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A note on the relation between Lindelöf and $\aleph_1$-compact spaces

Creede [1, 2] demonstrates that if S is a semistratifiable space, then the statement that S is Lindelöf is equivalent to the statement that S is $\aleph_1$-compact. It will be shown in this paper that this holds for a much weaker space.

All theorems which hold for semistratifiable spaces hold for both semimetric and stratifiable spaces since the concept of semistratifiable spaces is a generalization of both semimetric and stratifiable spaces. This generalization is due to E. A. Michael.

DEFINITION 1. The statement that a topological space S is *semistratifiable* means that there is a function G from the product of the collection of closed subsets of S with the natural numbers into the collection of open sets in S such that

(i) $$\bigcap_{n=1}^{\infty} G(A, n) = A \quad \text{for each closed set } A,$$

(ii) $$G(A_1, n) \subset G(A_2, n) \text{ whenever } A_1 \subset A_2.$$

DEFINITION 2. The statement that K is a *minimal cover* of the point set M means that K is an open cover of M and if g is an element of K, then $\{h\colon h \in K \text{ and } h \neq g\}$ is not a cover of M.

DEFINITION 3. The statement that a point set M is *minimal cover refineable* means that if K is an open cover of M, then there is an open refinement of K which is a minimal cover of M.

DEFINITION 4. The statement that a topological space S is *F_σ-screenable* means that every open cover of S has a σ-discrete closed refinement which covers S.

DEFINITION 5. The statement that a space S is *$\aleph_1$-compact* means that no collection of uncountably many subsets of S is discrete.

DEFINITION 6. The statement that a topological space S is *semimetric* means that there is a distance function d defined on $S \times S$ such that

(i) for all points x and y in S, $d(x, y) = d(y, x) \geqslant 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$, and

(iii) if x is a point of S and M is a subset of S, $x \in \overline{M}$ if and only if $\inf\{d(x, y)\colon y \in M\} = 0$.

All other terms are as used in [4].

First, it will be shown that every subset of a semistratifiable space is minimal cover refineable. Then, it will be shown that in a minimal cover refineable space S, the statement that S is Lindelöf is equivalent to the statement that S is $\aleph_1$-compact.

THEOREM 1. *Every F_σ-screenable space is minimal cover refineable.*

Proof. Let S be an F_σ-screenable space and O any open cover of S. Let $\{F_i\}$ be a countable collection of discrete closed refinements of O such that $\{F_i\}$ covers S. For each integer $i > 0$, associate with each element f of F_i one and only one open set $o = g \setminus (F_i^* \setminus f)$, where g is an element of O and $f \subset g$. Let O_i' be the collection of elements of O associated with F_i. Let $O_1 = O_1'$. For each integer $i > 1$, let

$$O_i = \{o\colon o' \in O_i', o = o' \setminus \bigcup_{j=1}^{i-1} F_j^*, \text{ and } o \cap F_i^* \neq \emptyset\}.$$

Then the collection of open sets $\{o$: for some integer $i > 0$, $o \in O_i\}$ is a minimal cover of S which refines O and Theorem 1 is established.

THEOREM 2. *Every subset of a semistratifiable space is minimal cover refineable.*

Proof. Creede [1, 2] demonstrates that a semistratifiable space is both F_σ-screenable and hereditarily semistratifiable. By Theorem 1, this is sufficient to show that a semistratifiable space and every subset of it is minimal cover refineable.

THEOREM 3. *A minimal cover refineable space is Lindelöf if and only if it is $\aleph_1$-compact.*

Proof. Since Lindelöf implies no collection of uncountably many subsets is discrete, only sufficiency must be shown. Let S be a minimal cover refineable space with the property that no collection of uncountably many subsets of the space is discrete and O any open cover of S. Let O' be a refinement of O which is a minimal cover of S. Let $H = \{h$: h is the set of all the points in some $g \in O'$ which are in no other element of $O'\}$. The collection H is discrete, so both H and O' have at most countably many members. Therefore, S is Lindelöf and Theorem 3 is established.

The following theorem is an application of Theorem 3:

THEOREM 4. *A hereditarily separable minimal cover refineable space is Lindelöf.*

Proof. Let D be a discrete collection. Let C be a countable point set such that $C \subset D^*$ and $\overline{D^*} \subset \overline{C}$. Each element of D contains a point of C, so D must be a countable collection. Then, by Theorem 3, Theorem 4 is established.

In [3], R. W. Heath points out that there is a separable semimetric space which is not Lindelöf. One such example is given by McAuley [5].

Creede [1, 2] demonstrates that in a semistratifiable space hereditary separability is equivalent to Lindelöf. This is not the case in minimal cover refineable spaces.

THEOREM 5. *There is a minimal cover refineable Lindelöf T_4 space which is not separable.*

Proof. Let Ω denote the first uncountable ordinal. Consider the collection $[0, \Omega]$ of ordinals with the order topology. This space is compact and T_4. Hence, it is minimal cover refineable and Lindelöf, but it is not separable.

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