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The method of synergistic models

In this paper we prove the consistency of the axiom of Choice (C) with the set-theoretical axioms of Zermelo–Fraenkel (ZF), by means of Synergistic models, without necessitating the construction of a model for the entire ZF and C. Also, based on the method of Synergistic models we prove that the axiom of Regularity is consistent with ZF + C.

As usual, by ZF we mean the axiom system consisting of the axioms of Extensionality (E), Powerset (P), Sumset (S), Infinity (I) and the axiom scheme of Replacement (R). The underlying logic being the first order predicate calculus without equality.

DEFINITION. An axiom system is called *Synergistic* if and only if it consists of the axioms E, P, S, I and any finite (possibly zero) number of instances of the axiom scheme R. A model is called *Synergistic* if and only if it is a model for a Synergistic axiom system.

Clearly, in order to prove that a statement (such as C) is consistent with ZF it is sufficient to prove that the statement is consistent with every Synergistic axiom system. Thus, the proof of the consistency of a statement with ZF can be given by constructing only Synergistic models in which the statement is valid *without* necessitating the construction of a model for the entire ZF and the statement under consideration.

As expected, we assume that ZF (i.e., the axioms E, P, S, I and the axiom scheme R) is a consistent system. Observing that the Completeness theorem for First-Order Theories is proved without use of the axiom of Choice, by the Completeness theorem, we let (K, ϵ) denote a model for ZF.

In what follows, by a "*set*" we mean an object of model (K, ϵ) and by "*is an element of*" we refer to "ϵ" appearing in (K, ϵ). Naturally, all other set-theoretical items are introduced by their usual definitions. In particular, every ordinal number which we use is an object of model (K, ϵ).

Let u be an ordinal number and A_u a set (naturally, of model (K, ϵ)). For every element m of A_u, we let $P_u(m)$ denote the set of precisely those subsets of m that can be found in A_u, i.e.,

$$x \in P_u(m) \quad \textit{if and only if} \quad x \in A_u \quad \textit{and} \quad x \subseteq m. \tag{1}$$

We refer to $P_u(m)$ as "*the powerset of m relativized to* A_u".
Thus, if $A_0 = \{0, 1, 2, \ldots, \omega\}$, then

$$P_0(3) = \{0, 1, 2, 3\} = 4.$$

Moreover, for every element m of A_u, we let $S_u(m)$ denote the set of precisely those elements of the elements of m that can be found in A_u, i.e.,

(2) $x \in S_u(m)$ *if and only if* $x \in A_u$ *and* $y \in A_u$ *and* $x \in y \in m$.

We refer to $S_u(m)$ as "*the sumset of m relativized to* A_u".
Furthermore, for every element m of A_u and every binary predicate $F(x, y)$, if $F(x, y)$ is functional in x on m when all the bound variables of $F(x, y)$ are *restricted to* A_u, we denote by

(3) $$F_u[m]$$

the set of precisely those mates y of the elements x of m such that $y \in A_u$.
We refer to $F_u[m]$ as "*the image of m under* $F(x, y)$ *relativized to* A_u".
Finally, motivated by [1], for every non-empty element m of A_u, if A_u is a transitive set (i.e., every element of A_u is also a subset of A_u), and, if w is a well ordering (naturally, in model (K, ϵ)) of A_u, we denote by

(4) $$C_u^w(m)$$

the set whose elements are precisely the *w-first* elements of the non-empty elements of m, and we set:

(5) $$C_u^w(\emptyset) = C_u^w(\{\emptyset\}) = \emptyset.$$

Clearly, $C_u^w(m)$ is a *choice-set* of m.
Next, based on (1), (2), (3), (4) and (5), we let:

$$\begin{aligned} P_u &= \{x \mid x = P_u(m) \text{ for some } m \in A_u\}, \\ S_u &= \{x \mid x = S_u(m) \text{ for some } m \in A_u\}, \\ F_u &= \{x \mid x = F_u[m] \text{ for some } m \in A_u\}, \\ C_u^w &= \{x \mid x = C_u^w(m) \text{ for some } m \in A_u\}. \end{aligned} \tag{6}$$

We observe that the existence of each P_u, S_u, F_u, C_u^w as a set of model (K, ϵ) is ensured by the fact that (K, ϵ) is a model for the axioms E, P, S and the axiom scheme R.

PROPOSITION 1. *The axiom of Choice is consistent with the axioms of Extensionality, Powerset, Sumset, Infinity and the axiom scheme of Replacement.*

Proof. As mentioned above, in order to prove Proposition 1, it is sufficient to show that the axiom of Choice is consistent with every Synergistic system. Consider the Synergistic system with finitely many instances of the axiom scheme of Replacement given by the finitely many binary predicates:

$$F^1(x, y), \ldots, F^n(x, y). \tag{7}$$

For every ordinal u, based on (6), (7) and by virtue of the Transfinite Induction, we define the set A_u by:

$$\begin{aligned} A_0 &= \{0, 1, 2, \ldots, \omega\}, \\ A_{u+1} &= A_u \cup P_u \cup S_u \cup F_u^1 \cup \ldots \cup F_u^n \cup C_u^{w(u)}, \\ A_u &= \bigcup_{v<u} A_v \text{ if } u \text{ is a limit ordinal}, \end{aligned} \tag{8}$$

where $w(u)$ in $C_u^{w(u)}$ is defined as follows:

$w(0)$ is the well ordering of A_0 as indicated by configuration:

$$\{0, 1, 2, \ldots, \omega\}. \tag{9}$$

Next, if $w(u)$ is the well ordering of A_u as indicated by configuration

$$\{a_0, a_1, a_2, \ldots\},$$

then $w(u+1)$ is the well ordering of A_{u+1} as indicated by configuration:

$$\begin{aligned} &\{a_0, a_1, a_2, \ldots \\ &P_u(a_0), P_u(a_1), P_u(a_2), \ldots \\ &S_u(a_0), S_u(a_1), S_u(a_2), \ldots \\ &F_u^1(a_0), F_u^1(a_1), F_u^1(a_2), \ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ &F_u^n(a_0), F_u^n(a_1), F_u^n(a_2), \ldots \\ &C_u^{w(u)}(a_0), C_u^{w(u)}(a_1), C_u^{w(u)}(a_2), \ldots\}, \end{aligned} \tag{10}$$

where repetitions are omitted. And,

$$w(u) = \bigcup_{v<u} w(v) \text{ if } u \text{ is a limit ordinal.} \tag{11}$$

From the formation of A_u, given by (1) to (11), it follows readily that for every ordinal u, every element of A_u is also a subset of A_u (i.e., A_u is a *transitive* set). Thus, $w(u)$ also well orders every element of every element of A_u. But this, clearly ensures that every element d of A_u has a *choice-set*

$$C_u^{w(u)}(d). \tag{12}$$

Moreover, from (6) and (8) it follows that

$$C_u^{w(u)}(d) \in A_{u+1}. \tag{13}$$

Again, we observe that for every ordinal u it is the case that A_u is a set of the model (K, ϵ), and, we emphasize that a reason for this is that in the formation of A_{u+1} only finitely many predicates are used.

Next, we define a model (S, ϵ) as follows:

(14) *x is a set of (S, ϵ) if and only if $x \in A_u$ for some ordinal u,*

where A_u is given by (8). We note that, as indicated in (14), the elementhood relation "ϵ" in (S, ϵ) is the same as in (K, ϵ).

We show below that (S, ϵ), as given by (14) is a Synergistic model for the Synergistic axiom system, with the finite number of instances of the axiom scheme of Replacement which corresponds to the finitely many binary predicates listed in (7). Moreover, we show that the Synergistic model (S, ϵ) is also a model for the axiom of Choice.

First of all, let us observe that, as mentioned above, A_u is a transitive set for every ordinal u. Therefore, *for every set x of (S, ϵ)* we have:

$$y \in x \text{ if and only if } y \in x \text{ in } (K, \epsilon). \tag{15}$$

Consequently,

$$x = y \text{ in } (S, \epsilon) \text{ if and only if } x = y \text{ in } (K, \epsilon). \tag{16}$$

However, since the axiom of Extensionality is valid in (K, ϵ), from (15) and (16) it follows that:

(i) *the axiom of Extensionality is valid in (S, ϵ).*

Now, for every set x of (K, ϵ), let

$$r(x) \tag{17}$$

denote *the smallest ordinal number such that*

$$x \in A_{r(x)}. \quad \text{Otherwise,} \quad r(x) = 0. \tag{18}$$

Next, let s be a set of (S, ϵ) and let $\mathfrak{P}(s)$ be the powerset of s in (K, ϵ). Using notation (17), let

$$p = \text{lub}\{r(x) |\ x \in \mathfrak{P}(s)\}. \tag{19}$$

But then, from (1), (6), (8), (18) and (19) it follows that there exists an element $P(s)$ of A_{p+1} such that $P(s)$ is the set of all the subsets of s that can be found in (S, ϵ). Thus, we have:

(ii) *the axiom of Powerset is valid in (S, ϵ).*

Next, let s be a set of (S, ϵ). Let $\bigcup s$ be the sumset of s in (K, ϵ). Since, for every ordinal u, the set A_u is transitive, using notation (17), from (2),

(6), (8) and (18) it follows that $\bigcup s$ is an element of $A_{r(s)+1}$. Thus, we have:

(iii) *the axiom of Sumset is valid in* (S, ϵ).

From A_0 in (8), we see that ω and all its elements are sets of (S, ϵ). But then from (15) and (16), we have:

(iv) *the axiom of Infinity is valid in* (S, ϵ).

Next, let s be a set of (S, ϵ) and let $F^i(x, y)$ with $i = 1, \ldots, n$ be functional in x on s, when all the bound variables of $F^i(x, y)$ range unrestrictedly over any A_u, i.e., when all the quantifiers appearing in $F^i(x, y)$ refer to (S, ϵ). However, since the axiom scheme of Replacement is valid in (K, ϵ) it follows that the set h of precisely those mates of the elements of s that can be found in (S, ϵ), exists in (K, ϵ). Thus, $r = \text{lub}\{r(y) |\ y \epsilon h\}$ exists. But then from Löwenheim–Skolem theorem [2] it follows that there exists an ordinal u such that $u \geqslant \max\{r, r(s)\}$ and

$$F^i(a, b) \text{ in } (S, \epsilon) \text{ if and only if } F^i(a, b) \text{ restricted to } A_u.$$

Based on the above, from (3), (6), (8) and (18) it then follows that there exists a set $F^i[s]$ of A_{u+1} such that $F^i[s]$ is the set of precisely the mates, in (S, ϵ), of all the elements of s with respect to $F^i(x, y)$, for $i = 1, \ldots, n$. Thus, we have:

(v) *the instances of the axiom scheme of Replacement corresponding to* $F^i(x, y)$, *for* $i = 1, \ldots, n$ *are valid in* (S, ϵ).

Finally, let us observe that in view of (12) and (13), we have:

(vi) *the axiom of Choice is valid in* (S, ϵ).

From (i) to (vi) it follows that (S, ϵ) is indeed a Synergistic model in which the axiom of Choice is valid. But since (7) is an arbitrary list of finitely many predicates, we see that the axiom of Choice is consistent with every Synergistic axiom system. However, this, as explained earlier, implies the conclusion of Proposition 1.

Next, let us recall that based on the axiom of Choice, the axiom of Regularity can be stated as: *there exists no infinite descending* ϵ*-chain such as*:

$$\ldots \epsilon e_4 \epsilon e_3 \epsilon e_2 \epsilon e_1. \tag{20}$$

Now, we prove the following proposition also established in [3]:

PROPOSITION 2. *The axiom of Regularity is consistent with the axioms of Extensionality, Powerset, Sumset, Infinity, Choice and the axiom scheme of Replacement.*

Proof. Let us consider the model (S, ϵ) described by (14). In the proof of Proposition 1, we showed that (S, ϵ) is a Synergistic model in which the axiom of Choice is valid.

We show below, that in (S, ϵ), the axiom of Regularity is also valid. Assume on the contrary, and, in view of (20), let u be the smallest ordinal such that there exists an infinite descending ϵ-chain such as

$$\ldots \epsilon s_4 \epsilon s_3 \epsilon s_2 \epsilon s_1 \quad \textit{with} \quad s_1 \epsilon A_u. \tag{21}$$

In view of (8), we see that

$$u = r(s_1) = v+1 \quad \textit{for some ordinal } v.$$

However, again, from (8) it follows that

$$s_2 \epsilon A_v \quad \textit{with} \quad v < u$$

so that

$$\ldots \epsilon s_4 \epsilon s_3 \epsilon s_2 \quad \textit{with} \quad s_2 \epsilon A_v.$$

Clearly, the above, in view of (21), contradicts the choice of u. Hence, our assumption is false and the axiom of Regularity is valid in the Synergistic model (S, ϵ). But since (7) is an arbitrary list of finitely many predicates, we see that the conjunction of the axioms of Choice and Regularity is consistent with every Synergistic axiom system. However, this, as explained earlier, implies the conclusion of Proposition 2.

References

[1] A. Mostowski, *Thirty Years of Foundational Studies*, New York 1966, p. 84.
[2] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, New York 1966, p. 81.
[3] J. von Neumann, *Über eine Widerspruchfreiheitsfrage in der axiomatischen Mengenlehre*, Journ. Reine Angew. Math. 160 (1929), p. 227-241.

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