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Integer points on conics

The aim of this note is to prove a result quoted in my survey article on Diophantine equations [2] which although quite elementary does not occur explicitly in the literature.

THEOREM. *If a non-degenerate conic*

$$F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

contains integer points, then one of them satisfies

$$\max\{|x|, |y|\} < (3H)^{300H^3}, \qquad H = \max\{|a|, |b|, |c|, |d|, |e|, |f|\}$$

(a, b, c, d, e, f *integers*).

Proof. Consider first the case of a parabola, i.e. $\delta = b^2 - 4ac = 0$. Then $ax^2 + bxy + cy^2 = \gamma(\alpha x + \beta y)^2$, α, β, γ integers. Let $\alpha x + \beta y = t$, $\alpha e - \beta d = \varepsilon$. Since $F(x, y) = 0$ is not degenerate, $\varepsilon \neq 0$. The equation $F(x, y) = 0$ is equivalent to the system

$$\alpha x + \beta y = t, \qquad dx + ey = -\gamma t^2 - f.$$

The necessary and sufficient condition of solvability of this system in integers x, y are the divisibilities

$$\varepsilon \mid \alpha\gamma t^2 + dt + \alpha f, \qquad \varepsilon \mid \beta\gamma t^2 + et + \beta f.$$

If these divisibilities are satisfied for some t, they are also satisfied for some t_0 in the interval $0 \leqslant t < |\varepsilon|$. Since

$$|\gamma|\,\varepsilon^2 = |e^2 a - bed + ed^2| \leqslant 3H^3$$

we obtain

$$|t_0| < |\varepsilon| < \sqrt{3}H^{3/2}, \qquad |\gamma t_0^2 + f| < |\gamma|\,\varepsilon^2 + |f| \leqslant 4H^3,$$

$$\max\{|x_0|, |y_0|\} \leqslant H|t_0| + \sqrt{H}\,|\gamma t_0^2 + f| < 6H^{7/2}.$$

Consider now the conics with centre. The transformation

$$X = \delta x + be - 2cd, \qquad Y = \delta y + bd - 2ae \tag{1}$$

reduces the equation $F(x, y) = 0$ to the form

$$aX^2+bXY+cY^2 = 4\delta\Delta, \tag{2}$$

where

$$\Delta = \begin{vmatrix} a & \frac{1}{2}b & \frac{1}{2}d \\ \frac{1}{2}b & c & \frac{1}{2}e \\ \frac{1}{2}d & \frac{1}{2}e & f \end{vmatrix}.$$

4Δ is an integer and one verifies easily that $0 < |\Delta| < 2H^3$. Consider the three possible cases.

Ellipse ($\delta < 0$). It follows from (2) that

$$|\delta|\, Y^2 \leqslant 16a|\delta\Delta|, \qquad |\delta|\, X^2 \leqslant 16c|\delta\Delta|,$$

thus $\max\{|X|, |Y|\} < 6H^2$ and in view of (1), $\max\{|x|, |y|\} < 9H^2$.

Hyperbola with rational asymptotes ($\delta > 0$ is a perfect square). Let u, v be the solution of the equation

$$u^2-4\delta^3v^2 = 4 \tag{3}$$

in the least positive integers and set

$$\varepsilon = \left(\frac{u_0+2\delta\sqrt{\delta}v_0}{2}\right)^2 = \frac{u_0^2+4\delta^3v_0^2}{4} + \delta\sqrt{\delta}u_0v_0.$$

If X_0, Y_0 is a solution of (2) and satisfies the congruences implied by (1) namely

$$X_0 \equiv be-2cd, \qquad Y_0 \equiv bd-2ae \bmod \delta,$$

then also every solution X, Y defined by the formula

$$2aX+(b+\sqrt{\delta})\, Y = (2aX_0+(b+\sqrt{\delta})\, Y_0)\,\varepsilon^n$$

satisfies these congruences. Therefore, we can assume that

$$4\sqrt{|a\delta\Delta|} \leqslant |2aX_0+(b+\sqrt{\delta})\, Y_0| < 4\sqrt{|a\delta\Delta|}\,\varepsilon,$$

whence in view of (2)

$$|2aX_0+(b-\sqrt{\delta})\, Y_0| \leqslant 4\sqrt{|a\delta\Delta|}.$$

We get

$$|Y_0| \leqslant 2\sqrt{|a\Delta|}\,(\varepsilon+1)$$

and by symmetry

$$|X_0| \leqslant 2\sqrt{|c\Delta|}\,(\varepsilon+1).$$

In virtue of Schur's estimate [3] of the least solution of (3)

$$\varepsilon+1<(4\delta^3)^{2\sqrt{4\delta^3}}+1<(500H^6)^{20\sqrt{5}H^3}+1<(3H)^{270H^3}$$

and we get by (1)

$$\begin{aligned}\max\{|x_0|,|y_0|\}&\leqslant 3H^2+\max\{|X_0|,|Y_0|\}\\&\leqslant 3H^2+2\sqrt{2}H^2(3H)^{270H^3}<(3H)^{300H^3},\end{aligned}$$

q.e.d.

Remark 1. Theorem remains true also for non-degenerate conics.

Remark 2. The estimate for $\max\{|x|,|y|\}$ obtained in the case of ellipse and hyperbola with rational asymptotes holds not only for some integer point but for every integer point on the conic. In this sense the estimation (by $20H^4$) is sharp as the following example shows. The hyperbola

$$tx^2+(2t+1)xy+(t+1)y+2tx=0$$

has an integer point $(-4t^4-8t^3+4t-1,\ 4t^4+4t^3-4t^2-2t+1)$. On the other hand there are no reasons to believe that the estimation given in the theorem is sharp. The example of the equation

$$x^2-2^{2t+1}(2y+1)^2=1,$$

where solution in the absolutely least integers is

$$x=\frac{(1+\sqrt{2})^{2t}+(1-\sqrt{2})^{2t}}{2};\qquad y=\frac{(1+\sqrt{2})^{2t}-(1-\sqrt{2})^{2t}}{2^{t+2}\sqrt{2}}-\tfrac{1}{2}$$

shows only that H^3 in the exponent cannot be replaced by a power of H lower than $H^{1/2}$. If instead of Schur's estimate for the least solution of (3) one uses a better estimate due to Hua [1] one gets

$$\max\{|x|,|y|\}<(5H)^{200H^3}.$$

References

[1] L. K. Hua, *On the least solution of Pell's equation*, Bull. Amer. Math. Soc. 48 (1942), p. 731-735.

[2] A. Schinzel, *Równania diofantyczne*, Wiadom. Mat. 12, p. 227-232.

[3] I. Schur, *Einige Bemerkungen zu der vorstehenden Arbeit des Herrn G. Pólya: Uber die Verteilung der quadratischen Reste und Nichtreste*, Nachr. Göttingen (1918), p. 30-36.

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Errata to the paper "Integer points on conics"*

On p. 134 after the heading "Hyperbola with rational asymptotes ($\delta > 0$ is a perfect square)" the following lines have been omitted:

We have

$$aX^2 + bXY + cY^2 = (\alpha_1 X + \beta_1 Y)(\alpha_2 X + \beta_2 Y),$$

where $|\alpha_1|$, $|\alpha_2|$, $|\beta_1|$, $|\beta_2|$ integers $\leqslant H$. It follows from (1) that

$$|\alpha_i X + \beta_i Y| \leqslant 4\delta|\Delta| \quad (i = 1, 2),$$

thus $\max\{|X|, |Y|\} \leqslant 8\delta|\Delta|$ and in view of (1)

$$\max\{|x|, |y|\} \leqslant 8|\Delta|H + 3H^2 \leqslant 16H^4 + 3H^2 < 20H^4.$$

Hyperbola with irrational asymptotes ($\delta > 0$ is not a perfect square).

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* Prace Mat. 16 (1972), p. 133-135.