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Some theorems about difference inequalities

In this note we present some comparative theorems for systems of discrete difference inequalities. In the first item we discuss the solution of system linear difference equations with certain initial conditions. Whereas the second one gives two comparative theorems for linear difference inequalities. It is to be observed that Theorem 1 is an analogon of the theorem on the differential inequalities treated in [2] (p. 187-188). In the third item there is formulated a theorem for non-linear system of discrete difference inequalities. In fact, our Theorem 3 includes two theorems, because the terms in brackets concern the second theorem, which regards opposite difference inequalities. Roughly speaking Theorem 3 is an adaptation of the theorem on ordinary differential inequalities to the case of difference inequalities [1]. But Corollary 3 to Theorem 3 is a certain analogy of Theorem 15.1, given by Szarski [5] (p. 47-48).

It is to be observed, that the authors: Chorlton [3] (p. 201-203), Levy and Lessman [4] (p. 245-247), give comparative theorems for linear difference inequalities of the second order. But we do not meet any comparative theorems for linear difference inequalities of the higher order or for systems of difference inequalities.

1. Let be given a system of linear difference equations

$$x_i(n+1) = \sum_{k=1}^{m} a_{ik}(n)\cdot x_k(n)+f_i(n) \quad \text{for } n = 1, \ldots, m, \tag{1}$$

which may be presented in the matrix notations as follows:

$$x(n+1) = A(n)\cdot x(n)+f(n), \tag{2}$$

where we have

$$A(n) = [a_{ik}(n)]_m \quad \text{and} \quad f(n) = [f_i(n)]_{m,1}$$

are given functions and

$$x(n) = [x_1(n), \ldots, x_m(n)] = [x_i(n)]'_{m,1} \quad (1)$$

s the function we look for.

We shall now consider the solution of system (2) with the following initial conditions

$$x(0) = \mathring{x}. \tag{3}$$

If we express the values the function $x(n)$ in (2) by the values $A(n)$, $f(n)$ and by $\mathring{x}$, then we obtain

$$\begin{aligned} x(1) &= A(0)\cdot x(0)+f(0) = A(0)\cdot\mathring{x}+f(0),\\ x(2) &= A(1)\cdot x(1)+f(1) = A(1)\cdot A(0)\cdot x+A(1)\cdot f(0)+f(1),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and generally

$$\begin{aligned} x(n) = \prod_{j=0}^{n-1} A(n-1-j)\cdot\mathring{x}+\prod_{j=1}^{n-1} A(n-1-j+1)\cdot f(0)+\\ +\prod_{j=2}^{n-1} A(n-1-j+2)\cdot f(1)+\ \ldots\ +A(n-1)\cdot f(n-2)+f(n-1) \end{aligned} \tag{4}$$

or

$$x(n) = \prod_{j=0}^{n-1} A(n-1-j)\cdot\mathring{x}+\sum_{s=1}^{n-1}\prod_{j=s}^{n-1} A(n-1-j+s)\cdot f(s-1)+f(n-1). \tag{5}$$

In the case when the coefficient matrix of the difference equations is constant, the solution of the system

$$x(n+1) = A\cdot x(n)+f(n) \tag{2'}$$

with initial conditions

$$x(0) = \mathring{x} \tag{3'}$$

appears like

$$x(n) = A^n\cdot x+\sum_{s=0}^{n-1} A^{n-1-s}\cdot f(s). \tag{5'}$$

From the above result we obtain

COROLLARY 1. *The system of linear homogeneous difference equations*

$$x(n+1) = A(n)\cdot x(n) \tag{6}$$

with initial conditions

$$x(0) = 0 \tag{7}$$

has no solutions except zero

$$x(n) = 0(n). \tag{8}$$

(1) $[x_i(n)]'_{m,1}$ denotes the matrix which is transposed to the matrix $[x_i(n)]_{1,m}$.

2. THEOREM 1. *Let be given the system of linear difference equations*

$$x(n+1) = A(n)\cdot x(n)+f(n) \quad \text{at } A(n) \geqslant 0 \tag{2''}$$

and initial conditions (3). *We assume furthermore, that the solutions of the system of difference inequalities*

$$y(n+1) \leqslant A(n)\cdot y(n)+f(n) \tag{9}$$

satisfies the initial conditions

$$y(0) = \mathring{x}. \tag{10}$$

Then there occur the inequalities

$$y(n) \leqslant x(n) \quad \text{for } n = 1, 2, \ldots \tag{11}$$

Proof. We denote $y(n+1)$ by

$$y(n+1) = A(n)\cdot y(n)+f(n)-h(n), \quad \text{where } h(n) \geqslant 0.$$

Then we obtain, by (5),

$$\begin{aligned} y(n) &= \prod_{j=0}^{n-1} A(n-1-j)\cdot y(0)+ \\ &\quad +\sum_{s=1}^{n-1}\prod_{j=s}^{n-1} A(n-1-j+s)\cdot[f(s-1)-h(s-1)]+f(n-1)-h(n-1) \\ &= \prod_{j=0}^{n-1} A(n-1-j)\cdot\mathring{x}+\sum_{s=1}^{n-1}\prod_{j=s}^{n-1} A(n-1-j+s)\cdot f(s-1)+f(n-1)- \\ &\quad -\Big[\sum_{s=1}^{n-1}\prod_{j=s}^{n-1} A(n-1-j+s)\cdot h(s-1)+h(n-1)\Big] \leqslant x(n) \end{aligned}$$

because by $A(n) \geqslant 0$ and $h(n) \geqslant 0$ we have

$$\sum_{s=1}^{n-1}\prod_{j=s}^{n-1} A(n-1-j+s)\cdot h(s-1)+h(n-1) \geqslant 0.$$

Remark. If we assume,

$$y(n+1) \geqslant A(n)\,y(n)+f(n), \tag{9'}$$

$A(n) \geqslant 0$ for the system of difference inequalities and initial conditions (10), then we obtain the inequalities

$$y(n) \geqslant x(n) \quad \text{for } n = 1, 2, \ldots \tag{11'}$$

From Theorem 1 we obtain the following

COROLLARY 2. *Let be given the system of linear difference equations* (2'') *and positive initial conditions* (3). *Furthermore, if the solutions of the system of linear inequalities*

$$|y(n+1)| \leqslant A(n)\cdot|y(n)|+f(n), \quad \textit{where } f(n) \geqslant 0 \tag{12}$$

satisfy the initial conditions

$$y(0) = \mathring{x} > 0, \tag{13}$$

then the following inequalities hold

$$|y(n)| \leqslant x(n) \quad \text{for } n = 1, 2, \ldots \tag{14}$$

The analogy implies the following theorem:

THEOREM 2. *Let be given the system of linear difference equations* (2'') *with initial conditions* (3) *and the system of linear difference equations*

$$y(n+1) = B(n) \cdot y(n) + g(n) \tag{15}$$

with initial conditions

$$y(0) = \mathring{x}. \tag{16}$$

Moreover, if we assume that

$$0 \leqslant B(n) \leqslant A(n) \quad \text{and} \quad g(n) \leqslant f(n) \quad \text{and} \quad f(n) \geqslant 0, \tag{17}$$

then the solutions of system (2'') *with initial conditions* (3) *satisfy the inequalities*

$$y(n) \leqslant x(n) \quad \text{for } n = 1, 2, \ldots \tag{18}$$

3. Let be given the system of the difference equations

$$x_i(n+1) = f_i\big(n, x_1(n), x_2(n), \ldots, x_m(n)\big), \quad i = 1, 2, \ldots, m. \tag{19}$$

This system takes the following form in a vector notation:

$$x(n+1) = f\big(n, x(n)\big), \tag{20}$$

where

$$x(n) = [x_1(n), \ldots, x_m(n)] = [x_i(n)]'_{m,1}$$

but

$$f(n) = [f_1(n), \ldots, f_m(n)]' = [f_i(n)]'_{m,1}.$$

An analogous argument as in system (2) with initial conditions (3) implies that if the function $f(t, u_1, \ldots, u_m)$ is defined in the domain

$$Q = \{(t, u_1, \ldots, u_m, 0 \leqslant t < +\infty, -\infty < u_i < +\infty, i = 1, \ldots, m\} \tag{21}$$

for every $n \geqslant 0$

$$\sup_{i=1,\ldots,m} |f_i(n)| < K = \text{const}, \tag{22}$$

then the system of equations (20) with initial conditions

$$x(0) = \mathring{x} \tag{23}$$

has the solution of the form

$$x(n) = f^n(0, \mathring{x}), \tag{24}$$

where

$$f^0(0, \mathring{x}) = \mathring{x}, \quad f^{k+1}(0, \mathring{x}) = f\big(k, f^k(0, \mathring{x})\big) \tag{25}$$

for any non-negative integer k.

THEOREM 3. *Let the function $f(t, u) = f(t, u_1, \ldots, u_m)$ defined in the domain* (21) *be increasing in every variable u_i. If the function $z(n)$ satisfies the inequality*

$$z(n+1) < f\big(n, z(n)\big) \quad (z(n+1) > f\big(n, z(n)\big)) \tag{26}$$

and the function $x(n)$ satisfies the equation

$$x(n+1) = f\big(n, x(n)\big) \tag{27}$$

and if for any certain fixed $j \geqslant 0$ there holds

$$x(j) = z(j), \tag{28}$$

then the following inequalities are velid

$$\begin{aligned} &z(n) > x(n) \quad (z(n) < x(n)) \quad \text{for } 0 \leqslant n < j \ (^2), \\ &z(n) < x(n) \quad (z(n) > x(n)) \quad \text{for } n > j. \end{aligned} \tag{29}$$

Proof. We obtain from (26), (27) and from (28)

$$0 < f\big(j, z(j)\big) = z(j+1) = f\big(j, x(j)\big) - z(j+1) = x(j+1) - z(j+1),$$

which yields

$$x(j+1) > z(j+1).$$

Thus, the second one from inequalities (29) is satisfied for $n = j+1$.

If we assume that the inequality holds for $n = j+k$, then the function $f(t, u)$ being with respect u increasing we obtain from (26) and (27)

$$x(j+k+1) = f\big(j+k, x(j+k)\big) > f\big(j+k, z(j+k)\big) > z(j+k+1).$$

Thus we obtain by mathematical induction the inequality $x(n) > z(n)$ for every $n > j$.

It remains to show the first of inequalities (29). Let us suppose that the inequality does not hold. That means $x(j-1) \geqslant z(j-1)$. Then from (28) we obtain because of the monotony of $f(t, u)$

$$0 = z(j) - x(j) < f\big(j-1, x(j-1)\big) - f\big(j-1, x(j-1)\big) = 0.$$

Thist leads to contradiction. Thus we have $x(j-1) < z(j-1)$.

(^{2}) In the case $j = 0$ we examine only the second of inequalities (29).

If we recall that the function $f(t, u)$ is monotone, then we obtain the inequality $x(n) < z(n)$ for every $0 \leqslant n < j$.

Analogously we conclude the theorem in the case when the second of inequalities (26) is satisfied.

From Theorem 3 we obtain the following

COROLLARY 3. *If function $f(t, u)$ is non-negative and increasing with regard to u in domain* (21) *and the solution of the system of difference equations*

$$x(n+1) = f(n, x(n)) \tag{30}$$

satisfies initial conditions

$$x(0) = \mathring{x} > 0 \tag{31}$$

and, moreover, if the solutions of the system of difference inequalities

$$|y(n+1)| < f(n, |y(n)|) \tag{32}$$

satisfy initial conditions

$$y(0) = \mathring{x}, \tag{33}$$

then the following inequalities hold:

$$|y(n)| < x(n) \quad \text{for } n = 1, 2, \ldots \tag{34}$$

In order to obtain the Corollary 3, it sufficies to apply Theorem 3 with initial conditions (29) for $j = 0$ and to assume $z(n) = |y(n)|$.

References

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