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On quasi-embedability of dendroids

Given metric spaces X, Y and a number $\varepsilon > 0$, we say that a continuous function $f\colon X \to Y$ is an *ε-mapping* provided $\operatorname{diam} f^{-1}(y) < \varepsilon$ for $y \in Y$. Thus each one-to-one mapping is an ε-mapping for each $\varepsilon > 0$. We say that X is *quasi-embedable* in Y provided there exists, for each $\varepsilon > 0$, an ε-mapping of X into Y. The space X is said to be *embedable* in Y provided there exists an embedding of X into Y, i.e. a homeomorphism mapping X onto a subset of Y. Clearly, the embedability implies the quasi-embedability. It is not difficult to see that the quasi-embedability does not imply the embedability even for finite dendrites (i.e. 1-dimensional acyclic connected polyhedra). In fact, let us call an *n-star* any metric space representable as the union of n arcs such that each two of them have exactly one point in common (the *center*) being an end point of all. Let X be a 4-star and let Y be the union of two 3-stars whose common part is a point being an end point of both; then X is quasi-embedable in Y, but X is not embedable in Y.

Let D_u denote the universal dendrite, i.e. a dendrite in which all dendrites are embedable (see [4], p. 318). It has been proved by McCord [3] that there exists a universal curve in the class of curves quasi-embedable in D_u. On the other hand, it follows from a theorem of Rogers [5] that if D is a finite dendrite and D is not an arc, then there exists no universal curve in the class of curves quasi-embedable in D. This contrast will be augmented by the result of the present paper, dealing with arcwise connected curves. Observe that each curve X quasi-embedable in a dendrite is acyclic, i.e. all continuous mappings of X into a simple closed curve are homotopic to a constant mapping. Acyclic arcwise connected curves are called *dendroids*, and it has been proved by Cook [2] that each dendroid is quasi-embedable in D_u. We are going to show that each dendroid quasi-embedable in a finite dendrite is a finite dendrite itself. Actually, a stronger result will be obtained (see the theorem below).

LEMMA 1. *If X is a 3-star, then there exists a number $\varepsilon > 0$ such that, for each ε-mapping f of X, the set $f(X)$ contains a 3-star.*

Proof. Let p_0 denote the center of X and let $X = p_0p_1 \cup p_0p_2 \cup p_0p_3$ be the decomposition of X into arcs such that $p_0p_i \cap p_0p_j = \{p_0\}$ for $i \neq j$. Put

$$\varepsilon = \operatorname{Min}\{\operatorname{dist}(p_i, p_0p_j)\colon\ i \neq j,\ i, j = 1, 2, 3\}$$

and suppose f is an ε-mapping of X. Let $q_i = f(p_i)$ for $i = 0, 1, 2, 3$ and let q_0q_i be an arc contained in $f(p_0p_i)$ for $i = 1, 2, 3$. Then q_i does not belong to q_0q_j for $i \neq j$. Suppose on the contrary that $f(X)$ contains no 3-star. Thus $f(X)$ is either an arc or a simple closed curve (see [4], p. 267). Consequently, the union $q_0q_1 \cup q_0q_2$ is an arc q_1q_2 and since q_3 does not belong to q_1q_2, we conclude that either q_1 or q_2 belongs to q_0q_3, a contradiction.

Given a metric space X, let us denote by $c(X)$ the cardinality of the set of all points of X which are centers of 3-stars contained in X.

LEMMA 2. *If a metric space X is quasi-embedable in a metric space Y such that $c(Y)$ is finite, then $c(X) \leqslant c(Y)$.*

Proof. Let m be a positive integer such that X contains m points wich are centers of 3-stars. Then there exist pairwise disjoint 3-stars $X_1, \ldots, X_m$ contained in X. Put

$$\varepsilon = \operatorname{Min}\{\operatorname{dist}(X_i, X_j)\colon\ i \neq j,\ i, j = 1, \ldots, m\}$$

and take an ε-mapping f of X into Y. The sets $f(X_1), \ldots, f(X_m)$ are pairwise disjoint and each of them contains a 3-star, by Lemma 1. Hence $m \leqslant c(Y)$.

THEOREM. *If a dendroid X is quasi-embedable in a metric space Y such that $c(Y)$ is finite and there exists an integer n such that Y contains no n-star, then X is a finite dendrite.*

Proof. We have $n \geqslant 3$ and we can assume that X is not an arc. Then it follows from Lemma 2 that $c(X) \leqslant c(Y)$ and $c(X)$ is a positive integer. Let a be the center of a 3-star contained in X. Since $c(X)$ is finite, there exists an arc $aa' \subset X$ such that no interior point of aa' is the center of a 3-star. We can assume that aa' is the maximum arc possessing the latter property since, in dendroids, the closure of each one-to-one continuous image of the real line is an arc (see [1], p. 18). Let $\boldsymbol{A}$ be the collection of all maximum arcs aa' obtained in such a way. Thus X is the union of arcs belonging to $\boldsymbol{A}$ and if $A \in \boldsymbol{A}$, then at least one end point of A is the center of a 3-star contained in X. Moreover, if $A, A' \in \boldsymbol{A}$ and $A \neq A'$, then $A \cap A'$ is either empty or contains only one point being an end point of both A and A'. It remains to prove that $\boldsymbol{A}$ is finite.

Suppose on the contrary that $\boldsymbol{A}$ is infinite. Since $c(X)$ is finite, there exist a point $a_0 \in X$ and arcs $a_0a_i \in \boldsymbol{A}$ such that $a_0a_i \cap a_0a_j = \{a_0\}$ for $i \neq j$

and $i, j = 1, 2, \ldots$ Let us consider the integer $m = c(Y)(n-2)+n$. Clearly, we have $m \geqslant 4$. Put

$$\varepsilon = \operatorname{Min}\{\operatorname{dist}(a_i, a_0 a_j)\colon\ i \neq j,\ i, j = 1, \ldots, m\}$$

and take an ε-mapping f of X into Y. Let $b_i = f(a_i)$ for $i = 0, \ldots, m$ and let $b_0 b_i$ be an arc contained in $f(a_0 a_i)$ for $i = 1, \ldots, m$. Then b_i does not belong to the union B_i of all the arcs $b_0 b_j$, where, $i \neq j$ and $j = 1, \ldots, m$. The end b_0 of the arc $b_0 b_i$ belongs to B_i for $i = 1, \ldots, m$. Let b_i' be the first point of $b_0 b_i$ which belongs to B_i when going from b_i to b_0. Thus $b_i \neq b_i'$ for $i = 1, \ldots, m$. We can assume that the points $b_1', \ldots, b_m'$ are indexed in such a manner that $b_0 \neq b_i'$ for $i = 1, \ldots, m_0$ and $b_0 = b_i'$ for $i = m_0 + 1, \ldots, m$ provided $m_0 < m$. If $m_0 < m$, then

$$b_0 b_{m_0+1} \cup \ldots \cup b_0 b_m \subset Y$$

is an $(m - m_0)$-star, whence $m - m_0 < n$ and $m_0 > m - n = c(Y)(n-2) \geqslant 1$. If $m_0 = m$, then $m_0 > c(Y)(n-2)$ too. Any case m_0 is an integer satisfying the inequalities

(i) $$1 \leqslant c(Y)(n-2) < m_0 \leqslant m.$$

For $i = 1, \ldots, m_0$, the subarc $b_i' b_i$ of the arc $b_0 b_i$ has exactly one point in common with the set B_i. This is the point b_i' and there exists an index $j = 1, \ldots, m$ such that $i \neq j$ and b_i' belongs to $b_0 b_j$. Thus $b_0 \neq b_i' \neq b_j$ and the arc $b_i' b_i$ has only the point b_i' in common with each of the arcs $b_i' b_0$, $b_i' b_j$ being subarcs of the arc $b_0 b_j$. Hence

$$b_i' b_i \cup b_i' b_0 \cup b_i' b_j \subset Y$$

is a 3-star whose center is b_i'. We conclude that the set consisting of the points b_i' (where $i = 1, \ldots, m_0$) has a cardinality not greater than $c(Y)$. However, some of these points can coincide. Let $1 \leqslant i_1 < \ldots < i_k \leqslant m_0$ be integers such that $b_{i_1}' = \ldots = b_{i_k}'$ and let us denote the latter point by b. Then $b \neq b_0$ and the subarc bb_0 of the arc $b_{i_1} b_0$ has only the point b in common with each of the arcs $bb_{i_1}, \ldots, bb_{i_k}$ being subarcs of the arcs $b_0 b_{i_1}, \ldots, b_0 b_{i_k}$, respectively. It follows that

$$bb_0 \cup bb_{i_1} \cup \ldots \cup bb_{i_k} \subset Y$$

is a $(k+1)$-star, whence $k+1 < n$ and therefore $k \leqslant n-2$. We conclude that $m_0 \leqslant c(Y)(n-2)$, which contradicts (i).

COROLLARY. *If a dendroid X is quasi-embedable in a finite dendrite, then X is a finite dendrite.*

References

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