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Additive functionals on abelian semigroups*

1. Introduction. In this paper we investigate the problem of separation of the subadditive and superadditive functionals on a semigroup. It is proved in Theorem 1 that an additive functional separating these two types of functionals always exists. This theorem is, what we prove, an equivalent form of the generalization, due to R. Kaufman, of the well-known Mazur–Orlicz Theorem ([5]).

Some steps are made also toward determining the conditions for extension of additive functionals dominated on semigroups by subadditive functionals. This what is done is far from the final solution, however, as similar problems have recently been considered in the literature we feel justified publishing it.

Throughout the paper we deal with semigroups which are commutative. Thus, from now on "semigroup" will mean "abelian semigroup". We reserve the symbol S for the semigroup, composition in S denoted $s+t$.

2. Separation Theorem. With these remarks we proceed to proving Theorem 1 which is the main result of this paper. The proof is analogous to that of Kaufman's theorem [3]. In fact it is, as we shall prove it later, another form of this theorem.

Recently Asimow and Ellis ([1], Theorem 2) announced a similar result, but the hypotheses they assumed were much stronger than it is done here. In some sense, therefore, the following theorem can be regarded as an improvement of their theorem.

THEOREM 1. *Let on a semigroup S be defined two real functionals ω and L: $S \to [-\infty, \infty]$ at least one of which is finite, and such that*

(i) $\omega(s_1+s_2) \leqslant \omega(s_1)+\omega(s_2)$ *for* $s_1, s_2 \in S$,

(ii) $L(s_1+s_2) \geqslant L(s_1)+L(s_2)$ *for* $s_1, s_2 \in S$,

(iii) $L(s) \leqslant \omega(s)$, $s \in S$.

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Then there exists an additive functional ξ on S such that

$$L(s) \leqslant \xi(s) \leqslant \omega(s) \quad \text{for all } s \text{ in } S.$$

Proof. Suppose that ω is finite. In case L is finite the proof is analogous. If $L \equiv -\infty$, then L would be the additive functional. We may then assume that $L \not\equiv -\infty$. Again, if $L = \omega$ on S, then both L and ω would already be additive. Let us exclude this and assume that there exists an element a_0 and a number r such that

$$L(a_0) < r < \omega(a_0).$$

It is an easy matter to show that at least one of the two following inequalities:

$$\omega(ma_0+s) \geqslant mr+L(s), \quad m \geqslant 1,\ s \in S, \tag{1}$$

$$nr+\omega(s) \geqslant L(na_0+t), \quad n \geqslant 1,\ t \in S, \tag{2}$$

must always hold. We omit the proof since it has been explicitly given in [3].

Let us suppose that (1) holds. Then we can construct a functional L' such that $L' \geqslant L$, $L'(a_0) \geqslant r$ and satisfying conditions (ii) and (iii). In what follows we adopt the convention $0 \cdot s+t \equiv t$. The functional L' is defined by the formula

$$L'(t) = \sup\{mr+L(s)\}, \tag{3}$$

where supremum is taken over all representations

$$t = s+ma_0, \quad m \geqslant 0,\ s, t \in S. \tag{4}$$

If there is no s in (4), we replace $L(s)$ in (3) by 0. Obviously $L'(s) \geqslant L(s)$. In fact, if $s \neq a_0$, then $n = 0$ yields this inequality, if $s = a_0$, then according to our convention $a_0 = 1 \cdot a_0+0 \cdot s = t$ and it follows that $L'(a_0) \geqslant r > L(a_0)$. It remains to show that L' satisfies also (ii) and (iii). First we prove the superadditivity of L'. Let $t_1, t_2 \in S$ and

$$\begin{aligned} t_1 &= na_0+s_1, \\ t_2 &= ma_0+s_2, \end{aligned} \qquad m, n \geqslant 0,\ s_1, s_2 \in S.$$

Then $t_1+t_2 = (s_1+s_2)+(m+n)a_0$, thus

$$L(s_1)+nr+L(s_2)+mr \leqslant L(s_1+s_2)+(m+n)r \leqslant L'(t_1+t_2)$$

and finally

$$L'(t_1)+L'(t_2) \leqslant L'(t_1+t_2).$$

Now, let $t = na_0+s$, $n \geqslant 0$. In view of (1) we have

$$L(s)+nr \leqslant \omega(s+na_0) = \omega(t),$$

what implies that

$$L'(t) \leqslant \omega(t) \quad \text{for all } t \in S.$$

From the above argumentation it follows that if (1) holds, then L is not the maximal element in the family $\{L'\}$ of all superadditive functionals satisfying $L' \leqslant \omega$.

Now, if (2) holds, then by similar procedure we show that the functional

$$\omega'(t) = \inf\{\omega(s)+nr\},$$

where infimum is taken again over all representations (4), is subadditive, satisfies (i) and (iii), and also $\omega'(a_0) < \omega(a_0)$. It leads us to conclusion that if inequality (2) holds, then ω is not a minimal element in the family $\{\omega'\}$ of all subadditive functionals such that $\omega' \geqslant L$.

But there exist both minimal element ω_0 in the family $\{\omega'\}$ and maximal element L_0 in $\{L'\}$. These functionals are equal on S, and so $\xi = \omega_0 = L_0$ is the required additive functional. We shall prove this.

Let Ω denote the class of all superadditive functionals L' on S such that $L \leqslant L' \leqslant \omega$. Let, further, $\{L'_a\colon a \in A\}$ be any chain in $(\Omega, \leqslant)$. Define

$$\mathfrak{L}(s) = \sup_{a \in A} L'_a(s) \qquad s \in S.$$

Obviously, for every $a \in A$, $L'_a \leqslant \mathfrak{L} \leqslant \omega$. To show that $\mathfrak{L} \in \Omega$ it suffices to demonstrate that $\mathfrak{L}$ is superadditive on S.

Let $s_1, s_2 \in S$. Then, since the set $\{L'_a\colon a \in A\}$ is linearly ordered with respect to a, for any $a_1, a_2 \in A$ we can find an $a \in A$ such that $L'_{a_1} \leqslant L_a$ and $L'_{a_2} \leqslant L_a$. Thus

$$L'_{a_1}(s_1)+L'_{a_2}(s_2) \leqslant L'_a(s_1)+L'_a(s_2) \leqslant L'_a(s_1+s_2),$$

from what we see that

$$\mathfrak{L}(s_1+s_2) \geqslant \mathfrak{L}(s_1)+\mathfrak{L}(s_2).$$

Now, the Zorn–Kuratowski Lemma implies that there exists the maximal element L_0 in Ω.

Analogously, in the class $\mathfrak{M}$ of all subadditive functionals ω' on S such that $L_0 \leqslant \omega' \leqslant \omega$ there exists the minimal element ω_0. From what has already been said it is evident that $L_0 = \omega_0$.

It is easy to prove that if S is a linear space and ω (or L) is positively homogeneous (i.e. $\omega(\lambda x) = \lambda\omega(x)$, $\lambda \geqslant 0$), then ξ is homogeneous. Indeed, it has been proved by Kaufman.

We have mentioned that Theorem 1 is an equivalent form of Kaufman's theorem. We prove this equivalence here.

First, we formulate the Kaufman's theorem.

Let S be a semigroup with a real functional ω subject to two conditions:

(i') $\infty > \omega(s) \geqslant -\infty$ for $s \in S$,

(ii') $\omega(s)+\omega(t) \geqslant \omega(s+t)$ for $s, t \in S$.

In addition to ω there is given a real functional L on S, restricted as follows:

(iii') $\infty > L(s) \geqslant -\infty, s \in S, L \not\equiv -\infty$,

(iv') If $s_1, \ldots, s_n$ is a finite sequence in S

$$\omega(s_1+\ldots+s_n) \geqslant \sum_{i=1}^{n} L(s_1).$$

There exists an additive functional ξ on such that $\omega \geqslant \xi \geqslant L$.

COROLLARY 1. *The Kaufman's Theorem and Theorem* 1 *are equivalent.*

Proof. Since the fact that Kaufman's Theorem implies Theorem 1 is obvious we prove only the other implication. Let the functionals L and ω satisfy (iv'). We define another functional L_1 on S:

$$L_1(s) = \sup\left\{\sum_{i=1}^{m} L(s_i) : s = s_1+\ldots+s_n,\ s_1, \ldots, s_n \in S\right\}.$$

It is evident that $L(s) \leqslant L_1(s) \leqslant \omega(s)$ for every s in S. We shall show that L_1 is superadditive on S. Indeed, let $s_1, \ldots, s_m \in S$ and let $s_i = s_1^i+ +\ldots+s_n^i$ be any decomposition of the element s_i $(i = 1, \ldots, m)$. Then

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} L(s_j^i)\right) \leqslant L_1(s_1+\ldots+s_m),$$

and hence

$$\sum_{i=1}^{m} L_1(s_i) \leqslant L_1\left(\sum_{i=1}^{m} s_i\right).$$

It is clear that $L_1 \leqslant \omega$. From Theorem 1 we deduce now that there exists an additive functional ξ such that $L_1 \leqslant \xi \leqslant \omega$.

3. Extension Theorem. Now we should like to determine the conditions for extension of an additive functional on a semigroup. That is we consider the following problem. Let S_0 be a subsemigroup of a semigroup S, ξ_0 be an additive functional on S_0 and ω — subadditive functional on S. Let further $\xi_0 \leqslant \omega$ on S_0. We ask what conditions must be satisfied by S_0, ω and ξ_0 in order that there existed an additive functional ξ on S such that $\omega \geqslant \xi$ on S and $\xi = \xi_0$ on S_0 (denoted by $\xi/S_0 = \xi_0$). This problem was dealt by R. Kaufman in his another paper [4]. He, however, considered multiplicative functionals. Of course, our present discussion falls somewhat

short of achieving this goal and will, in fact be restricted to some rather general remarks regarding extension of additive functionals. Before continuing, let us consider the first question one might ask: can every additive functional on a subsemigroup of a certain semigroup be extended to the whole semigroup with the same subadditive majorant? The answer is, of course, no and as examples of such situations abound we feel we do not need to present it here.

THEOREM 2. *Let on S be given a finite subadditive functional ω and on S_0 an additive ξ_0 such that $\xi_0(s) \leqslant \omega(s)$ for all s in S_0. Each of the two following conditions assures existence of an additive functional ξ on S such that $\xi \leqslant \omega$ and $\xi|_{S_0} = \xi_0$:*

(i) $s_1 \in S_0,\ s_1 + s_2 \in S_0 \Rightarrow s_2 \in S_0$,

(ii) $s_1 \in S_0,\ s_2 \notin S_0 \Rightarrow \omega(s_1 + s_2) \leqslant \xi_0(s_1) + \omega(s_2)$.

Proof. We shall only sketch the proof, since it inherits much from the proof of Theorem 1. First we note that if $\xi_0 \equiv -\infty$, then defining $\xi(s) = -\infty$ $(s \in S \setminus S_0)$ we have the required additive functional on S. So let us assume that $\xi_0 \not\equiv -\infty$. We define now another functional $\bar{\xi}$ on S:

$$\bar{\xi}(s) = \begin{cases} \xi_0(s) & s \in S_0, \\ -\infty & s \in S \setminus S_0. \end{cases}$$

Clearly, $\bar{\xi}$ is superadditive on S and $\bar{\xi} \leqslant \omega$. From Theorem 1 it follows that there exists an additive ξ^* satisfying $\bar{\xi} \leqslant \xi^* \leqslant \omega$. Besides $\xi^*|_{S_0} = \xi_0^* \geqslant \xi_0$. If it were $\xi_0^* = \xi_0$, then ξ^* would already be the extension of ξ_0. So we may suppose, similarly as in the proof of Theorem 1, that there exist a point a_0 in S_0 and a number r such that $\xi^*(a_0) > r > \xi_0(a_0)$. We shall show that there is another additive functional ξ' on S such that

$$\xi' \leqslant \xi^*, \qquad \xi' \neq \xi^*, \qquad \xi' \geqslant \bar{\xi}.$$

It is easy to verify that

(5) $\qquad \xi^*(s) + mr \geqslant \bar{\xi}(ma_0 + s)$ holds for any $m \geqslant 0$, $s \in S$.

Indeed, we know that

$$mr + \xi^*(s) \geqslant m\bar{\xi}(a_0) + \bar{\xi}(s).$$

If $ma_0 + s \in S_0$, then, by (i) $s \in S_0$, and $\bar{\xi}(ma_0 + s) = m\bar{\xi}(a_0) + \bar{\xi}(s)$ hence (5) is satisfied, if $ma_0 + s \notin S_0$, then $\bar{\xi}(ma_0 + s) = -\infty$ and (5) is again satisfied.

Now, the functional

$$\varphi(t) = \inf\{\xi^*(s) + mr\colon\ t = ma_0 + s,\ m \geqslant 0,\ s \in S\}$$

is easily seen to be subadditive on S and such that $\varphi(a_0) \leqslant r < \xi^*(a_0)$, $\bar{\xi} \leqslant \varphi < \xi^*$. In view of Theorem 1 there exists an additive functional ξ' having required properties, what by application of Zorn–Kuratowski Lemma establishes existence of a minimal element ξ in the class

$$\Omega = \{\xi' \colon \xi' \text{ is additive on } S,\ \omega \geqslant \xi' \geqslant \bar{\xi}\}.$$

It is obvious that $\xi|_{S_0} = \xi_0$, that is, condition (i) implies existence of the additive extension of ξ_0.

The proof in case (ii) is even simplier. It suffices to verify that

$$\lambda(s) = \begin{cases} \xi_0(s), & s \in S_0, \\ \omega(s), & s \in S \setminus S_0 \end{cases}$$

is then subadditive on S. In fact, let $s = s_1 + s_2$ and

(a) $s_1, s_2 \in S_0$, then $s_1 + s_2 \in S_0$ and we have

$$\lambda(s) = \xi_0(s) = \xi_0(s_1) + \xi_0(s_2) = \lambda(s_1) + \lambda(s_2),$$

(b) $s_1 \in S_0$, $s_2 \notin S_0$, $s_1 + s_2 \in S_0$, then by (ii)

$$\lambda(s_1) + \lambda(s_2) = \xi_0(s_1) + \omega(s_2) \geqslant \omega(s_1 + s_2) \geqslant \xi_0(s_1 + s_2) = \lambda(s),$$

(c) $s_1, s_2 \notin S_0$, $s_1 + s_2 \in S_0$,

$$\lambda(s) = \xi_0(s) \leqslant \omega(s) \leqslant \omega(s_1) + \omega(s_2) = \lambda(s_1) + \lambda(s_2),$$

(d) $s_1, s_2, s_1 + s_2 \notin S_0$,

$$\lambda(s) = \omega(s) \leqslant \omega(s_1) + \omega(s_2) = \lambda(s_1) + \lambda(s_2),$$

(e) $s_1 \in S_0$, $s_2 \notin S_0$, $s_1 + s_2 \notin S_0$,

$$\lambda(s) = \omega(s) \leqslant \xi_0(s_1) + \omega(s_2) = \lambda(s_1) + \lambda(s_2).$$

Now, between $\bar{\xi}$ and λ there lies an additive functional ξ which is, obviously, the extension of ξ_0. The proof is completed.

Every linear space satisfies condition (i) of the theorem. Therefore it is an immediate generalization of the famous Hahn–Banach Theorem, it has also obvious connections with the Theorem of Bonsall ([2], Theorem 1).

COROLLARY 2. *Let $S_0 \subset S$ be a subsemigroup satisfying condition: $s_1 \in S_0$, $s_2 \notin S_0 \Rightarrow s_1 + s_2 \in S_0$. If ξ_0 is an additive functional on S_0, ω — subadditive finite on S, both related as in Theorem 2, then ξ_0 has an additive extension $\xi \leqslant \omega$ to the whole S iff*

$$\xi_0(s_1 + s_2) \leqslant \xi_0(s_1) + \omega(s_2).$$

Proof. Necessity is trivial, and sufficiency follows from (b) in the proof of Theorem 2.

EXAMPLE. Let $S = R$ (the set of real numbers), $S_0 = R^+ = \{x \in R: x \geqslant 0\}$, ω and ξ_0 let be in addition positively homogeneous. Then ξ_0 can be extended to R (with ω as a majorant) iff for $x_1 \geqslant 0$, $x_2 < 0$ such that $x_1 + x_2 < 0$

$$\omega(x_1 + x_2) \leqslant \xi_0(x_1) + \omega(x_2). \tag{6}$$

To prove sufficiency we must show that (6) implies $\xi_0(x_1) + \omega(x_2) \geqslant \xi_0(y_1 + y_2)$ whenever $y_1 \geqslant 0$, $y_2 < 0$ and $y_1 + y_2 \geqslant 0$. In fact, $\xi_0(y_1 + y_2) = \xi_0(y_1) - \xi_0(-y_2)$. But in view of (6) $\omega(-y_2 - y_1) \leqslant \xi_0(-y_2) + \omega(-y_1)$ so that $-\xi_0(-y_2) \leqslant \omega(-y_1) - \omega(-y_2 - y_1) \leqslant \omega(y_2)$. Thus $\xi_0(y_1 + y_2) \leqslant \xi_0(y_1) + \omega(y_2)$, and therefore, from the proof of (ii), Theorem 2 we see that the extension ξ exists.

Necessity is easily established by the following argumentation:

$$\xi(x_1) + \omega(x_2) = -\xi(-x_1) + \omega(x_2) \geqslant -\omega(-x_1) + \omega(x_2).$$

But $-x_1 = \lambda x_2$ $(\lambda \geqslant 0)$, that is $\omega(-x_1) = \lambda\omega(x_2)$ and further $\xi(x_1) + \omega(x_2) \geqslant (1 - \lambda)\omega(x_2)$. Similarly $x_1 + x_2 = \mu x_2$ $(\mu \geqslant 0)$ and $\omega(x_1 + x_2) = \mu\omega(x_2)$. Now, $\lambda + \mu = (-x_1 + x_1 + x_2)/x_2 = 1$, hence (6) follows.

Added in proof. Recently H. Nakano proved a theorem [*On the Hahn–Banach Theorem*, Bull. Acad. Polon. Sci. 19 (1971), p. 743-745] concerning extension of additive functionals in vector spaces, which is a direct consequence of our Theorem 1. Nakano's Theorem states that if ξ_0 is a linear functional on a vector subspace of a vector space X such that ξ_0 is dominated by an additive functional ξ satisfying condition:

(*) $\xi(\lambda_n x) \to 0$

whenever $\{\lambda_n\}$ is a sequence of scalars converging to zero,

then ξ_0 admits a linear extension to the whole space. Indeed, if ξ_0 has the above properties, then ξ_0 completed by $-\infty$ in the points of X not belonging to a subspace, is a superadditive functional on the whole space. The additive functional whose existence follows now from Theorem 1 is, clearly, an extension of ξ_0. Condition (*) assures its homo-geneity.

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