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Approximation of functions of two variables in $L^{*\varphi}$-metrics*

1. By $\varphi(u)$ and $\varphi^*(u)$ we shall denote the convex functions complementary in the sense of Young, defined as in the paper [2] or [3], p. 16–17.

The symbol $L^{\varphi^*}(Q)$ will be used for the Orlicz class of all real functions $g(x, y)$ of two variables defined and measurable in the rectangle $Q = [a, b; c, d]$, satisfying the condition

$$\varrho(g, \varphi^*) = \int\int_Q \varphi^*(|g(x, y)|)\,dx\,dy < \infty.$$

By $L^{*\varphi}(Q)$ we shall denote the Orlicz space of the real functions of two variables, $f(x, y)$, defined and measurable in the rectangle Q such that for every $g \in L^{\varphi^*}(Q)$ the integral $\int\int_Q f(x, y)g(x, y)\,dx\,dy$ is finite, and the norm is defined as

$$\|f\|_\varphi = \|f(x, y)\|_\varphi = \sup\left|\int\int_Q f(x, y)g(x, y)\,dx\,dy\right|,$$

where the supremum is taken over all functions g for which $\varrho(g, \varphi^*) \leqslant 1$ ([3], p. 76–87).

In the space $L^{*\varphi}(Q)$ we can also introduce the Luxemburg's norm

$$\|f\|_{(\varphi)} = \|f(x, y)\|_{(\varphi)} = \inf \varepsilon,$$

where the infimum is taken over all positive ε's such that

$$\varrho\left(\frac{f}{\varepsilon}, \varphi\right) = \int\int_Q \varphi\left(\frac{|f(x, y)|}{\varepsilon}\right)dx\,dy \leqslant 1.$$

As in the case of Orlicz spaces of functions of one variable, both these norms are equivalent. This follows at once from the inequality

$$\|f\|_{(\varphi)} \leqslant \|f\|_\varphi \leqslant 2\,\|f\|_{(\varphi)}. \tag{1}$$

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$L^{*\varphi}(Q)$ is reflexive if and only if the function φ and its complementary function satisfy the Δ_2 — condition for large u ([3], p. 255).

By $L^{*\varphi}$ we shall denote the space of all 2π — periodic functions of two variables with respect to each variable separately belonging to $L^{*\varphi}(Q)$, where $Q = [-\pi, \pi; -\pi, \pi]$.

Let us introduce in the space $L^{*\varphi}$, the following modulus of smoothness of the function $f(x, y)$,

$$\omega_{r,s}(u, v)_\varphi = \omega_{r,s}(u, v; f)_\varphi = \sup_{|h|\leqslant u, |\delta|\leqslant v} \|\Delta^{r+s}_{h,\delta} f(x, y)\|_\varphi,$$

$$\omega^{(1)}_r(u)_\varphi = \omega^{(1)}_r(u; f)_\varphi = \sup_{|h|\leqslant u} \|\Delta^r_h f(x, y)\|_\varphi,$$

$$\omega^{(2)}_s(v)_\varphi = \omega^{(2)}_s(v; f)_\varphi = \sup_{|\delta|\leqslant v} \|\Delta^s_\delta f(x, y)\|_\varphi,$$

where

$$\Delta^{r+s}_{h,\delta} f(x, y) = \sum_{j=0}^{r} \sum_{k=0}^{s} (-1)^{j+k} \binom{r}{j} \binom{s}{k} f[x+(r-j)h, y+(s-k)\delta],$$

$$\Delta^r_h f(x, y) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} f[x+(r-j)h, y],$$

$$\Delta^s_\delta f(x, y) = \sum_{k=0}^{s} (-1)^k \binom{s}{k} f[x, y+(s-k)\delta].$$

It is easy to prove that, for each $\lambda, u \geqslant 0$,

$$\omega^{(l)}_r(\lambda u)_\varphi \leqslant (\lambda+1)^r \omega^{(l)}_r(u)_\varphi \qquad (l = 1, 2;\ r = 1, 2, \ldots). \tag{2}$$

As in [10], p. 122–123, let $\sigma[f]$ be the double Fourier series of the function $f(x, y)$ and $\bar{\sigma}[f, x]$, $\bar{\sigma}[f, y]$, $\bar{\sigma}[f, x, y]$ its conjugate series. Their partial sums we shall denote by

$$S^{(i)}_{mn}(x, y; f) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_{jk} A^{(i)}_{jk}(x, y) \qquad (i = 1, 2, 3, 4),$$

and the Cesàro (C, α, β) means by $\sigma^{(i,\alpha,\beta)}_{mn}(x, y; f)$, respectively (see [10], p. 159).

Also, the Riesz means

$$S^{(i,r,s)}_{mn}(x, y; f)$$

$$= \sum_{j=0}^{m} \sum_{k=0}^{n} \left(1 - \frac{j^r}{(m+1)^r}\right)\left(1 - \frac{k^s}{(n+1)^s}\right) \lambda_{jk} A^{(i)}_{jk}(x, y) \qquad (i = 1, 2, 3, 4)$$

will be considered further.

Let us introduce the conjugate functions $\bar{f}_i(x, y)$ $(i = 1, 2, 3)$ as in [10], p. 123.

It can easily be observed ([9], p. 471) that if $f(x, y) = S^{(1)}_{mn}(x, y; f)$, then $\bar{f}_i(x, y) = S^{(i+1)}_{mn}(x, y; f)$ $(i = 1, 2, 3)$.

Denote by

$$E_{mn}(f)_\varphi = \inf_{U_{mn}} \|f(x, y) - U_{mn}(x, y)\|_\varphi,$$

the best approximation of the function $f(x, y)$ by trigonometric polynomials $U_{mn}(x, y)$ of degree $\leqslant m$ with respect to x and of degree $\leqslant n$ with respect to y, in the metric $L^{*\varphi}$.

Write $B_\nu(\alpha, \beta, \ldots)$ $(\nu = 1, 2, \ldots)$ for the suitable positive constants depending on the indicated parameters only.

Section 2 contains some auxiliary results. In Section 3 the main theorems are given.

2. Considering an arbitrary Orlicz space $L^{*\varphi}$, we have

LEMMA 1. *For any trigonometric polynomial $T_{mn}(x, y)$ of the order $\leqslant m$ with respect to x and of the order $\leqslant n$ with respect to y and for each non-negative integers r and s*

$$\left\| \frac{\partial^{r+s} T_{mn}(x, y)}{\partial x^r \partial y^s} \right\|_{(\varphi)} \leqslant \left(\frac{m}{2 \sin mh} \right)^r \left(\frac{n}{2 \sin n\delta} \right)^s \| \Delta^{r+s}_{2h, 2\delta} T_{mn}(x, y) \|_{(\varphi)}$$

whenever $0 < h < \pi/m$, $0 < \delta < \pi/n$ (cf. [6], p. 232).

Proof. Consider first a trigonometric polynomial $T_m(x)$ of the order $\leqslant m$ with respect to x. Choose a complex-valued function $\nu(t)$ of bounded variation over $\langle -m, m \rangle$ such that

$$T_m(x) = \int_{-m}^{m} e^{ixt} d\nu(t) \quad \text{for all } x \in (-\infty, \infty).$$

Write

$$\mu_r(t) = \left(\frac{t}{2 \sin ht} \right)^r, \quad \psi_r(t) = (2i \sin ht)^r \quad (|t| \leqslant m, 0 < h < \pi/m),$$

and

$$f(x) = \int_{-m}^{m} \psi_r(t) e^{ixt} d\nu(t), \quad F(x) = \int_{-m}^{m} \mu_r(t) \psi_r(t) e^{ixt} d\nu(t).$$

As it is well known ([6], p. 229–230), for $x \in (-\infty, \infty)$,

$$F(x) = \sum_{k=-\infty}^{\infty} d_k f\left(\frac{k\pi}{m} + x \right), \quad \sum_{k=-\infty}^{\infty} |d_k| = \mu_r(m),$$

$$F(x) = T^{(r)}_m(x), \quad f(x) = \Delta^r_{2h} T_m(x - rh).$$

Hence, by Jensen's inequality,

$$\varphi\left(\frac{|F(x)|}{\mu_r(m)}\right) \leqslant \sum_{k=-\infty}^{\infty} \frac{|d_k|}{\mu_r(m)}\, \varphi\left(\left|f\left(\frac{k\pi}{m}+x\right)\right|\right),$$

i.e.

$$\int_{-\pi}^{\pi} \varphi\left(\frac{|T_m^{(r)}(x)|}{\mu_r(m)}\right) dx \leqslant \int_{-\pi}^{\pi} \varphi\left(|\Delta_{2h}^r T_m(x)|\right) dx. \tag{3}$$

Passing to the trigonometric polynomial $T_{mn}(x, y)$ and using (3), we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi\left(\left|\frac{1}{\mu_r(m)\mu_s(n)}\,\frac{\partial^r}{\partial x^r}\left[\frac{\partial^s T_{mn}(x, y)}{\partial y^s}\right]\right|\right) dx\, dy$$

$$\leqslant \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi\left(\left|\frac{1}{\mu_s(n)}\,\Delta_{2h}^r\left[\frac{\partial^s T_{mn}(x, y)}{\partial y^s}\right]\right|\right) dx\, dy$$

$$\leqslant \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi\left(|\Delta_{2\delta}^s \Delta_{2h}^r T_{mn}(x, y)|\right) dx\, dy.$$

Consequently,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi\left(\left|\frac{1}{\mu_r(m)\mu_s(n)}\cdot\frac{\partial^{r+s} T_{mn}(x, y)}{\partial x^r \partial y^s}\right|\right) dx\, dy$$

$$\leqslant \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi\left(|\Delta_{2h,2\delta}^{r+s} T_{mn}(x, y)|\right) dx\, dy$$

and this implies the desired assertion.

For reflexive Orlicz spaces $L^{*\varphi}$ the following two lemmas hold.

LEMMA 2. *If* $f(x, y) \in L^{*\varphi}$, *then*

$$\|\bar{f}_i(x, y)\|_{(\varphi)} \leqslant B_i(\varphi)\, \|f(x, y)\|_{(\varphi)} \qquad (i = 1, 2, 3).$$

The proof is analogous to that of ([4], p. 1375–1376). The functions $\varphi(u)$ and $M_1(u) = \int_0^u \frac{\varphi(t)}{t} dt$ satisfy the condition Δ_2 ([3], p. 62, 36); therefore there exist constants a, b and $u_0 \geqslant 0$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b \quad \text{for } u \geqslant u_0,$$

where $M_1'(u)$ is the right-hand side derivative of the function $M_1(u)$, ([3], p. 37, 40).

If

$$\varphi_1(u) = \begin{cases} M_1(u_0)\dfrac{u^a}{u_0^a} & \text{for} \quad 0 \leqslant u \leqslant u_0, \\ M_1(u) & \text{for} \quad u \geqslant u_0, \end{cases}$$

then the functions φ_1 and φ are equivalent ([3], p. 36–37) and

$$1 < a < \frac{u\varphi_1'(u)}{\varphi_1(u)} < b \quad \text{for} \quad u \geqslant 0. \tag{4}$$

Choose a pair of positive numbers α, β such that $1 < \alpha < a < b < \beta < \infty$.

The partial integration gives

$$\int_u^\infty \frac{\varphi_1'(t)}{t^\beta}\,dt = \beta \int_u^\infty \frac{\varphi_1(t)}{t^{\beta+1}}dt - \frac{\varphi_1(u)}{u^\beta},$$

$$\int_1^u \frac{\varphi_1'(t)}{t^\alpha}dt = \alpha \int_1^u \frac{\varphi_1(t)}{t^{\alpha+1}}dt + \frac{\varphi_1(u)}{u^\alpha} - \varphi_1(1).$$

Taking into account the above identities and (4), we have

$$\int_u^\infty \frac{\varphi_1(t)}{t^{\beta+1}}\,dt \leqslant \frac{1}{\beta - b} \cdot \frac{\varphi_1(u)}{u^\beta},$$

$$\int_1^u \frac{\varphi_1(t)}{t^{\alpha+1}}\,dt \leqslant \frac{1}{a - \alpha} \cdot \frac{\varphi_1(u)}{u^\alpha}.$$

Thus, the function φ_1 satisfies the conditions of Marcinkiewicz Theorem on the interpolation of operators ([8], p. 175). Moreover (see [5]; [10], p. 124), operators $\bar{f}_i = T_i f$ $(i = 1, 2, 3)$ are of the strong type (a, a), (β, β). Therefore, according to mentioned above interpolation theorem, there exist positive constants $C_i = C_i(\varphi_1)$ such that

$$\iint_Q \varphi_1\big(|\bar{f}_i(x, y)|\big)\,dx\,dy \leqslant C_i \iint_Q \varphi_1\big(|f(x, y)|\big)\,dx\,dy + C_i \quad (i = 1, 2, 3),$$

for all $f(x, y) \in L^{*\varphi_1}$ ([8], p. 175, see also [1], Theorems 32, 33). Since φ and φ_1 are equivalent there exist constants $K_i = K_i(\varphi) > 1$ such that

$$\iint_Q \varphi\big(|\bar{f}_i(x,y)|\big)\,dx\,dy \leqslant K_i \iint_Q \varphi\big(|f(x, y)|\big)\,dx\,dy + K_i \quad (i = 1, 2, 3).$$

By convexity of φ and the last inequality,

$$\iint_Q \varphi\left(\frac{|\bar{f}_i(x,y)|}{(1+K_i)K_i\|f(x,y)\|_{(\varphi)}}\right)dx\,dy \leqslant \frac{1}{1+K_i}\iint_Q \varphi\left(\frac{|\bar{f}_i(x,y)|}{K_i\|f(x,y)\|_{(\varphi)}}\right)dx\,dy$$

$$\leqslant \frac{1}{1+K_i}\left\{K_i\iint_Q \varphi\left(\frac{|f(x,y)|}{K_i\|f(x,y)\|_{(\varphi)}}\right)dx\,dy + K_i\right\}$$

$$\leqslant \frac{1}{1+K_i}\left\{\iint_Q \varphi\left(\frac{|f(x,y)|}{\|f(x,y)\|_{(\varphi)}}\right)dx\,dy + K_i\right\} \leqslant 1.$$

Hence

$$\|\bar{f}_i(x,y)\|_{(\varphi)} \leqslant (1+K_i)K_i\|f(x,y)\|_{(\varphi)} = B_i(\varphi)\,\|f(x,y)\|_{(\varphi)},$$

and the proof is completed.

LEMMA 3. *If* $f(x,y) \in L^{*\varphi}$, *then*

$$\|f(x,y) - S^{(1)}_{mn}(x,y;f)\|_\varphi \leqslant B_4(\varphi)E_{mn}(f)_\varphi.$$

The proof runs as in the case of functions of one variable (see [2] and [7]).

3. Given an arbitrary Orlicz space $L^{*\varphi}$, we have

THEOREM 1. *If* $f \in L^{*\varphi}$, *then, for* $m, n \geqslant 1$,

$$E_{mn}(f)_\varphi \leqslant B_5(r,s)\left\{\omega^{(1)}_r\left(\frac{1}{m}\right)_\varphi + \omega^{(2)}_s\left(\frac{1}{n}\right)_\varphi\right\},$$

where $\omega^{(1)}_r(u)\varphi$, $\omega^{(2)}_s(v)\varphi$ *are the partial modulus of smoothness of* f (cf. [6], p. 288).

Proof. Consider the trigonometric polynomial

$$J^{r,s}_{mn}(x,y) = J^{r,s}_{mn}(x,y;f)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi^r_m(t_1)\Phi^s_n(t_2)\sum_{j=1}^{r}\sum_{k=1}^{s}(-1)^{j+k}\binom{r}{j}\binom{s}{k} f(x+jt_1, y+kt_2)\,dt_1\,dt_2,$$

where

$$\Phi^r_m(t) = \frac{1}{\pi\gamma^r_m}\cdot\left(\frac{\sin\frac{mt}{2}}{m\sin\frac{t}{2}}\right)^{2r+4}, \qquad \int_{-\pi}^{\pi}\Phi^r_m(t)\,dt = 1.$$

Then,

$$J_{mn}^{r,s}(x,y)-f(x,y)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\sum_{j=0}^{r}\sum_{k=0}^{s}(-1)^{j+k}\binom{r}{j}\binom{s}{k} f(x+jt_1, y+kt_2)\,dt_1\,dt_2 -$$

$$- \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\sum_{j=0}^{r}(-1)^{j}\binom{r}{j} f(x+jt_1, y)\,dt_1\,dt_2 +$$

$$+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\sum_{k=0}^{s}(-1)^{k-1}\binom{s}{k} f(x, y+kt_2)\,dt_1\,dt_2 = \sum_{i=1}^{3} I_i(x,y).$$

If $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi^*(|g(x,y)|)\,dx\,dy \leqslant 1$, the Fubini's theorem and inequality (2) yield

$$\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} I_1(x,y)g(x,y)\,dx\,dy\right| \leqslant \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\times$$

$$\times\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{r}\sum_{k=0}^{s}(-1)^{j+k}\binom{r}{j}\binom{s}{k} f(x+jt_1, y+kt_2)g(x,y)\,dx\,dy\right| dt_1\,dt_2$$

$$\leqslant 2^r \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{s}(-1)^{k}\binom{s}{k} f(x, y+kt_2)g(x,y)\,dx\,dy\right| dt_1 dt_2$$

$$\leqslant 2^{r+1}\int_{0}^{\pi} \Phi_n^s(t)\,\omega_s^{(2)}(t)_\varphi\,dt \leqslant 2^{r+1}\,\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi \int_{0}^{\pi} \Phi_n^s(t)\,n^s\left(t+\frac{1}{n}\right)^s dt$$

$$\leqslant 2^{r+s+1}\,\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi \left\{\int_{0}^{1/n} \Phi_n^s(t)\,dt + n^s \int_{1/n}^{\pi} \Phi_n^s(t)\,t^s\,dt\right\} \leqslant B_6(r,s)\,\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi.$$

Analogously,

$$\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} I_2(x,y)g(x,y)\,dx\,dy\right|$$

$$\leqslant \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Phi_m^r(t_1)\Phi_n^s(t_2)\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{r}(-1)^{j}\binom{r}{j} f(x+jt_1, y)g(x,y)\,dx\,dy\right| dt_1\,dt_2$$

$$\leqslant 2\int_{0}^{\pi} \Phi_m^r(t)\,\omega_r^{(1)}(t)_\varphi\,dt \leqslant B_7(r)\,\omega_r^{(1)}\left(\frac{1}{m}\right)_\varphi$$

and

$$\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} I_3(x,y)g(x,y)\,dx\,dy\right| \leqslant B_7(s)\,\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi.$$

Hence

$$\|J_{mn}^{r,s}(x,y)-f(x,y)\|_\varphi \leqslant \sum_{i=1}^{3}\|I_i\|_\varphi \leqslant B_5(r,s)\left\{\omega_r^{(1)}\left(\frac{1}{m}\right)_\varphi+\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.$$

In the next theorems, $L^{*\varphi}$ is a reflexive Orlicz space.

THEOREM 2. *If $f(x,y)\in L^{*\varphi}$ and r, s are positive odd integers, then, for each $m, n \geqslant 1$,*

$$\|S_{mn}^{(1,r,s)}(x,y;f)-f(x,y)\|_\varphi \leqslant B_8(r,s,\varphi)\left\{\omega_r^{(1)}\left(\frac{1}{m}\right)_\varphi+\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.$$

Proof. It can easily be verified,

$$\begin{aligned}
S_{mn}^{(1,r,s)}(x,y;f)&-S_{mn}^{(1)}(x,y;f)\\
&=-\frac{1}{(m+1)^r}\sum_{j=1}^{m}\sum_{k=0}^{n}j^r\lambda_{jk}A_{jk}^{(1)}(x,y)-\\
&\qquad-\frac{1}{(n+1)^s}\sum_{j=0}^{m}\sum_{k=1}^{n}k^s\lambda_{jk}A_{jk}^{(1)}(x,y)+\\
&\qquad+\frac{1}{(m+1)^r(n+1)^s}\sum_{j=1}^{m}\sum_{k=1}^{n}j^rk^sA_{jk}^{(1)}(x,y)\\
&=-\frac{1}{(m+1)^r}\frac{\partial^r S_{mn}^{(2)}(x,y;f)}{\partial x^r}-\frac{1}{(n+1)^s}\frac{\partial^s S_{mn}^{(3)}(x,y;f)}{\partial y^s}+\\
&\qquad+\frac{1}{(m+1)^r(n+1)^s}\frac{\partial^{r+s}S_{mn}^{(4)}(x,y;f)}{\partial x^r\partial y^s}.
\end{aligned}$$

Therefore, in view of inequalities (1), Lemmas 1–3 and Theorem 1

$$\begin{aligned}
&\|S_{mn}^{(1,r,s)}(x,y;f)-S_{mn}^{(1)}(x,y;f)\|_\varphi\\
&\leqslant B_9(r,s)\{\|\Delta_{\pi/m}^r S_{mn}^{(2)}(x,y;f)\|_\varphi+\|\Delta_{\pi/n}^s S_{mn}^{(3)}(x,y;f)\|_\varphi+\|\Delta_{\pi/m\,\pi/n}^{r+s}S_{mn}^{(4)}(x,y;f)\|_\varphi\}\\
&\leqslant B_{10}(r,s,\varphi)\{\|\Delta_{\pi/m}^r S_{mn}^{(1)}(x,y;f)\|_\varphi+\|\Delta_{\pi/n}^s S_{mn}^{(1)}(x,y;f)\|_\varphi+\|\Delta_{\pi/m\,\pi/n}^{r+s}S_{mn}^{(1)}(x,y;f)\|_\varphi\}\\
&\leqslant B_{11}(r,s,\varphi)\left\{\omega_r^{(1)}\left(\frac{1}{m}\right)_\varphi+\omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.
\end{aligned}$$

By Lemma 3 and Theorem 1 we get the similar estimate for $\|S_{mn}^{(1)}(x,y;f)-f(x,y)\|_\varphi$. Hence the proof is completed.

In particular, if $r=s=1$

$$\|\sigma_{mn}^{(1,1,1)}(x,y;f)-f(x,y)\|_\varphi \leqslant B_{12}(\varphi)\left\{\omega_1^{(1)}\left(\frac{1}{m}\right)_\varphi+\omega_1^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.$$

Further, an argument similar to that of [10], p. 184 leads to

THEOREM 3. *If* $f(x, y) \in L^{*\varphi}$ *and* $\alpha, \beta \geqslant 1$, *then, for each positive integers* m, n,

$$\|\sigma_{mn}^{(1,\alpha,\beta)}(x, y; f) - f(x, y)\|_\varphi \leqslant B_{13}(\alpha, \beta, \varphi) \left\{\omega_1^{(1)}\left(\frac{1}{m}\right)_\varphi + \omega_1^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.$$

Inequalities (1), Lemma 2 and Theorem 2 give

THEOREM 4. *If* $f(x, y) \in L^{*\varphi}$ *and* r, s *positive odd integers, then, for* $i = 2, 3, 4$,

$$\|S_{mn}^{(i,r,s)}(x, y; f) - \bar{f}_{i-1}(x, y)\|_\varphi \leqslant B_{14}(r, s, \varphi) \left\{\omega_r^{(1)}\left(\frac{1}{m}\right)_\varphi + \omega_s^{(2)}\left(\frac{1}{n}\right)_\varphi\right\}.$$

Indeed,

$$\|S_{mn}^{(i,r,s)}(x, y; f) - \bar{f}_{i-1}(x, y)\|_\varphi \leqslant 2B_{i-1}(\varphi) \| S_{mn}^{(1,r,s)}(x, y; f) - f(x, y)\|_\varphi$$

and the result follows.

From this we easily get an analogue of Theorem 35 (part 2) of [10], p. 185 for $\alpha, \beta \geqslant 1$.

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