas. Let $f\colon M \to R$ be a m-function. We will denote by M_a the manifold with corners

$$M_a = f^{-1}\big((-\infty, a]\big)$$

and by $(\partial M)_a$ the manifold with boundary

$$(\partial M)_a = M_a \cap \partial M.$$

For $b > a$ we have the diagram of inclusions

$$\begin{array}{ccc} (\partial M)_a & \xrightarrow{i_1} & (\partial M)_b \\ \downarrow{\scriptstyle \alpha_a} & & \downarrow{\scriptstyle \alpha_b} \\ M_a & \xrightarrow{i_2} & M_b \end{array}$$

and let $\Phi_b^a = (i_1, i_2)$: $\alpha_a \to \alpha_b$.

We are going to prove some lemmas which we will use later in order to compute the homology groups of the maps Φ_b^a in some cases.

LEMMA 3.1. *Let f: $M \to R$ be a m-function, $a < b$ and $f^{-1}([a, b])$ contain no critical point of f. Then $f^{-1}([a, b])$ and $f^{-1}(a) \times I$ are diffeomorphic as the manifolds with corners.*

Proof. We can assume $a = 0$, $b = 1$. Let

$$\tilde{M} = f^{-1}([0, 1]), \quad M_0 = f^{-1}(0), \quad M_1 = f^{-1}(1),$$
$$M_2 = \partial M \cap \tilde{M}, \quad V_0 = \partial M_0, \quad V_1 = \partial M_1.$$

Let W be an open neighborhood of M_2 in $\tilde{M}$ such that there is a diffeomorphism

$$h\colon\ M_2 \times [0, 1) \to W$$

such that

$$h(x, 0) = x,$$
$$h\big(V_i \times [0, 1)\big) = W \cap M_i \quad \text{for } i = 0, 1.$$

Following the proof of a collar neighborhood theorem one can show that such W and h do exist.

Let $\mathfrak{z}$ be a gradient-like vector field on M_2 for $f|_{M_2}$ and let η be a vector field on W induced from $\mathfrak{z}$ by projection on M_2. $\eta(f|_W)$ is a smooth function on W and $\eta(f|_W)|_{M_2} > 0$. Let U be the open neighborhood of M_2 in W such that $\eta(f|_W)|_U > 0$. Since M_2 is compact there is $t_0 \in (0, \frac{1}{2})$ such that $M_2 \subset h\big(M_2 \times [0, t_0)\big) \subset U$. Let μ: $[0, 1) \to [0, 1)$ be a smooth function such that

$$\mu(t) = \begin{cases} t & \text{for } \frac{2}{3}t_0 \leqslant t < 1, \\ 0 & \text{for } 0 \leqslant t \leqslant \frac{1}{3}t_0. \end{cases}$$

Consider a function $\tilde{f}$: $\tilde{M} \to [0, 1]$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in M \setminus h\big(M_2 \times [0, \frac{2}{3}t_0)\big), \\ f\big(h(p_1(x), \mu(p_2(x)))\big) & \text{for } x \in W, \end{cases}$$

where

$$p_1\colon\ W \to M_2, \quad p_2\colon\ W \to [0, 1)$$

are projections. $\tilde{f}$ has no critical points on $\tilde{M}$, moreover,

$$\tilde{f}\,|_{M_2} = f|_{M_2}, \quad \tilde{f}^{-1}(0) = M_0, \quad \tilde{f}^{-1}(1) = M_1.$$

For $x \in h\big(M_2 \times [0, \frac{1}{4}t_0)\big)$ we have

$$\eta(\tilde{f}\,|_U)_x = \mathfrak{z}(f|_{M_2})_{p_1(x)} > 0.$$

Let ω be a gradient-like vector field for $\tilde{f}\,|_{\tilde{M}\setminus M_2}$ on the paracompact manifold $\tilde{M}\setminus M_2$. Consider

$$U_1 = \tilde{M}\setminus h(M_2 \times [0, \tfrac{1}{5}t_0]), \qquad U_2 = h\big(M_2 \times [0, \tfrac{1}{4}t_0)\big).$$

$\{U_1, U_2\}$ is an open covering of M and let $\{\lambda_1, \lambda_2\}$ be the associated partition of unity.

Let γ be the vector field on $\tilde{M}$ given by

$$\gamma = \lambda_1\omega + \lambda_2\eta.$$

It is a gradient-like vector field for $\tilde{f}$ on $\tilde{M}$, moreover, the integral curves of γ which have a point in common with M_2 are contained in M_2. The rest of the proof follows Milnor [3], p. 21, 22.

LEMMA 3.2. *Let f be a m-function on M and let $p_0 \in \partial M$ be a non-degenerate critical point of $f|_{\partial M}$ of index λ and $f(p_0) = 0$. Then there is a neighborhood $\tilde{U}$ of p_0 in M and some coordinate system*

$$\tilde{\varphi}\colon\ \tilde{U} \to R^n$$

such that:

(i) $f(p) = \tilde{\varphi}_n(p)$ *for* $p \in \tilde{U}$,

(ii) *the manifold* $\tilde{\varphi}(\tilde{U} \cap \partial M)$ *is given by the equation*

$$y_n = -y_1{}^2 - \ldots - y_\lambda{}^2 + y_{\lambda+1}{}^2 + \ldots + y_{n-1}{}^2.$$

Proof. There is a neighborhood U of p_0 in M and a map

$$\varphi\colon\ U \to R^n_-$$

such that $\varphi(p_0) = 0$ and let $V = \varphi(U) \subset R^n_- \subset R^n$. The map $f' = f\varphi^{-1}$: $V \to R$ can be extended over an open subset $\tilde{V}$ of R^n in such a way that $V = \tilde{V} \cap R^n$, f' has no critical points in $\tilde{V}$ and $f'|_{\tilde{V}\cap R^{n-1}}$ has just one non-degenerate critical point $0 \in R^{n-1}$.

This implies that there is a change of the coordinate system

$$\psi'\colon\ V' \to R^n,$$

V' an open subset of $\tilde{V}$ containing 0 and $\psi'(0) = 0$, such that on the set $\hat{V} = \psi'(V')$ we have

$$f'\psi'^{-1}(x_1, \ldots, x_n) = x_n$$

and

$$\frac{\partial}{\partial x_n}(\psi'^{-1}_n)_x \neq 0 \quad \text{on } \hat{V}.$$

The submanifold $\psi'(R^{n-1} \cap V')$ has in $\hat{V}$ the equation

$$(\psi'^{-1})_n(x_1, \ldots, x_n) = 0.$$

From the inverse function theorem we get a function θ defined in some neighborhood of 0 in R^{n-1} such that

$$x_n = \theta(x_1, \ldots, x_{n-1})$$

is the equation of $\psi'(R^{n-1} \cap V')$ in some neighborhood W of 0 in $\hat{V}$. Let $\mathrm{pr}: R^n \to R^{n-1}$ be the standard projection on the first $n-1$ coordinates. One can easly check that

$$\theta \mathrm{pr}|_{W \cap \psi'(R^{n-1} \cap V')} = x_n|_{W \cap \psi'(R^{n-1} \cap V')}$$

and since $\mathrm{pr}|_{W \cap \psi'(R^{n-1} \cap V')}$ is a diffeomorphism on some open neighborhood of 0, 0 is a non-degenerate critical point of θ with the index λ. Let W' be an open neighborhood of 0 in R^{n-1} and $\psi'': W' \to R^{n-1}$ a change of coordinate system such that

$$\theta\psi''(y_1, \ldots, y_{n-1}) = -y_1{}^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 \ldots + y_{n-1}^2.$$

Let ψ be the change of coordinate system given by

$$\overline{\psi} = (\psi'' \mathrm{pr}, id): \mathrm{pr}^{-1}(W') \to R^n.$$

Putting $\tilde{U} = \varphi^{-1}\left(V \cap \psi'^{-1}(\mathrm{pr}^{-1}(W'))\right)$ and $\tilde{\varphi} = \overline{\psi} \circ \psi' \circ \varphi$ we get the chart that we need.

LEMMA 3.3 *If $f: M \to R$ is a m-function and $p_0 \in \partial M$ is a critical point of index (λ, ε) and there is no more critical points with the same critical value $f(p_0) = c_0$, then there exists $\eta > 0$ such that*

(i) *if $\varepsilon = 1$, then $M_{c_0-\eta}$ is a strong deformation retract of $M_{c_0+\eta}$,*

(ii) *if $\varepsilon = -1$, then there is λ-dimensional disc $D^\lambda \subset M_{c_0+\eta}$ such that $D^\lambda \cap M_{c_0-\eta} = \partial D^\lambda$, and $M_{c_0-\eta} \cup D^\lambda$ is a strong deformation retract of $M_{c_0+\eta}$.*

Proof. We can assume that $f(p_0) = 0$. Let $\tilde{U}$ be a neighborhood of p_0 in M and $\tilde{\varphi}: \tilde{U} \to R^n$ the coordinate system as Lemma 3.2 says. Let V be an open subset of R^n containing $\tilde{V}$, $B_a \subset R^{n-1}$ a ball with a center 0 and radius a, $I_b \subset R^1$ a ball with a center 0 and radius b, such that $B_a \times I_b \subset V$ and $\overline{B_a \times I_b \cap \tilde{V}} \subset \tilde{V}$. Let $\varrho: R \to R$ be a smooth function such that

(i) there is $\delta \in (0, a^2)$ such that $\varrho(t) = 1$ for $t \in (-\infty, \delta)$,

(ii) $\varrho(t) = 0$ for $t \geqslant a^2$,

(iii) $B < \dfrac{d\varrho}{dt} \leqslant 0$, $B < 0$,

(iv) $0 \leqslant \varrho(t) \leqslant 1$, all t.

Let $\varepsilon > 0$ be such that $0 < 2\varepsilon < \min(\frac{1}{2}b, -\frac{1}{2}B, \delta)$ and let $\eta\colon R \to R$ be a smooth function such that

(v) $\eta(s) = 0$ for $s \in (-\infty, b] \cup [\varepsilon, +\infty)$

$$0 \leqslant \eta(s) < 2\varepsilon \quad \text{for all } s \in R,$$

(vi) there is $\gamma \in (0, \varepsilon)$ such that $\eta(s) = C$ for $s \in [-\varepsilon, \gamma]$, and $\varepsilon < C < 2\varepsilon$,

(vii) $\dfrac{d\eta}{ds} < \frac{1}{2}$ for all s,

(iix) $\dfrac{d\eta}{ds} \leqslant 0$ for $s \in [0, +\infty)$.

Let $F'\colon V \to R$ be defined by

$$F'(y_1, \ldots, y_n) = y_n - \varrho\big(\|\mathrm{pr}(y)\|^2\big)\eta(y_n),$$

and let $F\colon M \to R$ be defined by

$$F(p) = \begin{cases} f(p) & \text{for } p \in M \setminus \tilde{U}, \\ F'\big(\tilde{\varphi}(p)\big) & \text{for } p \in \tilde{U}. \end{cases}$$

We have

(I) $$F(p) \leqslant f(p)$$

and (v) implies

(II) $$F^{-1}\big((-\infty, \varepsilon]\big) = f^{-1}\big((-\infty, \varepsilon]\big).$$

As $F|_{M \setminus \tilde{\varphi}^{-1}(B_a \times I_b)} = f|_{M \setminus \tilde{\varphi}^{-1}(B_a \times I_b)}$, F (resp. $F|_{\partial M}$) has on $M \setminus \tilde{\varphi}^{-1}(B_a \times I_b)$ just the same critical points as f (resp. $f|_{\partial M}$). Let us consider critical points of F and $F|_{\partial M}$ on $\tilde{\varphi}^{-1}(B_a \times I_b)$. We have on V

$$\frac{\partial F'}{\partial y_i} = \begin{cases} -\dfrac{d\varrho}{dt} \cdot 2y_i \eta(y_n), & i = 1, \ldots, n-1, \\ 1 - \varrho\big(\|\mathrm{pr}(y)\|^2\big)\dfrac{d\eta}{ds}(y_n), & i = n. \end{cases}$$

Because of (iv) and (vii) $\dfrac{\partial F'}{\partial y_n} > \frac{1}{2}$ on V and F has no critical points in $\tilde{\varphi}^{-1}(B_a \times I_b)$. The normal vector to the surface $\tilde{\varphi}(\tilde{U} \cap \partial M)$ in a point $y = (y_1, \ldots, y_n)$ is

$$v_y = (-2y_1, \ldots, -2y_\lambda, 2y_{\lambda+1}, \ldots, 2y_{n-1}, -1)$$

and y is a critical point of $F'|_{\tilde{\varphi}(\tilde{U} \cap \partial M)}$ if and only if

$$(\partial F'_y) \| v_y.$$

If for some i, $1 \leqslant i \leqslant \lambda$, $y_i \neq 0$, then $v_y \| \partial F_y$ implies

$$-\frac{d\varrho}{dt}\big(\|\mathrm{pr}(y)\|^2\big)\eta(y_n) = 1 - \varrho\big(\|\mathrm{pr}(y)\|^2\big)\frac{d\eta}{ds}(y_n).$$

If for some i, $\lambda+1 \leqslant i \leqslant n-1$, $y_i \neq 0$, then $v_y \| \partial F_y$ implies

$$\frac{d\varrho}{dt}(\|\mathrm{pr}(y)\|^2)\eta(y_n) = 1-\varrho(\|\mathrm{pr}(y)\|^2)\frac{d\eta}{ds}(y_n).$$

But $1-\varrho(\|\mathrm{pr}(y)\|^2)\frac{d\eta}{ds}(y_n) > \frac{1}{2}$, while $\frac{d\varrho}{dt}(\|\mathrm{pr}(y)\|^2)\eta(y_n) \leqslant 0$ and $-\frac{d\varrho}{dt} < -B$ together with $\eta < 2\varepsilon < -\frac{1}{2}B$ implies $-\frac{d\varrho}{dt}(\|\mathrm{pr}(y)\|^2)\eta(y_n) < \frac{1}{2}$. Then 0 is the only critical point of $F'|_{\tilde{\varphi}(\tilde{U}\cap\partial M)}$.

Therefore p_0 is the only critical point of $F|_{\partial M\cap\tilde{U}}$. Now, (i) and (vi) imply that in some neighborhood of p_0 in M, $F = f-\mathrm{const}$. So p_0 is non-degenerate critical point of $F|_{\partial M}$ with the same index as f has.

We proved that $F\colon M \to R$ is a m-function on M, F has on M and on ∂M the same critical points as f each with the same index. Moreover, $M_\varepsilon^F = M_\varepsilon^f$ and $F(p_0) < -\varepsilon$. So $F^{-1}([-\varepsilon, \varepsilon])$ contains no critical point of F and by Lemma 3.1, $M_{-\varepsilon}^F$ is a strong deformation retract of M_ε^f.

Conditions (vi) and (iix) imply that if $y = (y_1, \ldots, y_n) \in B_a \times I_b$, $y_n \geqslant -\varepsilon$, $F'(y) \leqslant -\varepsilon$, then for each $y' \in B_a \times I_b$, $y' = (y_1, \ldots, y_{n-1}, y_n')$, $-\varepsilon \leqslant y_n' \leqslant y_n$, we have $F'(y') \leqslant -\varepsilon$.

If df_{p_0} is coming outside of M, then

$$\tilde{V} \subset \{(y_1, \ldots, y_n) \in R^n;\ y_n \leqslant -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2\}.$$

One can check that

$$\{y \in \tilde{V};\ F'(y) \leqslant -\varepsilon\} \subset \{y \in \tilde{V};\ y_n \leqslant -\varepsilon\} \cup (B_a \times I_b \cap \tilde{V})$$

hence

$$h_t(y_1, \ldots, y_n) = \begin{cases} (y_1, \ldots, y_n) & \text{for } y_n \leqslant -\varepsilon, \\ (y_1, \ldots, y_{n-1}, -\varepsilon t + (1-t)y_n) & \text{for } y_n \geqslant -\varepsilon \end{cases}$$

is a deformation of $\{y \in \tilde{V};\ F'(y) \leqslant -\varepsilon\}$ on $\{y \in \tilde{V};\ y_n \leqslant -\varepsilon\}$ rel. $\{y \in \tilde{V};\ y_n \leqslant -\varepsilon\}$. We showed that in this case $M_{-\varepsilon}^f$ is a strong deformation retract of $M_{-\varepsilon}^F$.

If df_{p_0} is coming inside of M, then

$$\tilde{V} \subset \{(y_1, \ldots, y_n) \in R^n;\ y_n \geqslant -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2\}.$$

Let $D^\lambda = \{(y_1, \ldots, y_n) \in R^n;\ y_{\lambda+1} = \ldots = y_{n-1} = 0$ and $-\varepsilon \leqslant y_n = -y_1^2 - \ldots - y_\lambda^2 \leqslant 0\}$. D^λ is diffeomorphic with a λ-dimensional disc and (i) and (vi) imply that $F(y) < -\varepsilon$ for all $y \in D^\lambda$. We have $D^\lambda \cap \{y \in \tilde{V};\ y_n \leqslant -\varepsilon\} = \partial D^\lambda$ as well.

Let $A = \{y \in R^n;\ y_n = -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2\}$ and consider the deformation

$$h_t(y_1, \ldots, y_n)$$

$$= \begin{cases} (y_1, \ldots, y_n) & \text{for} \quad -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2 \leqslant y_n \leqslant -\varepsilon, \\ (y_1, \ldots, y_{n-1}, & t(-y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2) + (1-t)y_n) \\ & \text{for} \quad -\varepsilon \leqslant -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2 \leqslant y_n, \\ (y_1, \ldots, y_{n-1}, & -t\varepsilon + (1-t)y_n) \\ & \text{for} \quad -y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_{n-1}^2 \leqslant -\varepsilon \leqslant y_n \end{cases}$$

of $\{y \in \tilde{V};\ F'(y) \leqslant -\varepsilon\}$ on $\{y \in \tilde{V};\ y_n \leqslant -\varepsilon\} \cup \big(A \cap \{y \in B_a \times I_b;\ F'(y) \leqslant -\varepsilon\}\big)$ keeping this last set fixed. With our assumption on ϱ and η it can be easily shown that there is a deformation of

$$\{y \in \tilde{V};\ y_n \leqslant -\varepsilon\} \cup \big(A \cap \{y \in B_a \times I_b;\ F'(y) \leqslant -\varepsilon\}\big)$$

on $\{y \in \tilde{V};\ y_n \leqslant -\varepsilon\} \cup D^\lambda$ rel. this last set. The composition of those two deformations gives the desired deformation of $M_{-\varepsilon}^F$ onto $M_{-\varepsilon}^f \cup D^\lambda$.

LEMMA 3.4 *If $p_0 \in \operatorname{Int} M$ is a critical point of f of index λ, $f(p_0) = c_0$ and $f^{-1}([c_0 - \varepsilon, c_0 + \varepsilon])$ contains no more critical points of f, then there is a λ-dimensional disc $D^\lambda \subset M_{c_0+\varepsilon}$, $D^\lambda \cap M_{c_0-\varepsilon} = \partial D^\lambda$ such that $M_{c_0-\varepsilon} \cup D^\lambda$ is a strong deformation retract of $M_{c_0+\varepsilon}$.*

Proof. As in the case of manifold without boundary. See [3].

4. The Morse inequalities. Let, as above, M be a manifold with boundary and $f\colon M \to R$ an m-function on M. Recall that Φ_b^a is the morphism in the category of pairs

$$\Phi_b^a\colon \quad \begin{array}{ccc} (\partial M)_a & \xrightarrow{i_1} & (\partial M)_b \\ \downarrow{\scriptstyle a_a} & \Rightarrow & \downarrow{\scriptstyle a_b} \\ M_a & \xrightarrow{i_2} & M_b \end{array}$$

We will compute the groups $H_n(\Phi_b^a)$ in the following special cases:

(A) $f^{-1}([a, b])$ contains just one critical point p_0 of f, $f(p_0) = c$, $a < c < b$, index $p_0 = (\lambda, +1)$,

(B) as in (A) with index $(\lambda, -1)$,

(C) as in (A) with index λ.

We have $H_n(\Phi_b^a) = H_n\big((\Phi_b^a)^T\big)$, $(\Phi_b^a)^T = (a_a, a_b)\colon i_1 \to i_2$ and there is an exact sequence

$$\ldots \to H_n(i_1) \xrightarrow{(a_a, a_b)_*} H_n(i_2) \to H_n\big((\Phi_b^a)^T\big) \to H_{n-1}(i_1) \to H_{n-1}(i_2) \to \ldots$$

But $H_n(i_1) = H_n\big((\partial M)_b, (\partial M)_a\big)$, $H_n(i_2) = H_n(M_b, M_a)$ and from Lemmas 3.1, 3.3, 3.4 and Theorem 3.14 of [2] we get:

Case (A)

$$H_n(M_b, M_a) = 0 \quad \text{for all } n,$$

$$H_n\big((\partial M)_b, (\partial M)_a\big) = \begin{cases} 0 & \text{for } n \neq \lambda, \\ Z & \text{for } n = \lambda, \end{cases}$$

thus

$$H_n(\Phi_b^a) = \begin{cases} 0 & \text{for } n \neq \lambda+1, \\ Z & \text{for } n = \lambda+1. \end{cases}$$

Case (B)

$$H_n(M_b, M_a) = \begin{cases} 0 & \text{for } n \neq \lambda, \\ Z & \text{for } n = \lambda, \end{cases}$$

$$H_n\big((\partial M)_b, (\partial M)_a\big) = \begin{cases} 0 & \text{for } n \neq \lambda, \\ Z & \text{for } n = \lambda \end{cases}$$

and $(\alpha_a, \alpha_b)_*$ is an isomorphism as the generator of $H_\lambda(\partial M_b, \partial M_a)$ and the generator of $H_\lambda(M_b, M_a)$ are given by the same disc D^λ. Then

$$H_n(\Phi_b^a) = 0 \quad \text{for all } n.$$

Case (C)

$$H_n(M_b, M_a) = \begin{cases} 0 & \text{for } n \neq \lambda, \\ Z & \text{for } n = \lambda, \end{cases}$$

$$H_n\big((\partial M)_b, (\partial M)_a\big) = 0 \quad \text{for all } n,$$

thus

$$H_n(\Phi_b^a) = \begin{cases} 0 & \text{for } n \neq \lambda, \\ Z & \text{for } n = \lambda. \end{cases}$$

Let $R_\lambda(\Phi_b^a) =$ the rank of $H_\lambda(\Phi_b^a)$ and define, following Milnor [2]

$$S_\lambda(\Phi_b^a) = R_\lambda(\Phi_b^a) - R_{\lambda-1}(\Phi_b^a) + R_{\lambda-2}(\Phi_b^a) - \ldots \pm R_0(\Phi_b^a).$$

Let $a < b < c$, we have $\Phi_c^a = \Phi_c^b \circ \Phi_b^a$ and from the exact sequence of composition (2) we get

$$S_\lambda(\Phi_c^a) \leqslant S_\lambda(\Phi_b^a) + S_\lambda(\Phi_c^b).$$

THEOREM 4.1. *We have two sequences of Morse inequalities*

$$(3_\lambda) \qquad S_\lambda(M, \partial M) \leqslant C_\lambda - C_{\lambda-1} + \ldots \pm C_0 + D_{\lambda-1}^+ - D_{\lambda-2}^+ + \ldots \pm D_0^+,$$

$$(4_\lambda) \qquad S_\lambda(M) \leqslant C_\lambda - C_{\lambda-1} + \ldots \pm C_0 + D_\lambda^- - D_{\lambda-1}^- + \ldots \pm D_0^-.$$

Proof. We can assume each critical value of f is reached in just one critical point. If f fails this assumption f can be changed without any change of critical points and their indexes to satisfy it. So there is a sequence $a_0 < a_1 < \ldots < a_k$ of regular values such that, for all i, $f^{-1}([a_{i-1}, a_i])$

contains just one critical point p_i and $M_{a_0} = \emptyset$, $M_{a_k} = M$. Then $H_*(\Phi^{a_0}_{a_k}) = H_*(M, \partial M)$ and inequalities (3_λ) follow from subadditivity of S_λ. For the proof of (4_λ) we use the filtration

$$\emptyset = M_{a_0} \subset M_{a_1} \subset \ldots \subset M_{a_k} = M.$$

COROLLARY 4.2.

$$\chi(M, \partial M) = \sum_{i=0}^{\dim M} (-1)^i C_i + \sum_{i=0}^{\dim M - 1} (-1)^{i-1} D_i^+. \tag{5}$$

5. An example and a problem. Let $f\colon S^n \to R$ be a Morse function with just two critical points p_0, p_1. We are going to show that $(f, \{p_i\}, \{\varepsilon_i\})$ has an extension on D^{n+1} with $C_\lambda = 0$ for all λ if ε_i are properely choosen.

With no loss of generality we can assume that

$$f\colon S^n \to [-1, 1]$$

and thus $p_0 = f^{-1}(-1)$, $p_1 = f^{-1}(1)$. Let $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. Consider D^{n+1} as the subset of R^{n+1} cosisting of points $x \in R^{n+1}$ with $\|x\| \leqslant 1$. We may assume that $p_0 = (0, \ldots, 0, -1)$ and $p_1 = (0, \ldots, 0, 1)$. Let $g\colon D^{n+1} \to [-1, 1]$ be an m-function which extends f. There is $\varepsilon > 0$ such that $\dfrac{dg}{dx_{n+1}} > 0$ in the set

$$g^{-1}([-1, -1+2\varepsilon] \cup [1-2\varepsilon, 1]).$$

Let $A = g^{-1}([-1, -1+\varepsilon] \cup [1-\varepsilon, 1])$, there exist a diffeomorphism $h\colon D^{n+1} \setminus A \to D^n \times [-1+\varepsilon, 1-\varepsilon]$ such that $g(h^{-1}(x, t)) = t$ for $x \in \partial D^n$ and a neighborhood U of $D^n \times \{-1+\varepsilon\} \cup D^n \times \{1-\varepsilon\}$ such that $\left.\dfrac{dgh^{-1}}{dt}\right|_{x,t} > 0$ for $(x, t) \in U$. It follows that there is a diffeomorphism

$$k\colon D^n \times [-1+\varepsilon, 1-\varepsilon] \to D^n \times [-1+\varepsilon, 1-\varepsilon]$$

which is identity on the boundary, is isotopic to the identity and

$$gh^{-1}k^{-1}(x, t) = t$$

in some smaller neighborhood V of $D^n \times \{-1+\varepsilon\} \cup D^n \times \{1-\varepsilon\}$. Let $g_1\colon D^{n+1} \setminus A \to R$ be defined by

$$g_1(y) = t \quad \text{for } y = h^{-1}k^{-1}(x, t).$$

The functions g and g_1 coincide on some neighborhood of $g^{-1}(\{-1+\varepsilon\} \cup \{1-\varepsilon\})$ in $D^{n+1} \setminus A$ and we can define a new C^∞ function with no critical points

$$g_2\colon D^{n+1} \to R$$

by

$$g_2(y) = \begin{cases} g(y) & \text{for } y \in A, \\ g_1(y) & \text{for } y \in D^{n+1} \setminus A. \end{cases}$$

It is clear that g_2 has desired properties.

In general, the following question remains open:

Question. What is the minimal number of critical points of an extension of $(f, \{y_i\}, \{\varepsilon_i\})$?

For the closed, compact, simply connected manifolds similar question was considered by Smale and the complete answer is given in [4], Theorem 6.1.

References

[1] P. J. Hilton, *Homotopy theory and duality*, Gordon and Beach, 1965.
[2] J. Milnor, *Lectures on Morse theory*, Princ. Univ. Press, 1963.
[3] — *Lectures on h-cobordism theorem*, Princ. Univ. Press, 1965.
[4] S. Smale, *On the structure of manifolds*, Amer. J. of Math. 84 (1962), p. 387-399.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA