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# Approximation of real functions by the double trigonometric polynomials

**1. Preliminaries.** Let $L_{2\pi}$ [resp. $C_{2\pi}$] be the class of all $2\pi$-periodic real-valued functions $f$ of two variables, Lebesgue-integrable over the square $Q = [-\pi, \pi;\ -\pi, \pi]$ [continuous everywhere]. Denote by $S_{k,k}(x, y; f)$ the square partial sums of the double Fourier series of $f \in L_{2\pi}$ ([6], p. 436), and write

$$U_n(x, y; f) = \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k}(x, y; f) \qquad (n = 1, 2, \ldots).$$

These means were first investigated by J. Marcinkiewicz in [1], p. 527-538.

A simple calculation leads to

$$U_n(x, y; f) = \frac{1}{n\pi^2} \iint_Q f(x+u, y+v)\, \Phi_n(u, v)\, du\, dv,$$

where

$$\Phi_n(u, v) = \sum_{\nu=0}^{n-1} D_\nu(u)\, D_\nu(v), \qquad D_\nu(u) = \tfrac{1}{2} + \sum_{l=1}^{\nu} \cos lu.$$

The inequalities $|D_\nu(u)| \leqslant \nu + \frac{1}{2}$ $(\nu = 0, 1, \ldots)$ imply

$$|\Phi_n(u, v)| < n^3 \quad \text{for } (u, v) \in Q. \tag{1}$$

By identity (3.2) of [1], p. 529,

$$|\Phi_n(u, v)| < \frac{3n\pi^2}{2\,|u-v|\,(u+v)} \quad \text{if} \quad \begin{cases} 0 \leqslant u \leqslant \pi,\ 0 \leqslant v \leqslant \pi, \\ 0 < u+v \leqslant 3\pi/2,\ u \neq v. \end{cases} \tag{2}$$

Applying (3.3) of [1], we obtain

$$|\Phi_n(u, v)| \leqslant \frac{n\pi^2}{4uv} \quad \text{if } 0 < u \leqslant \pi,\ 0 < v \leqslant \pi, \tag{3}$$

and

$$|\Phi_n(u, v)| \leqslant \frac{\pi^3}{8uv\,|u-v|} \quad \text{if } 0 < u \leqslant \pi,\ 0 < v \leqslant \pi,\ u \neq v. \tag{4}$$

It is known ([1], p. 528-532) that for any $f \in C_{2\pi}$,

$$\lim_{n\to\infty} U_n(x, y; f) = f(x, y)$$

uniformly in $(x, y) \in Q$. Basing on inequalities (1)-(4), we shall deduce some estimates concerning the speed of this convergence. Moreover, two related operators will be examined.

Consider an arbitrary $f \in L_{2\pi}$, defined in $Q$, and set

$$\Delta^*_{u,v} f(x, y) = f(x-u, y-v)+f(x+u, y-v)+$$
$$+f(x-u, y+v)+f(x+u, y+v)-4f(x, y),$$

$$\Delta^{(k,k)}_{u,v} f(x, y) = \sum_{i=0}^{k}\sum_{j=0}^{k}(-1)^{i+j}\binom{k}{i}\binom{k}{j} f(x+iu, y+jv),$$

$$\Delta^{(k)}_{u,0} f(x, y) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+iu, y),$$

$$\Delta^{(k)}_{0,v} f(x, y) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x, y+jv).$$

The following moduli of smoothness will be needed below:

$$\omega_*(s, t; f) = \sup_{|u|\leqslant s,\,|v|\leqslant t}\{\sup_{(x,y)\in Q}|\Delta^*_{u,v} f(x, y)|\},$$

$$\omega_{k,k}(s, t; f) = \sup_{|u|\leqslant s,\,|v|\leqslant t}\{\sup_{(x,y)\in Q}|\Delta^{(k,k)}_{u,v} f(x, y)|\},$$

$$\omega_k(s, 0; f) = \sup_{|u|\leqslant s}|\{\sup_{(x,y)\in Q}|\Delta^{(k)}_{u,0} f(x, y)|\},$$

$$\omega_k(0, t; f) = \sup_{|v|\leqslant t}\{\sup_{(x,y)\in Q}|\Delta^{(k)}_{0,v} f(x, y)|\}.$$

As well known ([5], p. 116, 126), for any bounded $f$ and for each non-negative $a, b, s, t$,

$$\omega_k(as, 0; f) \leqslant (1+a)^k \omega_k(s, 0; f), \tag{5}$$

$$\omega_k(0, bt; f) \leqslant (1+b)^k \omega_k(0, t; f) \tag{6}$$

whenever $k$ is a positive integer; moreover,

$$\omega_*(s, t; f) \leqslant 2\omega_2(s, 0; f)+2\omega_2(0, t; f) \leqslant 4\omega_1(s, 0; f)+4\omega_1(0, t; f). \tag{7}$$

Assuming that the partial derivative $f'_x(x, y)$ [resp. $f'_y(x, y)$] is bounded in $Q$ and that $k \geqslant 2$, we have

$$\omega_k(s, 0; f) \leqslant s\omega_{k-1}(s, 0; f'_x) \quad [\omega_k(0, t; f) \leqslant t\omega_{k-1}(0, t; f'_y)]. \tag{8}$$

In Sections 2-4 we shall signify by $C_\nu(\alpha)$, $C_\nu(\alpha, \beta)$ $(\nu = 1, 2, \ldots)$ the suitable positive constants depending on the indicated parameters, only. The symbols $C_\nu$ for the absolute constants will be used.

**2. Properties of the Marcinkiewicz means.** Let us start with the following

LEMMA 1. *Write*

$$I_n(\alpha) = \int_0^\pi \int_0^\pi u^\alpha |\Phi_n(u, v)| \, du \, dv$$

*for any positive* $\alpha$ *and* $n = 1, 2, \ldots$ *Taking* $n \geqslant 2$, *we have*

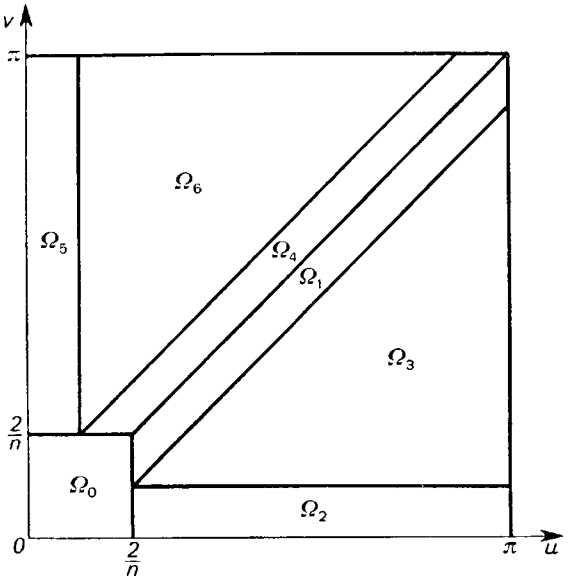
$$I_n(\alpha) \leqslant \begin{cases} C_1(\alpha) n^{1-\alpha} & \text{if } 0 \leqslant \alpha < 1, \\ C_2 \log^2 n & \text{if } \alpha = 1, \\ C_3(\alpha) \log n & \text{if } \alpha > 1. \end{cases}$$

Proof. Split the square $Q' = [0, \pi; 0; \pi]$ into seven domains $\Omega_j$ as shewn in Figure. Then,

$$I_n(\alpha) = \sum_{j=0}^{6} \iint_{\Omega_j} u^\alpha |\Phi_n(u, v)| \, du \, dv = I_n^0(\alpha) + I_n^1(\alpha) + \ldots + I_n^6(\alpha).$$

By (1), for any $\alpha \geqslant 0$,

$$I_n^0(\alpha) < n^3 \int_0^{2/n} \left\{ \int_0^{2/n} u^\alpha \, du \right\} dv = \frac{2^{\alpha+2}}{\alpha+1} n^{1-\alpha};$$



In view of (3),

$$I_n^1(\alpha) \leqslant \int_{2/n}^{\pi} \left\{ \int_{u-1/n}^{u} u^{\alpha} \frac{n\pi^2}{4uv}\, dv \right\} du = \int_{2/n}^{\pi} \frac{n\pi^2}{4}\, u^{\alpha-1} \log \frac{u}{u-1/n}\, du$$

$$\leqslant \frac{n\pi^2}{4} \int_{2/n}^{\pi} \frac{u^{\alpha-1}}{nu-1}\, du \leqslant \frac{\pi^2}{2} \int_{2/n}^{\pi} u^{\alpha-2}\, du.$$

Consequently,

$$I_n^1(\alpha) \leqslant \begin{cases} \dfrac{\pi^2 n^{1-\alpha}}{2^{2-\alpha}(1-\alpha)} & \text{when } 0 \leqslant \alpha < 1, \\ \dfrac{\pi^2}{2} \log \dfrac{n\pi}{2} & \text{when } \alpha = 1, \\ \dfrac{\pi^{\alpha+1}}{2(\alpha-1)} & \text{when } \alpha > 1. \end{cases}$$

Applying (2), we obtain

$$I_n^2(\alpha) < \int_{2/n}^{\pi} \left\{ \int_{0}^{1/n} u^{\alpha} \frac{3n\pi^2}{2(u-v)(u+v)}\, dv \right\} du$$

$$= \int_{2/n}^{\pi} \left\{ \frac{3n\pi^2}{4}\, u^{\alpha-1} \log \frac{u+1/n}{u-1/n} \right\} du \leqslant 3\pi^2 \int_{2/n}^{\pi} u^{\alpha-2}\, du.$$

Hence

$$I_n^2(\alpha) < \begin{cases} 3\, \dfrac{2^{\alpha-1}\pi^2}{1-\alpha}\, n^{1-\alpha} & \text{when } 0 \leqslant \alpha < 1, \\ 3\pi^2 \log \dfrac{n\pi}{2} & \text{when } \alpha = 1, \\ 3\, \dfrac{\pi^{\alpha+1}}{\alpha-1} & \text{when } \alpha > 1. \end{cases}$$

Further, by (4),

$$I_n^3(\alpha) \leqslant \int_{1/n}^{\pi-1/n} \left\{ \int_{v+1/n}^{\pi} u^{\alpha} \frac{\pi^3}{8uv(u-v)}\, du \right\} dv = \int_{1/n}^{\pi-1/n} \left\{ \frac{\pi^3}{8v} \int_{1/n}^{\pi-v} \frac{(t+v)^{\alpha-1}}{t}\, dt \right\} dv.$$

In the case $0 \leqslant \alpha < 1$,

$$I_n^3(\alpha) < \int_{1/n}^{\pi} \frac{\pi^3}{8v} \left\{ \int_{1/n}^{v} \frac{dt}{tv^{1-\alpha}} + \int_{v}^{\pi} \frac{dt}{t^{2-\alpha}} \right\} dv < \frac{\pi^3}{8} \left\{ \int_{1}^{\infty} \frac{\log z}{z^{2-\alpha}}\, dz + \frac{1}{(1-\alpha)^2} \right\} n^{1-\alpha}.$$

If $\alpha = 1$, then

$$I_n^3(\alpha) \leqslant \int_{1/n}^{\pi-1/n} \frac{\pi^3}{8v} \log n(\pi - v)\,dv < \frac{\pi^3}{8} \log^2 n\pi.$$

In the case $\alpha > 1$,

$$I_n^3(\alpha) < \int_{1/n}^{\pi} \frac{\pi^3}{8v} \left\{ \int_{1/n}^{v} \frac{(2v)^{\alpha-1}}{t}\,dt + \int_{v}^{\pi} \frac{(2t)^{\alpha-1}}{t}\,dt \right\} dv < \frac{\pi^{\alpha+2}}{2^{3-\alpha}(\alpha-1)} \log n\pi.$$

Finally

$$I_n^j(\alpha) \leqslant \iint_{\Omega_j} v^\alpha |\Phi_n(u, v)|\,du\,dv \quad \text{for } j = 4, 5, 6$$

and, by symmetry,

$$I_n^j(\alpha) \leqslant I_n^{j-3}(\alpha).$$

Thus, the proof is completed.

Passing to the Marcinkiewicz means $U_n(x, y; f)$, we shall denote by $H_n(x, y; f)$ the difference $U_n(x, y; f) - f(x, y)$.

THEOREM 1. *Given two positive numbers $\alpha, \beta$, we consider functions $f \in C_{2\pi}$ such that*

$$\omega_*(s, t; f) \leqslant 4(s^\alpha + t^\beta) \quad \text{for each } s, t \in \langle 0, \pi \rangle,$$

*and set $\gamma = \min(\alpha, \beta)$. Then,*

$$\max_{(x,y)\in Q} |H_n(x, y; f)| \leqslant \begin{cases} C_4(\alpha, \beta) n^{-\gamma} & \text{if } \gamma < 1, \\ C_5(\alpha, \beta) n^{-1} \log^2 n & \text{if } \gamma = 1, \\ C_6(\alpha, \beta) n^{-1} \log n & \text{if } \gamma > 1 \end{cases}$$

*for $n = 2, 3, \ldots$* (cf. [4], Theorem 2).

Proof. It is easily seen,

$$H_n(x, y; f) = \frac{1}{n\pi^2} \int_0^\pi \int_0^\pi \Delta^*_{u,v} f(x, y) \Phi_n(u, v)\,du\,dv.$$

Hence

$$|H_n(x, y; f)| \leqslant \frac{4}{n\pi^2} \int_0^\pi \int_0^\pi (u^\alpha + v^\beta) |\Phi_n(u, v)|\,du\,dv.$$

Now, the assertion follows at once from Lemma 1.

THEOREM 2. *If $f$ is of class $C_{2\pi}$, then*

$$\max_{(x,y)\in Q} |H_n(x, y; f)| \leqslant C_7 \left\{ \omega_1\left(\frac{\log^2 n}{n}, 0; f\right) + \omega_1\left(0, \frac{\log^2 n}{n}; f\right) \right\}$$

*for all integers $n \geqslant 2$.*

Proof. Clearly,

$$|H_n(x,y;f)| \leqslant \frac{4}{n\pi^2}\left\{\int_0^\pi\int_0^\pi \omega_1(u,0;f)|\Phi_n(u,v)|\,du\,dv + \right.$$

$$\left. + \int_0^\pi\int_0^\pi \omega_1(0,v;f)|\Phi_n(u,v)|\,du\,dv\right\} = \frac{4}{n\pi^2}\{A+B\}.$$

In view of (5) and (6),

$$A \leqslant \omega_1\left(\frac{\log^2 n}{n}, 0; f\right)\int_0^\pi\int_0^\pi\left(1+\frac{nu}{\log^2 n}\right)|\Phi_n(u,v)|\,du\,dv,$$

$$B \leqslant \omega_1\left(0, \frac{\log^2 n}{n}; f\right)\int_0^\pi\int_0^\pi\left(1+\frac{nv}{\log^2 n}\right)|\Phi_n(u,v)|\,du\,dv.$$

Next, we apply Lemma 1, as previously.

LEMMA 2. *Let $R$ be a rectangle defined by the inequalities*

(i) $\delta \leqslant u \leqslant \pi,\ 0 \leqslant v \leqslant h$

*or*

(ii) $0 \leqslant u \leqslant h,\ \delta \leqslant v \leqslant \pi,$

*where $\delta$, $h$ are in $(0,\pi\rangle$. Then, for any $f \in L_{2\pi}$,*

$$\lim_{n\to\infty}\frac{1}{n}\int\!\!\int_R f(x+u,y+v)\,\Phi_n(u,v)\,du\,dv = 0$$

*uniformly in $(x,y)$ on the plane.*

*The factor $f(x+u,y+v)$ can be replaced by $f(x-u,y-v)$, $f(x+u,y-v)$, $f(x-u,y+v)$, respectively.*

Proof in Case (i). If $\delta \leqslant u \leqslant \pi,\ 0 \leqslant v \leqslant \delta/2$, inequality (2) implies

$$\left|\frac{1}{n}\Phi_n(u,v)\right| \leqslant \frac{3\pi^2}{\delta^2} \quad \text{for } n = 1,2,\ldots$$

Considering $\delta \leqslant u \leqslant \pi,\ \delta/2 \leqslant v \leqslant h$ and applying (3), we get

$$\left|\frac{1}{n}\Phi_n(u,v)\right| \leqslant \frac{\pi^2}{2\delta^2} \quad \text{for } n = 1,2,\ldots$$

Since

$$\left|\frac{1}{n}\int_\delta^\xi\int_0^\eta \Phi_n(u,v)\,du\,dv\right| \leqslant \frac{1}{n}\sum_{\nu=0}^{n-1}\left|\int_\delta^\xi D_\nu(u)\,du\right|\left|\int_0^\eta D_\nu(v)\,dv\right|,$$

the relation

$$\lim_{n\to\infty}\frac{1}{n}\int_{\delta}^{\xi}\int_{0}^{\eta}\Phi_n(u,v)\,du\,dv=0$$

holds uniformly in $(\xi,\eta)\in R$.

Now, we conclude as in Section 2 of [4].

By Lemma 2, the following result can easily be obtained (cf. Theorem 1 of [4]).

THEOREM 3. *Suppose that $f$ is of class $L_{2\pi}$. Then*

(i) *the condition*

$$\lim_{(u,v)\to(0,0)}\Delta^{*}_{u,v}f(x,y)=0$$

*implies*

$$\lim_{n\to\infty}U_n(x,y;f)=f(x,y); \tag{9}$$

(ii) *if $f$ is continuous at every point of the rectangle $E=[a,b;\,c,d]$, where $a\leqslant b$, $c\leqslant d$, relation* (9) *holds uniformly in $E$.*

**3. Integrals of the Jackson type.** Write, for $n=1,2,\ldots$,

$$\Phi_n^2(u,v)=\{\Phi_n(u,v)\}^2,$$

$$1/\varrho_n=\iint_{Q}\Phi_n^2(u,v)\,du\,dv.$$

It is easily seen,

$$\Phi_n(u,v)=\frac{n}{4}+\sum_{k=1}^{n-1}(n-k)\{D_k(u)\cos kv+D_{k-1}(v)\cos ku\}$$

$$=\sum_{k=0}^{n-1}(n-k)\sum_{(p,q)\in e_k}\lambda_{p,q}\cos pu\cos qv,$$

where $\lambda_{0,0}=1/4$, $\lambda_{p,0}=\lambda_{0,q}=1/2$, $\lambda_{p,q}=1$ if $p,q\geqslant 1$, and $e_k$ denotes the set of all these pairs $p$, $q$ in which $p=k$, $0\leqslant q\leqslant k$ or $0\leqslant p\leqslant k$, $q=k$. By Parseval's identity ([6], p. 509),

$$\frac{1}{\pi^2}\iint_{Q}\Phi_n^2(u,v)\,du\,dv=\sum_{k=0}^{n-1}(n-k)^2\sum_{(p,q)\in e_k}\lambda_{p,q}$$

$$=\frac{n^2}{4}+\{2(n-1)^2+4(n-2)^2+\ldots+(2n-2)1^2\}=\frac{2n^4+n^2}{12}.$$

Hence

$$\varrho_n=\frac{12}{(2n^4+n^2)\pi^2}\qquad(n=1,2,\ldots).$$

Consider now the operators of Jackson's type

$$V_n(x, y; f) = \varrho_n \iint_Q f(x+u, y+v)\Phi_n^2(u, v)\,du\,dv \qquad (n = 1, 2, \ldots).$$

They are some trigonometric polynomials of order $2(n-1)$ in each variable, separately (cf. [2], p. 114-115).

THEOREM 4. *Given an arbitrary $f \in C_{2\pi}$, with partial moduli $\omega_2(s, 0) = \omega_2(s, 0; f)$, $\omega_2(0, t) = \omega_2(0, t; f)$, we have*

$$\max_{(x,y)\in Q} |V_n(x, y; f) - f(x, y)| \leqslant C_8 \left\{\omega_2\left(\frac{1}{n}, 0\right) + \omega_2\left(0, \frac{1}{n}\right)\right\} \tag{10}$$

*for $n = 1, 2, \ldots$* (cf. [2], p. 115-117, 141-142).

Proof. Clearly, the difference $V_n(x, y; f) - f(x, y)$ is equal to

$$J_n(x, y) = \varrho_n \int_0^\pi \int_0^\pi \Delta^*_{u,v} f(x, y) \cdot \Phi_n^2(u, v)\,du\,dv.$$

By (7), (5) and (6),

$$|J_n(x, y)| \leqslant 2\varrho_n \int_0^\pi \int_0^\pi \{\omega_2(u, 0) + \omega_2(0, v)\}\Phi_n^2(u, v)\,du\,dv$$

$$\leqslant 2\varrho_n \left\{\omega_2\left(\frac{1}{n}, 0\right) + \omega_2\left(0, \frac{1}{n}\right)\right\} \int_0^\pi \int_0^\pi (1+nu)^2 \Phi_n^2(u, v)\,du\,dv.$$

An argument similar to that of Lemma 1 leads to

$$\int_0^\pi \int_0^\pi u\Phi_n^2(u, v)\,du\,dv \leqslant C_9 n^3$$

and

$$\int_0^\pi \int_0^\pi u^2 \Phi_n^2(u, v)\,du\,dv \leqslant C_{10} n^2$$

for $n = 1, 2, \ldots$ Since

$$4\varrho_n \int_0^\pi \int_0^\pi \Phi_n^2(u, v)\,du\,dv = 1 \quad \text{and} \quad \varrho_n = O(1/n^4),$$

our assertion is established.

We note that if the partial derivative $f'_x(x, y)$ [resp. $f'_y(x, y)$] exists and is bounded in $Q$, the first [second] term in curly brackets of (10) can be replaced by

$$\frac{1}{n}\,\omega_1\left(\frac{1}{n}, 0; f'_x\right) \quad \left[\text{resp. } \frac{1}{n}\,\omega_1\left(0, \frac{1}{n}; f'_y\right)\right].$$

This follows at once from inequalities (8).

It is easy to see that Lemma 2 remains valid for the integrals with the kernel $\varrho_n \Phi_n^2(u, v)$ instead of $n^{-1}\Phi_n(u, v)$. Hence, considering the operators $V_n(x, y; f)$, the analogue of Theorem 3 can be stated, too.

LEMMA 3. *If* $(u, v) \in Q$, *then, for* $n = 1, 2, \ldots,$

$$\frac{\Phi_n^2(u, v)}{n^4} \leqslant \frac{C_{11} n^2}{(1+n^2u^2)(1+n^2v^2)} + \frac{C_{11} n^2}{\{1+n^2(u-v)^2\}\{1+n^2(u+v)^2\}}.$$

Proof. It is enough to show that for $(u, v) \in [0, \pi; 0, \pi]$ the assertion is true.

We shall consider the domains $\Omega_j$ defined in the proof of Lemma 1.

1° If $(u, v) \in \Omega_0$, then, by (1), $n^{-4}\Phi_n^2(u, v) < n^2$. Since $n^2u^2 \leqslant 4$, $n^2v^2 \leqslant 4$, we obtain

$$\frac{\Phi_n^2(u, v)}{n^4} < \frac{25n^2}{(1+n^2u^2)(1+n^2v^2)}.$$

2° In the case $(u, v) \in \Omega_1$ inequality (3) implies

$$\frac{\Phi_n^2(u, v)}{n^4} \leqslant \frac{\pi^4}{16n^2u^2v^2}.$$

Further, $2n^2u^2 \geqslant 4+n^2u^2$ and $2n^2v^2 \geqslant 1+n^2v^2$. Hence

$$\frac{\Phi_n^2(u, v)}{n^4} < \frac{n^2\pi^4}{4(1+n^2u^2)(1+n^2v^2)}.$$

3° Now let $(u, v) \in \Omega_2$. In view of (2),

$$\frac{\Phi_n^2(u, v)}{n^4} < \frac{9\pi^4}{4n^2(u-v)^2(u+v)^2}.$$

But $2n^2(u-v)^2 \geqslant 1+n^2(u-v)^2$, $2n^2(u+v)^2 \geqslant 4+n^2(u+v)^2$. Therefore,

$$\frac{\Phi_n^2(u, v)}{n^4} < \frac{9n^2\pi^4}{\{1+n^2(u-v)^2\}\{1+n^2(u+v)^2\}}.$$

4° Taking $(u, v) \in \Omega_3$ and applying (4), we get

$$\frac{\Phi_n^2(u, v)}{n^4} \leqslant \frac{\pi^6}{64n^4u^2v^2(u-v)^2} < \frac{n^2\pi^6}{16(1+n^2u^2)(1+n^2v^2)}.$$

By symmetry, the estimates of 2°-4° remains valid in the domains $\Omega_4 - \Omega_6$. Thus, the proof is completed.

Taking into account the last Lemma and reasoning as in [1], p. 537-538 and 552-555, we obtain

THEOREM 5. *Suppose that $f$ is of class $L_{2\pi}$. Then*

$$\lim_{n\to\infty} V_n(x, y; f) = f(x, y)$$

*almost everywhere on the plane.*

**4. Operators of the Stečkin type.** Let $k$ be a positive integer, and let $r$ be an even number such that $2r \geqslant k+2$. Putting

$$\Phi_n^r(u, v) = \Big\{\sum_{\nu=0}^{n-1} D_\nu(u) D_\nu(v)\Big\}^r,$$

choose the numbers $b_n$ satisfying the identities

$$4b_n \int_0^\pi \int_0^\pi \Phi_n^r(u, v)\, du\, dv = 1 \quad \text{for } n = 1, 2, \ldots$$

Inequalities (1)-(4) and $D_\nu(u) \geqslant (1+\nu)/2$ when $0 \leqslant u \leqslant 1/n$, $0 \leqslant \nu \leqslant n-1$ lead to

$$C_{12}(r) \leqslant n^{3r-2} b_n \leqslant C_{13}(r).$$

Given any $f \in L_{2\pi}$, we set

$$W_n(x, y; f) = b_n \iint_Q \Big\{\sum_{i=1}^{k}\sum_{j=1}^{k} (-1)^{i+j} \binom{k}{i}\binom{k}{j} f(x+iu, y+jv)\Big\} \Phi_n^r(u, v)\, du\, dv.$$

It can easily be observed, the integral

$$\iint_Q f(x+iu, y+jv)\Phi_n^r(u, v)\, du\, dv$$

coincides with a certain trigonometric polynomial of the order $[r(n-1)/i]$ in $x$, $[r(n-1)/j]$ in $y$, at most (see [5], p. 253-254, 222). Hence, $W_n(x, y; f)$ is a trigonometric polynomial of order less or equal to $r(n-1)$ in each variable, separately.

We shall present an analogue of the result announced in [3], p. 225-226.

THEOREM 6. *Suppose that $f$ is of class $C_{2\pi}$. Denote by $\omega_{k,k}(s, t)$, $\omega_k(s, 0)$, $\omega_k(0, t)$ its moduli of smoothness defined in Section 1. Then, for $n = 1, 2, \ldots$,*

$$\max_{(x,y)\in Q} |W_n(x, y; f) - f(x, y)| \leqslant C_{14}(k)\Big\{\omega_k\Big(\frac{1}{n}, 0\Big) + \omega_k\Big(0, \frac{1}{n}\Big)\Big\}.$$

Proof. Writing $T_n(x, y) = W_n(x, y; f) - f(x, y)$, we have

$$T_n(x, y) = b_n \iint_Q \Delta_{u,v}^{(k,k)} f(x, y) \cdot \Phi_n^r(u, v)\, du\, dv +$$

$$+ (-1)^k b_n \Big\{\iint_Q \Delta_{u,0}^{(k)} f(x, y) \cdot \Phi_n^r(u, v)\, du\, dv + \iint_Q \Delta_{0,v}^{(k)} f(x, y) \cdot \Phi_n^r(u, v)\, du\, dv\Big\}.$$

Consequently,

$$|T_n(x,y)| \leqslant 4b_n\Big\{\int_0^\pi\int_0^\pi \omega_{k,k}(u,v)\Phi_n^r(u,v)\,du\,dv+$$

$$+\int_0^\pi\int_0^\pi \omega_k(u,0)\Phi_n^r(u,v)\,du\,dv+\int_0^\pi\int_0^\pi \omega_k(0,v)\Phi_n^r(u,v)\,du\,dv\Big\}.$$

As well known,

$$\omega_{k,k}(u,v) \leqslant 2^{k-1}\{\omega_k(u,0)+\omega_k(0,v)\}.$$

In view of (5) and (6),

$$\omega_k(u,0) \leqslant (nu+1)^k\omega_k\Big(\frac{1}{n},0\Big), \qquad \omega_k(0,v) \leqslant (nv+1)^k\omega_k\Big(0,\frac{1}{n}\Big).$$

Therefore,

$$|T_n(x,y)| \leqslant 4b_n(2^{k-1}+1)\Big\{\omega_k\Big(\frac{1}{n},0\Big)+\omega_k\Big(0,\frac{1}{n}\Big)\Big\}G_n(k,r),$$

where

$$G_n(k,r) = \int_0^\pi\int_0^\pi (nu+1)^k\Phi_n^r(u,v)\,du\,dv$$

$$\leqslant 2^k\Big\{\int_0^\pi\int_0^\pi \Phi_n^r(u,v)\,du\,dv+n^k\int_0^\pi\int_0^\pi u^k\Phi_n^r(u,v)\,du\,dv\Big\}.$$

Proceeding now as in the proof of Lemma 1, we easily get the desired assertion.

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