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Uniform continuity and net behaviour

In this paper the following results are presented: (1) Uniform continuity with respect to an appropriate uniformity characterizes when (a) a net of functions is uniformly Cauchy (Theorem 2), (b) a pair of nets is asymptotic (Corollary 3), or (c) a net is Cauchy (Corollary 4). (2) A function is uniformly continuous precisely whenever it preserves (a) pairs of asymptotic nets (Theorem 5), (b) Cauchy nets (and has a totally bounded domain, Theorem 6), or (c) uniformly convergent nets (Theorem 7).

The reader is referred to Kelley [2] for definitions not given here. If a proof does not follow a theorem, we consider the proof straightforward from pertinent definitions. In the sequel, h and k are nets with directed set A and values in a uniform space $(X, \mathscr{U})$; f is a function from a uniform space $(X, \mathscr{U})$ to a uniform space $(Y, \mathscr{V})$.

DEFINITION 1. The nets h and k are *asymptotic* if for every U in $\mathscr{U}$ there is an a_U in A such that $a > a_U$ implies (ha, ka) is in U. The nets h and k *have asymptotic subnets* if there are subnets of h and k of the form $h \circ N$ and $k \circ N$, respectively, which are asymptotic.

Remark. Goetz [1] uses an axiomatization of the concept of asymptotic sequences to talk about uniform notions in a sequence space setting.

DEFINITION 2. If h and k are nets with domain A, the net of functions $h \# k$ with directed set A and domain A is defined by

$$h \# k(b, a) = h(a) \quad \text{if } a > b,$$
$$h \# k(b, a) = k(a) \quad \text{otherwise.}$$

LEMMA 1. *h and k are asymptotic if and only if $h \# k$ is uniformly convergent to k.*

DEFINITION 3. Let A be a directed set. The uniformity for A which has as a base all sets of the form

$$U(a) = \{(x, y): x = y \text{ or } (x > a \text{ and } y > a)\}$$

(see Kelley [2], p. 204, Prob. C) will be called the *directed uniformity for A*.

THEOREM 2. *Let G: $A \times X \to Y$ be a net of functions with directed set A, domain X, and values in a uniform space Y. Then G is uniformly Cauchy if and only if G is uniformly continuous with respect to the product uniformity on $A \times X$, where A has the directed and X the discrete uniformity.*

COROLLARY 3. *h and k are asymptotic if and only if $h \# k$ is uniformly continuous with respect to the product uniformity on $A \times A$, where the first coordinate space has the directed and the second coordinate space the discrete uniformity.*

Proof. Lemma 1 and Theorem 2.

COROLLARY 4. *h is Cauchy if and only if h is uniformly continuous with respect to the directed uniformity on A.*

THEOREM 5. *The following conditions are equivalent:*

(a) *f is uniformly continuous.*

(b) *h and k are asymptotic nets in X implies $f \circ h$ and $f \circ k$ are asymptotic nets in Y.*

(c) *h and k are asymptotic nets in X implies $f \circ h$ and $f \circ k$ have asymptotic subnets in Y.*

Proof. Since it is easy to see that (a) implies (b) and (b) implies (c), assume (c) holds and (a) does not. Then there is a V in $\mathscr{V}$ such that for every U in $\mathscr{U}$ there is a pair (hU, kU) in $\mathscr{U}$ such that $(f(hU), f(kU))$ is not in $\mathscr{V}$. If U is directed downward by inclusion, h and k are asymptotic nets in X such that $f \circ h$ and $f \circ k$ do not have asymptotic subnets in Y. This contradiction shows that (c) implies (a).

Remark. A function which preserves Cauchy nets is not in general a *uniformly* continuous function since *any* continuous function whose domain is complete has this property. However, we do have the following

THEOREM 6. *Let f be a function from a totally bounded uniform space X to a uniform space Y. Then f is uniformly continuous if and only if whenever h is a Cauchy net in X, $f \circ h$ is a Cauchy net in Y.*

Proof. That f is uniformly continuous implies f preserves Cauchy nets is straight-forward.

Assume f preserves Cauchy nets. We will use Theorem 5. Let h and k be asymptotic nets in X. Then h and k have asymptotic Cauchy subnets $h \circ N$ and $k \circ N$, respectively, since X is totally bounded.

We define a net as follows: Let the set $\{1, 2\}$ have the order $1 > 1$, $1 > 2$, $2 > 1$, and $2 > 2$. Let B be the directed set of $h \circ N$ and $k \circ N$ and give $B \times \{1, 2\}$ the product order. Let, for (b, c) in $B \times \{1, 2\}$, $j(b, c)$ be $h \circ Nb$ if $c = 1$ and $k \circ Nb$ if $c = 2$. Then j is a Cauchy net with $h \circ N$ and $k \circ N$ as subnets.

Thus by assumption $f \circ j$ is Cauchy and clearly the subnets of $f \circ j$, $f \circ h \circ N$ and $f \circ k \circ N$, must be asymptotic. Therefore f is uniformly continuous.

Remark. I wish to thank Professor R. D. Johnson of Georgia Tech. for the construction of the net j in the above proof.

THEOREM 7. *f is uniformly continuous if and only if whenever G: $A \times Z \to X$ is a uniformly convergent net of functions with directed set A, domain Z, and values in $X = \mathrm{dom} f$, $f \circ G$ is a uniformly convergent net.*

Proof. Without difficulty one sees that if f is uniformly continuous, it preserves uniformly convergent nets of functions.

Conversely, assume f preserves uniformly convergent nets of functions. Again we will use Theorem 5. Let h and k be asymptotic nets in X. Then $h \# k$ is uniformly convergent to k by Lemma 1, so by assumption $f \circ (h \# k)$ is uniformly convergent. But $f \circ (h \# k) = (f \circ h) \# (f \circ k)$ and is necessarily convergent to $f \circ k$ so by Lemma 1 $f \circ h$ and $f \circ k$ are asymptotic. Therefore f is uniformly continuous.

References

[1] A. Goetz, *On a notion of uniformity for L-spaces of Fréchet*, Colloq. Math. 9 (1962), p. 223-231.

[2] J. L. Kelley, *General topology*, Princeton, New Jersey, 1955.

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