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On the set of orders on a field*

1. Introduction. This paper is concerned with the analysis of the set of orders on a formally real field using category theory and sheaf theory as tools. We topologize the set of orders in a way similar to that used in the prime spectrum of algebraic geometry and define a presheaf of groups using the notion of inverse limits and find necessary and sufficient conditions that these presheaves be sheaves. Finally, we notice that, if $F \subset F'$ are fields, θ and θ' are the set of orders on F and F respectively, and Γ and Γ' are the associated presheaves, then there exists a natural continuous map ψ: $\theta' \to \theta$ and a presheaf homomorphism from Γ to the direct image of Γ' under ψ.

2. Definitions and notation. We shall denote an inverse system by $(\Gamma_\alpha, f_{\beta,\alpha}, I)$, where I is a partially ordered index set, Γ_α abelian groups, and $f_{\beta,\alpha}$: $\Gamma_\beta \to \Gamma_\alpha$ homomorphism which exist if $\beta \leqslant \alpha$. If no confusion is possible we shall shorten this to $(\Gamma_\alpha, f_{\beta,\alpha})$ or (Γ_α). (X, f_α, I) is said to be *for* the inverse system $(\Gamma_\alpha, f_{\beta,\alpha}, I)$ if f_α: $X \to \Gamma_\alpha$ and all triangles commute. The inverse limit of $(\Gamma_\alpha, f_{\beta,\alpha}, I)$ will be denoted $(X, f_\alpha, I) = \operatorname{invlim}(\Gamma_\alpha, f_{\beta,\alpha}, I)$ or $X = \operatorname{invlim}(\Gamma_\alpha)$. It should be noted that if $(\Gamma_\alpha, f_{\beta,\alpha}, I)$ is a coinitial sub-inverse system of $(\Gamma_\alpha, f_{\beta,\alpha}, I)$, then the objects of the inverse limits of these two systems are the same. Also, if I has a smallest element d, then $(\Gamma_d, f_{d,\alpha}, I) = \operatorname{invlim}(\Gamma_\alpha, f_{\beta,\alpha}, I)$. By a sheaf we shall mean a presheaf that satisfies the two sheaf axioms SI and SII.

3. Fields and division cones. Let F be a field and $\mathscr{D}$ the set of division cones in F, where a *division cone* is a set α that is closed under multiplication, addition, and division such that $\alpha \cap -\alpha = 0$ and $1 \in \alpha$. Note that the intersection of division cones is again a division cone.

DEFINITION 3.1. The *Harrison topology* on $\mathscr{D}$ is the topology for which $\{X_G = \{\alpha \in \mathscr{D}\colon \alpha \cap G = \emptyset\}\colon G$ is a finite subset of $F\}$ is a base [$X_G \cap X_H = H_{G \cup H}$ shows that is a base]. Let $\mathscr{H} \subseteq \mathscr{D}$ be a meet semi-

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lattice with the induced topology, where the meet operation is intersection and note that $\mathscr{D}$ is a poset with $\leqslant$ being taken as inclusion.

LEMMA 3.2. *If U is open in $\mathscr{H}$, then U is an ideal of $\mathscr{H}$, i.e., if $\alpha \in U$ and $\beta \in \mathscr{H}$ with $\beta \leqslant \alpha$, then $\beta \in U$.*

Proof. U open in $\mathscr{H}$ implies $U = (\bigcup X_G) \cap H$. Thus $\alpha \in U$ implies $\alpha \in X_G$ for some G and hence $\beta \in X_G$ since $\beta \subseteq \alpha$ implies $\beta \cap G = \emptyset$. This implies $\beta \in \bigcup X_G$ and thus $\beta \in U = (\bigcup X_G) \cap \mathscr{H}$ since $\beta \in \mathscr{H}$ by hypothesis.

DEFINITION 3.3. For $\alpha \in D$ let $\Gamma_\alpha = F^*/\alpha^*$, where $F^* = F \setminus \{0\}$ and $\alpha^* = \alpha \setminus \{0\}$. Both F^* and α^* are multiplicative groups since α is a division cone.

DEFINITION 3.4. For $\beta \leqslant \alpha$ define the homomorphism $\Gamma_{\beta,\alpha}\colon \Gamma_\beta \to \Gamma_\alpha$ by $\Gamma_{\beta,\alpha}(\mathrm{cls}_\beta x) = \mathrm{cls}_\alpha x$.

LEMMA 3.5. *Let U be an open subset of $\mathscr{H}$. Then, considering U as a poset, $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, U)$ is an inverse system.*

Proof. We need to prove that U is a directed (down) poset. Let α and β be elements of U. Then since U is an ideal $\alpha \cap \beta \in U$ with $\alpha \cap \beta \leqslant \alpha$ and $\alpha \cap \beta \leqslant \beta$. The rest of the properties for an inverse system are clear and the proof is omitted.

DEFINITION 3.6. For U an open subset of H define $\Gamma(U)$ to be X, where $(X, f_\alpha, U) = \mathrm{invlim}(\Gamma_\alpha, \Gamma_{\beta,\alpha}, U)$.

LEMMA 3.7. *If U and V are open subsets of $\mathscr{H}$ with $U \neq \emptyset$ contained in V, then $\Gamma(U) = \Gamma(V)$.*

Proof. We shall show that $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, U)$ is a coinitial sub-inverse system of $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, V)$. Then we have that $\Gamma(U) = \Gamma(V)$. Clearly $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, U)$ is a sub-inverse system of $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, V)$, thus all we have to show is that if α is an element of V, then there exists $\beta \in U$ such that $\beta \leqslant \alpha$. Let $\sigma \in U$; then $\sigma \cap \alpha \in U$ since U is an ideal so let $\beta = \sigma \cap \alpha$.

LEMMA 3.8. *If U and V are non-empty open subsets of $\mathscr{H}$, then $\Gamma(U) = \Gamma(V)$.*

Proof. $U \cap V$ is open and non-empty for $\alpha \in U$, $\beta \in V$ implies that $\alpha \cap \beta \in U \cap V$. Thus, using Lemma 3.7, we have

$$\Gamma(U) = \Gamma(U \cap V) = \Gamma(V).$$

DEFINITION 3.9. Let $\Gamma(\mathscr{H}) = \Gamma(U)$, where U is any non-empty open subset of $\mathscr{H}$. This is well-defined by Lemma 3.8.

THEOREM 3.10. *Let $\mathscr{H} \subseteq \mathscr{D}$ be a meet semi-lattice, $(X, \pi_\alpha, \mathscr{H}) = \mathrm{invlim}(\Gamma_\alpha, \Gamma_{\beta,\alpha}, \mathscr{H})$, $d = \bigcap \{\alpha\colon \alpha \in \mathscr{H}\}$, and $f\colon \Gamma_d \to X$ be defined by $f(\mathrm{cls}_d x) = (x_\alpha)$, where $x_\alpha = \mathrm{cls}_\alpha x$. Then we have that*

(i) *f is well-defined, maps $\Gamma_d \to X$, and is injective,*

(ii) *(Γ_d, f_α, H) is for $(\Gamma_\alpha, \Gamma_{\beta,\alpha}, \mathscr{H})$ and f is the unique map, $\Gamma_d \to X$, such that $\pi_\alpha f = f_\alpha$, where $f_\alpha = \Gamma_{d,\alpha}$,*

(iii) *if* $d \in H$, *then* $(\Gamma_d, f_a, \mathscr{H}) = \operatorname{invlim}(\Gamma_a, \Gamma_{\beta,a}, \mathscr{H})$ *and hence* f *is an isomorphism.*

Proof. (i) To show f is well-defined we need that $\operatorname{cls}_d x = \operatorname{cls}_d y$ implies $\operatorname{cls}_a x = \operatorname{cls}_a y$ for all $a \in \mathscr{H}$. $\operatorname{cls}_d x = \operatorname{cls}_d y$ implies $xy^{-1} \in a$ since $d \subseteq a$ and thus $\operatorname{cls}_a x = \operatorname{cls}_a y$. Now $\beta \subseteq a$ implies $\Gamma_{\beta,a}(\operatorname{cls}_\beta x) = \operatorname{cls}_a x$ and so f does map Γ_d into X. Suppose $f(\operatorname{cls}_d x) = 1$, then $\operatorname{cls}_a x = 1$ for all $a \in \mathscr{H}$ and so $x \in a$ for all a which implies that $x \in \bigcap a = d$ and so $\operatorname{cls}_d x = 1$. Thus f is injective.

(ii) For this part we need to show that if $\beta \leqslant a$, then $\Gamma_{\beta,a}\Gamma_{d,\beta} = \Gamma_{d,a}$ in view of the fact that d is a division cone, this is true since $\mathscr{D}$ is an inverse system. Thus $(\Gamma_d, f_a, \mathscr{H})$ is for $(\Gamma_a, \Gamma_{\beta,a}, \mathscr{H})$. Now,

$$\pi_a f(\operatorname{cls}_d x) = \pi_a(\operatorname{cls}_\beta x) = \operatorname{cls}_a x = \Gamma_{d,a}(\operatorname{cls}_d x) = f_a(\operatorname{cls}_d x).$$

Hence $\pi_a f = f_a$ and the uniqueness of f comes from the uniqueness part of the definition for invlim.

(iii) If $d \in \mathscr{H}$, then $(\Gamma_d, f_{d,a}, \mathscr{H}) = \operatorname{invlim}(\Gamma_a)$ and the uniqueness, up to isomorphism, of the object in invlim yields the fact that f is an isomorphism.

Remark 3.11. The conclusion to part (iii) of the previous theorem does not hold for an arbitrary meet semi-lattice $\mathscr{H} \subseteq \mathscr{D}$. A counterexample can be found upon examining the set of orders on the rationals with one indeterminate adjoined.

We shall now consider a more general situation. Again let F be a field and $\mathscr{D}$ the set of division cones in F; but from now on we will let $\mathscr{H}$ be any subset of D with the induced topology, and for U open in $\mathscr{H}$ let $\underset{\sim}{U}$ be the semi-lattice generated by U, i.e., the set of all finite intersections of elements of U.

As before let $\Gamma_a = F^*/a^*$, where $a \in \underset{\sim}{U}$ and for $\beta \subseteq a$ let $\Gamma_{\beta,a}\colon \Gamma_\beta \to \Gamma_a$ be defined by $\Gamma_{\beta,a}(\operatorname{cls}_\beta x) = \operatorname{cls}_a x$. But now we let $\Gamma(U) = X$, where $(X, f_a, \underset{\sim}{U}) = \operatorname{invlim}(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{U})$.

Remark 3.12. If U is open in $\mathscr{H}$, then there exists a V open in $\mathscr{D}$ with $V \cap \mathscr{H} = U$; however, $\underset{\sim}{U} \neq \underset{\sim}{V} \cap \mathscr{H}$. To see this, $\underset{\sim}{U}$ may be a singleton but $\underset{\sim}{V} \cap \mathscr{H}$ not a singleton. This also shows that $U \subseteq V$ does not imply that $\underset{\sim}{U}$ is coinitial in $\underset{\sim}{V}$ and so we may have $\Gamma(U) \neq \Gamma(V)$.

Lemma 3.13. *For* $U \subseteq V$ *there exists a unique* $P_{U,V}\colon \Gamma(V) \to \Gamma(U)$ *such that for all* $a \in U$, $f_{a,U}P_{U,V} = f_{a,V}$, *where*

$$\left(\Gamma(U), f_{a,U}, \underset{\sim}{U}\right) = \operatorname{invlim}(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{U})$$

and

$$\left(\Gamma(V), f_{a,V}, \underset{\sim}{V}\right) = \operatorname{invlim}(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{V}).$$

Proof. $U \subseteq V$ implies $\underset{\sim}{U} \subseteq \underset{\sim}{V}$ and hence $\left(\Gamma(V), f_a, \underset{\sim}{U}\right)$ is for $(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{U})$, where $f_a = f_{a,V}$ for $a \in \underset{\sim}{U}$. Thus by definition of invlim there exists a unique $P_{U,V}$ with the required property.

LEMMA 3.14. *Let $U \subseteq V \subseteq W$ be open subsets of $\mathscr{H}$ and $P_{U,V}$, $P_{V,W}$, $P_{U,W}$ be as in Lemma* 3.13; *then $P_{U,V}P_{V,W} = P_{U,W}$.*

COROLLARY 3.15. *Γ defines a presheaf on $\mathscr{H}$.*

THEOREM 3.16. (SI) *Let $\mathscr{H}$ be a set of division cones in a field F, U an open subset of $\mathscr{H}$, and $\bigcup\{V\colon V \in D\} = U$, where each V is an open subset of H. Suppose there exists a and b, elements of $\Gamma(U)$, such that $P_{V,U}(a) = P_{V,U}(b)$ for all $V \in D$, then $a = b$. That is to say the condition* SI *holds for the presheaf defined in Corollary* 3.1.

The proof is straightforward and will be omitted.

Remark 3.17. As we shall see later SII is not satisfied in general; however, there are special cases in which it is satisfied.

4. Fields and orders. In the preceding section we developed a presheaf on a set of division cones in a field F. We shall now restrict our attention to a set of orders, an order being a division cone with the property that $a \cup -a = F$, with the intention of describing the groups involved in the presheaf and finding necessary and sufficient conditions that the presheaf be a sheaf.

DEFINITION 4.1. Let $\mathscr{O}$ be the set of all orders on a field F, S a subset of $\mathscr{O}$, and x a non-zero element of F. The function $S_x\colon S \to \{1, -1\}$ defined by

$$S_x(a) = \delta, \qquad \text{where } x \in \delta a \text{ for } \delta = \pm 1$$

will be called the *order function of x with respect to S.*

Essentially, the order function of x with respect to S assigns to these a which are elements of S the sign of x and ignores a if it is not an element of S.

DEFINITION 4.2. Let $G_S = \{S_x\colon x \in F^* = F \setminus \{0\}\}$ and define an operation on G_S by $(S_xS_y)(a) = S_x(a)S_y(a)$. We shall call G_S equipped with this operation the *order group of S.* The following lemma shows that G_S is actually a group.

LEMMA 4.3. *If S is a set of orders on a field F and G_S is the order group of S, then*

(i) *if* 1 *is the multiplicative identity of F, then S_1 is the identity of G_S;*

(ii) *$S_x = S_x^{-1}$ for all $x \in F$;*

(iii) *$S_{xy} = S_xS_y$ for all x, y is F;*

(iv) *G_S is a commutative group;*

(v) *$G_S = \bigoplus_I Z_2$, where I is not necessarily finite;*

(vi) *$S_x = S_{x^{-1}}$ for all $x \in F$;*

(vii) *If $\{-1, 1\}$ is given the discrete topology and S the topology induced from the Harrison topology on $\mathscr{O}$, then S_x is continuous;*

(viii) *The topology on S induced by the Harrison topology is the weak topology for G_S.*

Proof. The proof of parts (i)–(iv) and (vi) is computational and will be omitted.

(v) By part (ii) we have that every element is its own inverse and hence G_S is a direct sum of copies of Z_2.

(vii) $S_x^{-1}(1) = \{a \in S\colon x \in a\} = \{a \in \mathcal{O}\colon x \in a\} \cap S$ which is an open set. Similarly for $S_x^{-1}(-1)$. Thus S_x is continuous.

(viii) If U is a basic open set in S, then U is the set of orders missing some finite set $G \subset F$.

$$U = \{a \in S\colon a \cap G = \emptyset\} = \bigcap_{x \in G} \{S_x^{-1}(-1)\}.$$

Thus S has the weak topology for G_S.

LEMMA 4.4. *Let S be a set of orders on a field and $\alpha = \bigcap \{\beta\colon \beta \in S\}$. Then $\varphi\colon F^*/\alpha^* \to G_S$ defined by $\varphi(\mathrm{cls}_\alpha x) = S_x$ is an isomorphism and hence $F^*/\alpha^* \cong G_S$.*

COROLLARY 4.5. *$F^*/\alpha^* \cong \oplus Z_2$, where α is as in Lemma 4.4.*

Remark 4.6. If $S = \{a_1, a_2, \ldots, a_n, \beta\}$, then to say that $\bigcap \{a_i\colon 1 \leqslant i \leqslant n\} \subseteq \beta$ is equivalent to saying that $S_x(a_i) = 1$ for $1 \leqslant i \leqslant n$ implies $S_x(\beta) = 1$.

DEFINITION 4.7. A set S of orders will be called *irredundant* if $\{a_1, a_2, \ldots, a_n, \beta\} \subseteq S$ with $\bigcap a_i \subseteq \beta$ implies $\beta = a_j$ for some j.

LEMMA 4.8. *If S is a set of orders in a field F, then there exists a maximal irredundant subset of S.*

Proof. Apply Zorn's lemma to the set of irredundant sets contained in S.

NOTATION 4.9. If S is a set of orders in a field F we shall let S' denote some maximal irredundant subset of S.

LEMMA 4.10. *If S is a set of orders on a field F with $\{a_1, \ldots, a_n, \beta_1, \ldots, \beta_k\} \subseteq S'$ such that $\bigcap a_i \subseteq \bigcap \beta_j$, then*

$$\{\beta_1, \ldots, \beta_k\} \subseteq \{a_1, \ldots, a_n\}.$$

Proof.

$$\bigcap a_i \subseteq \bigcap \beta_j \subseteq \beta_m \quad \text{for } 1 \leqslant m \leqslant k.$$

Thus, since S' is irredundant, there exists i with $\beta_m = a_i$ and so $\{\beta_1, \ldots, \beta_k\} \subseteq \{a_1, \ldots, a_n\}$.

COROLLARY 4.11. *If $S \subseteq T$ is irredundant with $\underset{\sim}{S} = T$, then $S = T$.*

LEMMA 4.12. *If $a \in S$, then there exists $\beta \in$ ' such that $\beta \subseteq a$.*

Proof. If $a \in S'$, then we may let $\beta = a$ thus we may assume that

$a \notin S'$. Hence S' is properly contained in $S' \cup \{a\}$ and so $S' \cup \{a\}$ is not irredundant by maximality of S. Thus there exists

$$B = \{a_1, a_2, \ldots, a_{n+1}\} \subseteq S' \cup \{a\}$$

with $a_i \neq a_j$ for $i \neq j$, and such that

$$\bigcap \{a_i\colon\ 1 \leqslant i \leqslant n\} \subseteq a_{n+1},$$

and $a = a_j$ for some j. If $a = a_{n+1}$, then let

$$\beta = \bigcap \{a_i\colon\ 1 \leqslant i \leqslant n\}$$

and we have $\beta \in S'$ with the property that $\beta \subseteq a$. Thus we may assume that $a \neq a_{n+1}$ and, without loss of generality, we may let $a = a_n$. Now, if

$$\bigcap \{a_i\colon\ i \leqslant n-1\} \cap a_{n+1} \subseteq a_n = a$$

we have the desired β, hence we may assume

$$\bigcap \{a_i\colon\ i \leqslant n-1\} \cap a_{n+1} \nsubseteq a.$$

Thus we have the following conditions:

(CI) $\bigcap \{a_i\colon\ i \leqslant n-1\} \cap a \subset a_{n+1}$,

(CII) $\bigcap \{a_i\colon\ i \leqslant n-1\} \nsubseteq a_{n+1}$ since S is irredundant,

and

(CIII) $\bigcap \{a_i\colon\ i \leqslant n-1\} \cap a_{n+1} \nsubseteq a$.

Using Remark 4.6 to translate these relationships we will see that this is impossible. (CII) restated means that there exists $x \in F^*$ such that

$$B_x(a_i) = \begin{cases} 1, & 1 \leqslant i \leqslant n-1, \\ -1, & i = n+1 \end{cases}$$

and then by (CI) we see that $B_x(a) = -1$. (CIII) yields the existence of a $y \in F^*$ such that

$$B_y(a_i) = \begin{cases} 1, & 1 \leqslant i \leqslant n-1 \text{ or } i = n+1, \\ -1, & 1 = n. \end{cases}$$

Therefore we have

$$B_{xy}(a_i) = \begin{cases} 1, & 1 \leqslant i \leqslant n, \\ -1, & i = n+1 \end{cases}$$

which contradicts (CI). Thus (CIII) must be false and so

$$\beta = \bigcap \{a_i\colon\ i \leqslant n-1\} \cap a_{n+1}$$

is an element of $\underset{\sim}{S}'$ contained in a.

COROLLARY 4.13. *If S is a set of orders on a field and U is open in S, then $\Gamma(U) = \Gamma(U')$.*

Remark 4.14. If $\mathcal{O}$ is the set of all orders on a field F, then θ' is a set of orders on F with the property that any order on F contains a finite intersection of orders in $\mathcal{O}'$ but intersections over different finite sets are different.

Remark 4.15. If $S \subset T$ are two sets of orders with $\alpha = \bigcap \{\beta\colon \beta \in S\}$ and $\sigma = \bigcap \{\beta\colon \beta \in T\}$, then the map ψ defined by $\psi(T_x) = S_x$ commutes the following square:

$$\begin{array}{ccc} F^*/\sigma^* & \xrightarrow{\text{can}} & F^*/\alpha^* \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ G_T & \xrightarrow[\psi]{} & G_S \end{array}$$

Thus we have converted our inverse systems in the preceding section to inverse systems where the objects are groups of functions and the morphisms are restriction mappings.

LEMMA 4.16. *If S is a set of orders on F, then $G_S \cong G_{S'}$.*

Proof. Let $\alpha = \bigcap \{\beta\colon \beta \in S\}$ and $\sigma = \bigcap \{\beta\colon \beta \in S'\}$. Then, by Lemma 4.12, $\alpha = \sigma$ and hence Lemma 4.4 yields that

$$G_S \cong F^*/\alpha^* = F^*/\sigma^* \cong G_{S'}.$$

LEMMA 4.17. *If S is a finite set of orders and $\alpha = \bigcap \{\beta\colon \beta \in S\}$, then*

$$F^*/\alpha^* \cong \bigoplus_1^n Z_2$$

where n is the number of elements in S'. Also $\Gamma_\alpha \cong G_S \cong F^*/\alpha^*$.

Proof.

$$F^*/\alpha^* \cong G_S \cong G_{S'} \cong \bigoplus_1^k Z_2$$

and $o(G_{S'}) \leqslant 2^n$ since $G_{S'}$ is a set of functions from S' to $\{-1, 1\}$. To show that $o(G_{S'}) = 2^n$, and hence that $k = n$, we need to show that every function from S' to $\{-1, 1\}$ is an order function of some $x \in F^*$ with respect to S'. Hence we let g be such a function and choose $x_i \in F^*$, $1 \leqslant i \leqslant n$, such that $S'_{x_i}(\alpha_j) = 1$ if $i \neq j$ and $S'_{x_i}(\alpha_i) = -1$. This can be done since S' is irredundant. Now let

$$y_i = \begin{cases} x_i & \text{if} \quad g(\alpha_i) = -1, \\ 1 & \text{if} \quad g(\alpha_i) = 1 \end{cases}$$

and

$$x = \prod \{y_i\colon 1 \leqslant i \leqslant n\};$$

then

$$S'_x(\alpha_j) \doteq \prod \{S'_{y_i}(\alpha_j)\colon 1 \leqslant i \leqslant n\} = g(\alpha_j).$$

DEFINITION 4.18. If $\alpha = \bigcap\{\beta_i\colon i \leqslant n\}$, where β_i is an order for each i, then we say $\{\beta_i\}$ *covers* α or $\{\beta_i\}$ is a cover for α.

LEMMA 4.19. *If S is irredundant and $\alpha \in \underset{\sim}{S}$, then there exists a unique cover for α contained in S.*

LEMMA 4.20. *Let S be an irredundant set of orders with $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq S$. Then, if $x_\alpha \in F^*$, $\alpha \in A$ there exists $x \in F^*$ such that $\mathrm{cls}_\alpha x = \mathrm{cls}_\alpha x_\alpha$ for all $\alpha \in A$, i.e., $x \in \alpha$ if and only if $x_\alpha \in \alpha$.*

Proof. Define $g\colon A = \{\alpha_1, \ldots, \alpha_n\} \to \{-1, 1\}$ by $g(\alpha) = 1$ if $x_\alpha \in \alpha$ and $g(\alpha) = -1$ if $x_\alpha \notin \alpha$. Then by the proof of Lemma 4.17 there exists $x \in F^*$ such that $A_x = g$. Clearly this x satisfies the desired properties.

THEOREM 4.21. *Let S be a set of orders and U an open subset of S; then $\Gamma(U) = \oplus Z_2$.*

Proof. For $\alpha \in U$ we have by Lemma 4.17 $\Gamma_\alpha \cong \oplus Z_2$ and $\Gamma(U) \subseteq \prod\{\Gamma_\alpha\colon \alpha \in U\} \subseteq \oplus Z_2$ and hence $\Gamma(U) \cong \oplus Z_2$.

Remark 4.22. It should be noted that there are examples of sets of orders U such that $\Gamma(U) \neq F^*/a^* \cong G_{U'} \cong G_U$ and hence $\Gamma(U)$ is not, in general, the order group for U.

THEOREM 4.23. *A set S of orders on a field F is irredundant if and only if the associated presheaf is a sheaf.*

Proof. We know from the previous section that the presheaf always satisfies SI hence all we have to do is prove that SII is satisfied if and only if S is irredundant.

We will first show that if S is irredundant, then SII is satisfied. Let U be an open subset of S and $\{V\colon V \in D\}$ be a collection of open sets such that $\bigcup V = U$. Let $\{a(V)\colon V \in D\}$ be a set with the properties that $a(V) \in \Gamma(V)$ and for each $(V, W) \in D \times D$

$$P_{V\cap W, V}\big(a(V)\big) = P_{V\cap W, W}\big(a(W)\big).$$

We must find $a \in \Gamma(U)$ such that $P_{V,U}(a) = a(V)$ for each $V \in D$. We know that $\Gamma(U) \subseteq \prod\{\Gamma_\alpha\colon \alpha \in U\}$ and $\Gamma(V) \subseteq \prod\{\Gamma_\alpha\colon \alpha \in V\}$, thus we may let the α-th projection of $a(V)$ be represented by $a(V)_\alpha = \mathrm{cls}_\alpha x_{V,\alpha}$, where $x_{V,\alpha} \in F^*$ and a will be defined if we define it at each coordinate. Let A be the unique cover for $\alpha \in \underset{\sim}{U}$ and let $V(\beta) \in D$ be an open set with $\beta \in V(\beta)$ for each β in A. By Lemma 4.20, there exists $x(\alpha) \in F^*$ such that $\mathrm{cls}_\beta x(\alpha) = \mathrm{cls}_\beta x_{V(\beta),\beta}$ for all $\beta \in A$. Let $a \in \prod\{\Gamma_\alpha\colon \alpha \in U\}$ be the element whose α-th projection is $a_\alpha = \mathrm{cls}_\alpha x(\alpha)$ and note that a_α is well defined since the cover for α is unique. Let $\alpha, \sigma \in \underset{\sim}{U}$ with $\alpha \subset \sigma$ and let A and B be the unique covers for α and σ respectively. Then $A \cup B$ is a cover for α hence $A \cup B = A$ and thus $B \subseteq A$. For each $\beta \in A$ $\mathrm{cls}_\beta x(\alpha) = \mathrm{cls}_\beta x_{V(\beta),\beta}$ and hence for each $\beta \in B$ $\mathrm{cls}_\beta x(\alpha) = \mathrm{cls}_\beta x_{V(\beta),\beta} = \mathrm{cls}_\beta x(\sigma)$. Thus $x(\alpha)x(\sigma)^{-1} \in \beta$ for each $\beta \in B$ and so $x(\alpha)x(\sigma)^{-1} \in \sigma$ which implies $\mathrm{cls}_\sigma x(\alpha) = \mathrm{cls}_\sigma x(\sigma)$ and thus $\Gamma_{\alpha,\sigma}(a_\alpha) = \Gamma_{\alpha,\sigma}\big(\mathrm{cls}_\alpha x(\alpha)\big) = \mathrm{cls}_\sigma x(\alpha) = \mathrm{cls}_\sigma x(\sigma) = a_\sigma$. Thus $a \in \Gamma(U)$.

Also let $\alpha \in \underset{\sim}{V}$ and $A \subseteq V$, hence $V(\beta) = V$ for all $\beta \in A$, be the unique cover for α. Then since $a(V) \in \Gamma(V)$ we have for each $\beta \in A$ that

$$\begin{aligned}\mathrm{cls}_\beta x_{V,\alpha} &= \Gamma_{\alpha,\beta}(\mathrm{cls}_\alpha x_{V,\alpha}) = \Gamma_{\alpha,\beta}\big(a(V)_\alpha\big) = a(V)_\beta = \mathrm{cls}_\beta x_{V,\beta} \\ &= \mathrm{cls}_\beta x_{V(\beta),\beta} = \mathrm{cls}_\beta x(\alpha).\end{aligned}$$

Thus $x_{V,\alpha}x(\alpha)^{-1} \in \beta$ for each $\beta \in A$ hence $x_{V,\alpha}x(\alpha)^{-1} \in \alpha$ and we have proved that $\mathrm{cls}_\alpha x_{V,\alpha} = \mathrm{cls}_\alpha x(\alpha)$ implying that $P_{V,U}(a) = a(V)$ [i.e. for each $\alpha \in \underset{\sim}{V}$

$$\big(P_{V,U}(a)\big)_\alpha = a_\alpha = \mathrm{cls}_\alpha x(\alpha) = \mathrm{cls}_\alpha x_{V,\alpha} = a(V)_\alpha].$$

For the converse suppose that S is not irredundant. Then there exists $\{a_i\colon 1 \leqslant i \leqslant n+1\} \subseteq S$ such that

$$\bigcap \{a_i, 1 \leqslant i \leqslant n\} \subseteq a_{n+1}, \qquad a_i \neq a_j$$

if $i \neq j$, and $\{a_1, a_2, \ldots, a_n\}$ is irredundant. For $0 \leqslant j \leqslant n$ there exists $x_j \in F^*$ such that $x_j \in a_{n+1}$, $x_j \notin a_j$, and $x_j \in a_{j+1}$ if $1 \leqslant j \leqslant n-1$; $x_n \in a_{n+1}$, $x_n \notin a_n$, and $x_n \in a_1$; $x_0 \in F^*$. Let $U_j = \{\beta \in S\colon x_j \in \beta\}$ if $1 \leqslant j \leqslant n$ and let

$$U_0 = \{\beta \in S\colon -x_j \in \beta \text{ for } 1 \leqslant j \leqslant n\}.$$

Then

$$\bigcup \{U_i\colon 0 \leqslant i \leqslant n\} = S, \qquad U_0 \cap (\bigcup \{U_i\colon 1 \leqslant i \leqslant n\}) = \emptyset,$$

and $\{x_i, x_j\} \cap U_i \cap U_j = \emptyset$ if $i \neq j$. Let $a_i \in \Gamma(U_i)$ be that element whose α-th projection is $\mathrm{cls}_\alpha x_i$ for every $\alpha \in U_i$ and note that if $\alpha \in \underset{\sim}{U_j}$ for some j, then $\mathrm{cls}_\alpha x_i = \mathrm{cls}_\alpha x_j$ since $\{x_i, x_j\} \subseteq \alpha$. Thus

$$P_{U_i \cap U_j, U_i}(a_i) = P_{U_i \cap U_j, U_j}(a_j).$$

Let $\beta = \bigcap \{a_i \in S\colon 1 \leqslant i \leqslant n+1\}$. If there exists $a \in \Gamma(S)$ such that $P_{U_i,U}(a) = a_i$, then the β-th projection of a would be represented by $\mathrm{cls}_\beta x$ with the property that

$$\Gamma_{\beta,\alpha_i}(\mathrm{cls}_\beta x) = \mathrm{cls}_{\alpha_i} x = \mathrm{cls}_{\alpha_i} x_i$$

for $1 \leqslant i \leqslant n+1$. But this implies that $x \notin a_i$ for $1 \leqslant i \leqslant n$ and $x \in a_{n+1}$ implying $-x \in a_i$ for $1 \leqslant i \leqslant n$ and $-x \notin a_{n+1}$ contradicting the hypothesis that $\bigcap \{a_i\colon 1 \leqslant i \leqslant n\} \subseteq a_{n+1}$. Therefore no such a exists and hence the associated presheaf is not a sheaf.

LEMMA 4.24. *Let $F' \subset F$ be two fields with θ' and θ the set of orders in F' and F respectively. The map $\psi\colon \theta \to \theta'$ defined by $\psi(\alpha) = \alpha \cap F'$ is continuous.*

Proof. U' a basic open set in θ' implies that there exists $G \subset F'$ such that $U' = \{a' \in \theta'\colon a' \cap G = \emptyset\}$. Hence $\psi^{-1}(U') = \{\alpha \in \theta\colon \alpha \cap G = \emptyset\}$ which is an open set in θ. Thus ψ is continuous.

NOTATION 4.25. For the rest of this section F' and F will be two fields with $F' \subset F$, $\mathcal{O}'$ and $\mathcal{O}$ will be the set of orders on F' and F respec-

tively, $\psi\colon \mathcal{O} \to \mathcal{O}'$ will be the map defined by $\psi(a) = a \cap F$, for an open subset U' of $\mathcal{O}'$ we will let $U = \psi^{-1}(U')$, for $a \in U$ we will let $a' = a \cap F'$, Γ' and Γ will be the presheaves associated with F' and F respectively, $(\Gamma'(U'), f'_{U',a'}, \underset{\sim}{U}') = \operatorname{invlim}(\Gamma'_{a'}, \Gamma_{\beta',a'}, \underset{\sim}{U}')$ and $(\Gamma(U), f_{U,a}, \underset{\sim}{U}) = \operatorname{invlim}(\Gamma_a, \Gamma_{\beta,a}, U)$, and finally ψ_* will denote the direct image of Γ under ψ. Note that $\psi_*(U') = \Gamma(U)$.

LEMMA 4.26. *If U' is open in $\mathcal{O}'$ and $a \in U$, then $a' \in U'$. Also, if $\{\alpha, \beta\} \subseteq \underset{\sim}{U}$ with $\beta \subseteq a$, then $\beta' \subseteq a'$.*

Proof. $a \in \underset{\sim}{U}$ implies that there exists a cover $\{a_1, \ldots, a_n\}$ for a contained in U. Thus

$$a' = a \cap F = (\bigcap\{a_i\colon 1 \leqslant i \leqslant n\}) \cap F = \bigcap(a_i \cap F) = \bigcap\{a'_i\colon 1 \leqslant i \leqslant n\} \in \underset{\sim}{U}'.$$

The second statement is clear.

LEMMA 4.27. *If U' is open in $\mathcal{O}'$ and $a \in \underset{\sim}{U}$, then $g_a\colon \Gamma'_{a'} \to \Gamma_a$ defined by $g_a(\operatorname{cls}_{a'} x) = \operatorname{cls}_a x$ is a homomorphism with the property that if $\beta \subseteq a$, then*

$$\Gamma_{\beta,a} g_\beta = g_a \Gamma'_{\beta',a'}.$$

LEMMA 4.28. *If V' is open in $\mathcal{O}'$, then there exists a unique $\theta_{V'}\colon \Gamma'(V') \to \Gamma(V) = \psi_*(V')$ with the property that $f_{V,a}\theta_{V'} = g_a f'_{V',a'}$ for all $a \in \underset{\sim}{V}$, where g_a is as in Lemma 4.27.*

Proof. If $\beta \in V$ with $\beta \subseteq a$, then

$$\Gamma_{\beta,a} g_\beta f'_{V',\beta'} = g_a \Gamma'_{\beta',a'} f'_{V',\beta'} = g_a f'_{V',a'}$$

and hence $(\Gamma'(V), g_a f'_{V',a'}, \underset{\sim}{V})$ is for $(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{V})$ implying the existence of a unique $\theta_{V'}\colon \Gamma'(V') \to \Gamma(V)$ with the property that $f_{V,a}\theta_{V'} = g_a f'_{V',a'}$ for all $a \in \underset{\sim}{V}$ by the definition of inverse limit.

THEOREM 4.29. *If Γ' and Γ are sheaves, then $\theta = \{\theta_{U'}\colon U'$ is open in $\theta'\}$ is a sheaf homomorphism from Γ' to ψ_*, where $\theta_{U'}$ is as in Lemma 4.28.*

Proof. We must show that if U' and V' are open subsets of θ' with the property that $U' \subseteq V'$, then $P_{U,V}\theta_{V'} = \theta_{U'} P'_{U',V'}$. We see that $\Gamma_{\beta,a} g_\beta f'_{V',\beta'} = g_a f'_{V',a'}$ for all $a \in \underset{\sim}{V}$ and hence for all $a \in \underset{\sim}{U}$ thus $(\Gamma'(V'), f'_{V',a'}, \underset{\sim}{U})$ is for $(\Gamma_a, \Gamma_{\beta,a}, \underset{\sim}{U})$ and hence by the definition of inverse limit there exists a unique $f\colon \Gamma'(V') \to \Gamma(U)$ such that $g_a f'_{V',a'} = f_{U,a} f$ for all $a \in U$. Now

$$F_{U,a} P_{U,V} \theta'_V = f_{V,a}\theta_{V'} = g_a f'_{V',a'}$$

for all $a \in \underset{\sim}{U}$ and so $P_{U,V}\theta_{V'} = f$. Also

$$f_{U,a}\theta_{U'} P'_{U',V'} = g_a f'_{U',a'} P'_{U',V'} = g_a f'_{V',a'}$$

for all $a \in \underset{\sim}{U}$ implying that $\theta_{U'} P'_{U',V'} = f$ and hence we have

$$P_{U,V}\theta_{V'} = \theta_{U'} P'_{U',V'}.$$

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