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Musielak–Orlicz–Sobolev spaces on arbitrary metric space

Noureddine Aissaoui, Youssef Akdim, and My Cherif Hassib

Summary. In this article we define Musielak–Orlicz–Sobolev spaces on arbitrary metric spaces with finite diameter and equipped with finite, positive Borel regular outer measure. We employ a Hajlasz definition, which uses a pointwise maximal inequality. We prove that these spaces are Banach, that the Poincaré inequality holds, and that the Lipschitz functions are dense. We develop a capacity theory based on these spaces. We study basic properties of capacity and several convergence results. As an application, we prove that each Musielak–Orlicz–Sobolev function has a quasi-continuous representative.

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Metric measure space;
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1. Introduction

The theory of Sobolev spaces was originally developed in the domains of $\mathbb{R}^N$ using the notion of distributional derivatives. To generalize this theory to metric spaces, P. Hajlasz showed in [8] that a p-integrable function u, $1 < p < \infty$, belongs to $W^{1,p}(\mathbb{R}^N)$ if and only if there exists a non-negative p-integrable function g such that

$$\big|u(x) - u(y)\big| \leqslant |x - y|\big|g(x) + g(y)\big|$$

Noureddine Aissaoui, Ecole Normale Superieure, B.P 5206, Ben Souda, Fez, Morocco (*e-mail*: dinoaissaoui@gmail.com)

Youssef Akdim, Faculty Poly-Disciplinary of Taza, Laboratory, LSI, Morocco (*e-mail*: akdimyoussef@yahoo.fr)

My Cherif Hassib, SMBA University, Faculty of Science and Technique, Fez. Laboratory, LSI, Taza, Morocco (*e-mail*: cherif_hassib@yahoo.fr)

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for almost every $x, y \in \mathbb{R}^n$. This inequality can be stated also in metric measure spaces if $|x-y|$ is replaced by the distance between the points x and y. This theory was generalized by N. Aissaoui in [4] to the Orlicz–Sobolev spaces and by Petteri Harjulehto, Peter Hasto and Mikko Pere in [9] to the variable exponent spaces. The objective of this paper is to develop this theory in the setting of Musielak–Orlicz–Sobolev spaces.

The average value of u will be denoted by

$$u_A = \frac{1}{\mu(A)} \int_A u d\mu = \fint_A u d\mu.$$

The paper is organized as follows. In section 2 we recall some important definitions and results on Musielak–Orlicz spaces $L_\varphi(X)$; in Section 3, we define the Musielak–Orlicz––Sobolev space on a metric space, and we show that: it is a Banach space, the Poincaré inequality holds, and Lipschitz-continuous functions are dense. In Section 4, we develop a capacity theory based on this space; we study basic properties of capacity, including monotonicity and countable subadditivity, as well as several convergence results. As an application we prove that each Musielak–Orlicz–Sobolev function has a quasi-continuous representative.

2. Preliminary

Let (X, Σ, μ) be a σ-finite, complete measure space. A real function $\varphi: X \times [0, \infty) \to [0, \infty)$ is said to be a Musielak–Orlicz function or a generalized N-function, on (X, Σ, μ) if:

(i) $\varphi(x, \cdot)$ is an N-function for every $x \in X$ [convex, increasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for every $t > 0$ and $\frac{\varphi(x,t)}{t} \to 0$ as $t \to 0$, $\frac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$];

(ii) $\varphi(\cdot, t)$ is a measurable function.

The Musielak–Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(x, 2t) \leqslant K\varphi(x, t) \quad \text{for all } x \in X \text{ and } t \geqslant 0.$$

The smallest such K is called the Δ_2-constant of φ. When the last inequality holds only for $t \geqslant t_0$ for some $t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity. By $L^0(X, \mu)$ we denote the space of all μ-measurable functions on X. We define the functional

$$\rho_\varphi(f) = \int_X \varphi\big(y, |f(y)|\big) d\mu(y), \quad f \in L^0(X, \mu).$$

The set

$$K_\varphi(X) = \big\{ f \in L^0(X, \mu) : \rho_\varphi(f) < \infty \big\}$$

is called the Musielak–Orlicz class.

The Musielak–Orlicz space $L_\varphi(X)$ is the vector space generated by $K_\varphi(X)$. We have

$$L_\varphi(X) = \left\{ f \in L^0(X,\mu) \,:\, \rho_\varphi\left(\frac{|f(x)|}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\}.$$

$K_\varphi(X)$ is a convex subset of $L_\varphi(X)$.

The equality $K_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfies the Δ_2-condition for all t or for large t, according to whether X has infinite measure or not.

Let

$$\varphi^*(x,s) = \sup\{st - \varphi(x,t) \,:\, t \geqslant 0\}, \quad x \in X.$$

Thus φ^* is the Musielak–Orlicz function complementary to φ in the sense of Young with respect to the variable s.

In the space $L_\varphi(X)$ we define the following two norms:

$$\|f\|_\varphi = \inf\left\{ \lambda > 0 \,:\, \int_X \varphi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu \leqslant 1 \right\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||f|||_\varphi = \sup_{\|v\|_{\varphi^*} \leqslant 1} \int_X |f(x)v(x)| d\mu$$

where φ^* is the Musielak–Orlicz function complementary to φ. These two norms are equivalent [11].

We say that a sequence of functions $u_n \in L_\varphi(X)$ is modular convergent to $u \in L_\varphi(X)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \rho_\varphi\left(\frac{u_n - u}{k}\right) = 0.$$

If φ satisfies the Δ_2-condition; then modular convergence coincides with norm convergence.

The space $L_\varphi(X)$ is reflexive if and only if φ and φ^* satisfy the Δ_2-condition for all t or for large t, according to whether X has infinite measure or not.

2.1. Lemma ([7]). *Let φ be a Musielak–Orlicz function and $f_n, f, g \in L^0(X,\mu)$*

(i) *If $f_n \to f$ a.e. then $\rho_\varphi(f) \leqslant \liminf_{n\to\infty} \rho_\varphi(f_n)$.*

(ii) *If $|f_n| \nearrow |f|$ a.e. then $\rho_\varphi(f) = \lim_{n\to\infty} \rho_\varphi(f_n)$.*

(iii) *If $f_n \to f$ a.e. $|f_n| \leqslant |g|$ a.e, and $\rho_\varphi(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ in $L_\varphi(X)$.*

2.2. Theorem ([7]). *Let $\varphi \in N(X,\mu)$*

(i) *$\|f\|_\varphi = \||f|\|_\varphi$ for all $f \in L_\varphi(X)$.*

(ii) *If* $f \in L_\varphi(X)$, $g \in L^0(X,\mu)$ *and* $0 \leqslant |g| \leqslant |f|$ *a.e. then* $g \in L_\varphi(X)$ *and* $\|g\|_\varphi \leqslant \|f\|_\varphi$.

(iii) *If* $f_n \to f$ *a.e. then* $\|f\|_\varphi \leqslant \liminf_{n\to\infty} \|f_n\|_\varphi$.

(iv) *If* $|f_n| \nearrow |f|$ *a.e. with* $f_n \in L_\varphi(X)$ *and* $\sup_n \|f_n\|_\varphi < \infty$ *then* $f \in L_\varphi(X)$ *and* $\|f_n\|_\varphi \nearrow \|f\|_\varphi$.

3. Musielak–Orlicz–Sobolev spaces on arbitrary metric space

Let (X,d) be a metric space with finite diameter ($\operatorname{diam}(X) = \sup_{x,y\in X} d(x,y) < \infty$) and equipped with a finite, positive Borel regular outer measure μ. The triplet(X,d,μ) will be fixed in the sequel and will be denoted by X.

Let $u\colon X \to [-\infty,\infty]$ be a μ-measurable functions defined on X. We denote by $D(u)$ the set of all μ-mesurable functions $g\colon X \to [0,\infty]$ such that

$$\big|u(x) - u(y)\big| \leqslant d(x,y)\big(g(x) + g(y)\big) \tag{1}$$

for every $x, y \in X \setminus F, x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g, and g is called the Hajlasz gradient of u.

Note that the right-hand side of (1) is always defined for $x \neq y$. For the points $x, y \in X$, $x \neq y$ such that the left-hand side of (1) is undefined, we may assume that the left-hand side is ∞.

3.1. Definition. Let φ be a Musielak–Orlicz function. The Dirichlet–Musielak–Orlicz space $L^1_{\varphi,\mu}(X)$ is the space of all μ-measurable functions u such that $D(u) \cap L_\varphi(X) \neq \varnothing$. This space is equipped with the semi-norm

$$\|u\|_{L^1_{\varphi,\mu}(X)} = \inf\big\{\|g\|_\varphi \,:\, g \in D(u) \cap L_\varphi(X)\big\}.$$

The Musielak–Orlicz–Sobolev space on an arbitrary metric space X is $M^1_{\varphi,\mu}(X) = L_\varphi(X) \cap L^1_{\varphi,\mu}(X)$, equipped with the norm

$$\|u\|_{M^1_{\varphi,\mu}(X)} = \|u\|_\varphi + \|u\|_{L^1_{\varphi,\mu}(X)}.$$

3.2. Lemma. *Let* φ *be a Musielak–Orlicz function such that* φ *and* φ^* *satisfy the* Δ_2*-condition. For every* $u \in M^1_{\varphi,\mu}(X)$ *there is a unique* $g \in D(u) \cap L_\varphi(X)$ *denoted by* g_u *such that*

$$\|g_u\|_\varphi = \|u\|_{L^1_{\varphi,\mu}(X)}.$$

Proof. Let $(g_i) \subset D(u) \cap L_\varphi(X)$ be a sequence such that

$$\|u\|_{L^1_{\varphi,\mu}(X)} = \lim_{i\to\infty} \|g_i\|_\varphi.$$

Since $L_\varphi(X)$ is reflexive, there is a subsequence denoted again by (g_i) and a function $g_u \in L_\varphi(X)$ such that $g_i \to g_u$ weakly in $L_\varphi(X)$. By Mazur's lemma there exist convex combinations $f_k = \sum_{i=k}^{m_k} \alpha_{k_i} g_i$ such that $f_k \to g_u$ strongly in $L_\varphi(X)$, where $\alpha_{k_i} \geqslant 0$ and $\sum_{i=k}^{m_k} \alpha_{k_i} = 1$.

Since every $g_i \in D(u) \cap L_\varphi(X)$, we have

$$\begin{aligned}\big|u(x) - u(y)\big| &= \sum_{i=k}^{i=m_k} \alpha_{k_i} \big|u(x) - u(y)\big| \\ &\leqslant \sum_{i=k}^{i=m_k} \alpha_{k_i} d(x;y)\big(g_i(x) + g_i(y)\big) \\ &= d(x;y)\big(f_k(x) + f_k(y)\big)\end{aligned}$$

for μ-almost all $x, y \in X$. Therefore $f_k \in D(u) \cap L_\varphi(X)$ for all k. On the other hand, there exists a subsequence of (f_k) which converges to g_u pointwise μ-almost everywhere in X, thus $g_u \in D(u) \cap L_\varphi(X)$.

Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ such that

$$\|g_i\|_\varphi \leqslant \|u\|_{L^1_{\varphi,\mu}(X)} + \varepsilon, \quad i \geqslant k.$$

For this k we have

$$\|f_k\|_\varphi \leqslant \sum_{i=k}^{m_k} \alpha_{k_i} \|g_i\|_\varphi.$$

Therefore

$$\|f_k\|_\varphi \leqslant \|u\|_{L^1_{\varphi,\mu}(X)} + \varepsilon.$$

Thus

$$\|g_u\|_\varphi = \|u\|_{L^1_{\varphi,\mu}(X)}.$$

Now suppose that there exist two minimal g_1 and g_2 with $g_1 \neq g_2$, so that $\|g_1 - g_2\|_\varphi > 0$.

Let $G_1 = \frac{g_1}{\|g_1\|_\varphi}$ and $G_2 = \frac{g_2}{\|g_2\|_\varphi}$. By the uniform convexity of the norm we have

$$\|G_1 + G_2\|_\varphi < 2;$$

then

$$\left\|\tfrac{1}{2}g_1 + \tfrac{1}{2}g_2\right\|_\varphi < \|g_1\|_\varphi$$

which is the contradiction since

$$\left(\tfrac{1}{2}g_1 + \tfrac{1}{2}g_2\right) \in D(u) \cap L_\varphi(X).$$

□

With simple verification, we obtain the following lemma.

3.3. Lemma. *Let $g_1 \in D(u_1)$, $g_2 \in D(u_2)$ and $\alpha, \beta \in \mathbb{R}$. If $g \geqslant |\alpha| g_1 + |\beta| g_2$ μ-a.e., then $g \in D(\alpha u_1 + \beta u_2)$.*

3.4. Lemma. *Let φ a Musielak–Orlicz function and (u_n) and (g_n) sequences of functions such that*

$$g_n \in D(u_n), \quad n \in \mathbb{N}.$$

If $u_n \to u$ in $L_\varphi(X)$ and $g_n \to g$ in $L_\varphi(X)$, then $g \in D(u)$.

Proof. There are subsequences of (u_n) and (g_n) which we denote again by (u_n) and (g_n) such that $u_n \to u$ and $g_n \to g$ μ-a.e. For all $n \in \mathbb{N}$, let F_n be the exceptional set for g_n, H a set of measure zero such that $u_n \to u$ and $g_n \to g$ on H^c, and $F = H \cup (\bigcup_n F_n)$. We have $\mu(F) = 0$ and

$$\big|u(x) - u(y)\big| \leqslant d(x,y)\big|g(x) + g(y)\big|, \quad x, y \in X \setminus F.$$

Therefore $g \in D(u)$. □

3.5. Theorem. *Let φ be a Musielak–Orlicz function. Then $(M^1_{\varphi,\mu}(X), \|\cdot\|_{M^1_{\varphi,\mu}})$ is a Banach space.*

Proof. It is clear that $M^1_{\varphi,\mu}(X)$ is a vector space.

Let (u_n) be a Cauchy sequence in $M^1_{\varphi,\mu}(X)$. Then (u_n) is a Cauchy sequence in $L_\varphi(X)$, and hence there exists a function $u \in L_\varphi(X)$ such that $u_n \to u$ in $L_\varphi(X)$. We can choose a subsequence denoted again (u_n) such that $\|u_{n+1} - u_n\|_{M^1_{\varphi,\mu}} < 2^{-n}$ and $u_n \to u$ μ-a.e. in X. There exist non-negative functions $g_n \in L_\varphi(X)$ such that $\|g_n\|_\varphi < 2^{-n}$ and

$$\big|(u_{n+1} - u_n)(x) - (u_{n+1} - u_n)(y)\big| \leqslant d(x,y)\big(g_n(x) + g_n(y)\big) \quad \mu\text{-a.e. in } X.$$

By adding the last inequalities, we have for $n > m$,

$$\big|(u_n - u_m)(x) - (u_n - u_m)(y)\big| \leqslant d(x,y)\left(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y)\right) \quad \mu\text{-a.e. in } X.$$

Letting $n \to \infty$ yields

$$\big|(u - u_m)(x) - (u - u_m)(y)\big| \leqslant d(x,y)\left(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y)\right) \quad \mu\text{-a.e. in } X.$$

Since

$$\left\|\sum_{k=m}^{\infty} g_k\right\|_\varphi \leqslant \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$$

the previous inequality implies that $(u - u_m) \in M^1_{\varphi,\mu}(X)$ and therefore $u \in M^1_{\varphi,\mu}(X)$.

On the other hand, we have

$$\|u - u_m\|_{L^1_{\varphi,\mu}(X)} \leqslant \left\| \sum_{k=m}^{\infty} g_k \right\|_{\varphi} \leqslant 2^{-m+1},$$

then $u_m \to u$ in $M^1_{\varphi,\mu}(X)$. □

3.6. Lemma. *Let φ be a Musielak–Orlicz function. The function u belongs to $M^1_{\varphi,\mu}(X)$ if and only if $u \in L_\varphi(X)$ and there exist sequences $(u_n) \subset L_\varphi(X)$ and $(g_n) \subset D(u_n) \cap L_\varphi(X)$ such that $u_n \to u$ μ-a.e. and $g_n \to g$ μ-a.e. for some $g \in L_\varphi(X)$.*

Proof. The same as in [10, Lemma 2.5]. □

3.7. Lemma. *Let φ be a Musielak–Orlicz function and u_1, $u_1 \in M^1_{\varphi,\mu}(X)$. If $g_1 \in D(u_1)$ and $g_2 \in D(u_2)$, then:*

(i)
$$u = \max(u_1, u_2) \in M^1_{\varphi,\mu}(X) \quad \text{and} \quad g = \max(g_1, g_2) \in D(u) \cap L_\varphi(X).$$

(ii)
$$v = \min(u_1, u_2) \in M^1_{\varphi,\mu}(X) \quad \text{and} \quad h = \max(g_1, g_2) \in D(v) \cap L_\varphi(X).$$

Proof. We only prove the case (i), as the proof of (ii) is similar.

Let F_1 and F_2 the exceptional sets for g_1 and g_2, respectively. Clearly $u, g \in L_\varphi(X)$. It remains to prove that $g \in D(u)$. Let $G = \{x \in X \setminus (F_1 \cup F_2) : u_1(x) \geqslant u_2(x)\}$. If $x, y \in G$, then

$$\big|u(x) - u(y)\big| = \big|u_1(x) - u_1(y)\big| \leqslant d(x,y)\big(g_1(x) + g_1(y)\big).$$

Analogously, for $x, y \in X \setminus G$, we obtain $|u(x) - u(y)| \leqslant d(x,y)(g_2(x) + g_2(y))$.

For the remaining cases, let $x \in G$ and $y \in X \setminus G$. If $u_1(x) \geqslant u_2(y)$, then

$$\big|u(x) - u(y)\big| = u_1(x) - u_2(y) \leqslant u_1(x) - u_1(y) \leqslant d(x,y)\big(g_1(x) + g_1(y)\big).$$

If $u_1(x) < u_2(y)$, then

$$\big|u(x) - u(y)\big| = u_2(y) - u_1(x) \leqslant u_2(y) - u_2(x) \leqslant d(x,y)\big(g_2(x) + g_2(y)\big).$$

The case $x \in X \setminus G$ and $y \in G$ follows by symmetry and hence

$$\big|u(x) - u(y)\big| \leqslant d(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in X \setminus (F_1 \cup F_2)$, with $\mu(F_1 \cup F_2) = 0$. □

3.8. Theorem (The Poincare inequality). *Let φ be a Musielak–Orlicz function such the function $x \mapsto 1$ belongs to $L_\varphi(X) \cap L_{\varphi^*}(X)$. If $u \in M^1_{\varphi,\mu}(X)$, then for every $g \in D(u) \cap L_\varphi(X)$*

$$\|u - u_X\|_\varphi \leqslant C(\varphi)\,\mathrm{diam}(X)\,\|g\|_\varphi.$$

Proof. Integrating the inequality $|u(x)-u(y)| \leqslant d(x,y)(g(x)+g(y))$ over y, we obtain

$$\left|u(x)-\fint_X u(y)d\mu(y)\right| \leqslant \fint_X |u(x)-u(y)|d\mu(y) \leqslant \operatorname{diam}(X)\left(g(x)+\fint_X g(y)d\mu(y)\right).$$

Applying Hölder's inequality, we have

$$|u(x)-u_X| \leqslant \operatorname{diam}(X)\left(g(x)+\frac{C}{\mu(X)}\|1\|_{\varphi^*}\|g\|_\varphi\right).$$

And finally by (ii) of Theorem 2.2,

$$\|u-u_X\|_\varphi \leqslant \operatorname{diam}(X)\left(\|g\|_\varphi+\frac{C}{\mu(X)}\|1\|_{\varphi^*}\|1\|_\varphi\|g\|_\varphi\right) \leqslant C(\varphi)\operatorname{diam}(X)\|g\|_\varphi.$$

□

3.9. Theorem. *Let φ a Musielak–Orlicz function satisfying the Δ_2-condition. For every $u \in M^1_{\varphi,\mu}(X)$ and every $\varepsilon > 0$ there exists a Lipschitz function $h \in M^1_{\varphi,\mu}(X)$ such that*

(i) $\mu\big(\{x \in X : u(x) \neq h(x)\}\big) \leqslant \varepsilon$.

(ii) $\|u-h\|_{M^1_{\varphi,\mu}(X)} \leqslant \varepsilon$.

Proof. We fix $u \in M^1_{\varphi,\mu}(X)$ and denote by $g \in L_\varphi(X)$ a Hajlasz gradient of u. We write

$$E_n = \big\{x \in X : |u(x)| \leqslant n \text{ and } g(x) \leqslant n\big\}.$$

We have $u, g \in L_\varphi(X)$, and φ satisfies the Δ_2-condition, hence $u, g \in K_\varphi(X)$. Furthermore, for all $y \in E_n^C$

$$\varphi(y,n) < \varphi\big(y, |u(y)|+g(y)\big).$$

Since $\lim_{n\to\infty} \frac{\varphi(y,n)}{n} = \infty$, we have that for all $A > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the following inequality holds $\frac{\varphi(y,n)}{n} > A$. Thus

$$\mu(E_n^c) \leqslant \frac{C}{n}\int_{E_n^C} \varphi(y,n)d\mu(y) \leqslant \frac{C}{n}\int_X \varphi\big(y,|u(y)|+g(y)\big)d\mu(y),$$

so that

$$\lim_{n\to\infty} \mu(E_n^c) = 0.$$

On the other hand, we have $0 \leqslant \chi_{E_n^C}\varphi(y,n) \leqslant \chi_{E_n^C}\varphi(y,|u(y)|+g(y))$, hence

$$\lim_{n\to\infty} \chi_{E_n^C}\varphi(y,n) = 0,$$

and since $\chi_{E_n^C}\varphi(y,n) < \varphi(y,|u(y)|+g(y))$, by dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{E_n^C} \varphi(y,n)d\mu(y) = 0.$$

The restriction $u|_{E_n}$ is Lipschitz with the constant $2n$. We can extend $u|_{E_n}$ by McShane extention [11] to all of X as a Lipschitz function by defining

$$\bar{u}(x) = \inf_{y\in E_n}\big\{u|_{E_n}(y) + 2n\,\mathrm{dist}(x,y)\big\}.$$

We put

$$u_n = (\operatorname{sign}\bar{u})\min\big(|\bar{u}|, n\big).$$

It is clear that u_n is Lipschitz with constant $2n$, $u|_{E_n} = u_n|_{E_n}$, $|u_n| \leqslant n$, and for all $n \geqslant 1$

$$\mu\big(\{x : u(x) \neq u_n(x)\}\big) \leqslant \mu(E_n^c) \to 0 \quad \text{as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned}\int_X \varphi\big(y, |u-u_n|(y)\big)d\mu(y) &= \int_{E_n^c} \varphi\big(y, |u-u_n|(y)\big)d\mu(y)\\ &\leqslant \int_{E_n^c} \varphi\big(y, |u|(y) + |u_n|(y)\big)d\mu(y).\end{aligned}$$

Since $\varphi(y,\cdot)$ is convex and φ satisfies the Δ_2-condition, then

$$\int_X \varphi\big(y, |u-u_n|(y)\big)d\mu(y) \leqslant \frac{C_\varphi}{2}\left[\int_{E_n^c} \varphi\big(y, |u|(y)\big)d\mu(y) + \int_{E_n^c} \varphi\big(y, |u_n|(y)\big)d\mu(y)\right],$$

thus

$$\int_X \varphi\big(y, |u-u_n|(y)\big)d\mu(y) \leqslant \frac{C_\varphi}{2}\left[\int_{E_n^c} \varphi\big(y, |u|(y)\big)d\mu(y) + \int_{E_n^c} \varphi(y,n)d\mu(y)\right].$$

Therefore $\int_X \varphi\big(y, |u-u_n|(y)\big)d\mu(y) \to 0$ as $n \to \infty$.

Since φ satisfies the Δ_2-condition, $\|u-u_n\|_\varphi \to 0$ as $n \to \infty$. It remains to estimate the gradient. Let

$$g_n = \begin{cases} 0 & \text{for } x \in E_n,\\ g(x) + 3n & \text{for } x \in E_n^c.\end{cases}$$

It is easy to check that

$$\big|(u-u_n)(x) - (u-u_n)(y)\big| \leqslant d(x,y)\big(g_n(x) + g_n(y)\big) \quad \text{for } \mu\text{-almost all } x, y \in X.$$

We have $\int_X \varphi(y, g_n(y))d\mu(y) = \int_{E_n^c} \varphi(y, g_n(y))d\mu(y)$. Since $\varphi(y,\cdot)$ is convex and φ satisfies the Δ_2-condition,

$$\int_{E_n^c} \varphi\big(y, g_n(y)\big)d\mu(y) \leqslant \frac{C_\varphi}{2}\left[\int_{E_n^c} \varphi\big(y, g(y)\big)d\mu(y) + \int_{E_n^c} \varphi(y,3n)d\mu(y)\right],$$

then $\|g_n\|_\varphi \to 0$ as $n \to \infty$. □

4. Capacity on Musielak–Orlicz–Sobolev space on metric space

4.1. Definition. Let T be a Borel σ-algebra of subsets of X and $C\colon T \to [0,\infty]$ a function.

(i) C is called a capacity if the following axioms are satisfied:
   a) $C(\varnothing) = 0$.
   b) $E \subset F \Rightarrow C(E) \leqslant C(F)$ for all E and F in T.
   c) for all sequences $(E_n) \subset T$

$$C\Big(\bigcup_n E_n\Big) \leqslant \sum_n C(E_n).$$

(ii) C is called an outer capacity if for all $E \in T$:

$$C(E) = \inf\big\{C(O) \,:\, O \supset E,\ O \text{ open}\big\}.$$

(iii) A property that holds true except perhaps on a set of capacity zero is said to be true C-quasi everywhere (abbreviated C-q.e).

(iv) Let f and (f_n) be real-valued functions finite C-q.e. We say that (f_n) converges to f in C-capacity if for any $\varepsilon > 0$

$$\lim_{n\to\infty} C\Big(\big\{x \,:\, \big|f_n(x) - f(x)\big| > \varepsilon\big\}\Big) = 0.$$

(v) Let f and (f_n) be real-valued functions finite C-q.e. We say that (f_n) converges to f C-quasi uniformly (abbreviated C-q.u.) if for any $\varepsilon > 0$ there exists $X \in T$ such that

$$C(X) < \varepsilon \quad \text{and} \quad (f_n) \text{ converges to } f \text{ uniformly on } X^c.$$

4.2. Definition. Let φ be a Musielak–Orlicz function. The Sobolev φ-capacity of the set $E \subset X$ is defined by

$$C_{\varphi,\mu}(E) = \inf_{u\in A_\varphi(E)} \|u\|_{M^1_{\varphi,\mu}},$$

where

$$A_\varphi(E) = \big\{u \in M^1_{\varphi,\mu}(X) \,:\, u \geqslant 1 \text{ on an open set containing } E \text{ and } u \geqslant 0\big\}.$$

If $A_\varphi(E) = \varnothing$ we set $C_{\varphi,\mu}(E) = \infty$. The functions belonging to $A_\varphi(E)$ are called the admissible functions for E.

4.3. Remark. In the defition of $C_{\varphi,\mu}(E)$ we may restrict ourselves to those admissible functions u for which $0 \leqslant u \leqslant 1$.

Proof. If $A'_\varphi(E) = \big\{u \in A_\varphi(E) : 0 \leqslant u \leqslant 1\big\}$, then $A'_\varphi(E) \subset A_\varphi(E)$ implies

$$C_{\varphi,\mu}(E) \leqslant \inf_{u\in A'_\varphi(E)} \|u\|_{M^1_{\varphi,\mu}}.$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that $\|u\|_{M^1_{\varphi,\mu}} < C_{\varphi,\mu}(E) + \varepsilon$. Then by Lemma 3.7, we have $v = \max(0, \min(u, 1))$ belongs to $A'_\varphi(E)$. Therefore

$$\inf_{\omega \in A'_\varphi(E)} \|\omega\|_{M^1_{\varphi,\mu}} \leqslant \|v\|_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\inf_{\omega \in A'_\varphi(E)} \|\omega\|_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E).$$

This completes the proof. □

4.4. Theorem. *Let φ be a Musielak–Orlicz function. $C_{\varphi,\mu}$ is an outer capacity.*

Proof. It is obvious that $C_{\varphi,\mu}(\varnothing) = 0$ and $C_{\varphi,\mu}(E) \leqslant C_{\varphi,\mu}(F)$ if $E \subset F$. Let $(E_i) \subset X$ be such that $\sum_i C_{\varphi,\mu}(E_i) < \infty$. Then $C_{\varphi,\mu}(E_i) < \infty$ for each $i \in \mathbb{N}$.

Therefore for every $\varepsilon > 0$ there exist $u_i \in A_\varphi(E_i)$ and $g_{u_i} \in D(u_i) \cap L_\varphi(X)$ so that

$$\|u_i\|_\varphi + \|g_{u_i}\|_\varphi \leqslant C_{\varphi,\mu}(E_i) + \varepsilon 2^{-i}.$$

We show that $v = \sup_i u_i$ is an admissible function for $\bigcup_{i=1}^\infty E_i$ and $g = \sup_i g_{u_i} \in D(v) \cap L_\varphi(X)$. Let $v_k = \max_{1 \leqslant i \leqslant k} u_i$. By Lemma 3.7 the function $g_{v_k} = \max_{1 \leqslant i \leqslant k} g_{u_i}$ belongs to $D(v_k) \cap L_\varphi(X)$. Since $v_k \to v$ μ-a.e. and $g_{v_k} \to g$ μ-a.e., Lemma 3.6 gives $v \in M^1_{\varphi,\mu}$.

Clearly $v \geqslant 1$ in a neighbourhood of $\bigcup_{i=1}^\infty E_i$. Then

$$C_{\varphi,\mu}\left(\bigcup_{i=1}^\infty E_i\right) \leqslant \|v\|_{M^1_{\varphi,\mu}} \leqslant \sum_{i=1}^\infty \big(\|u_i\|_\varphi + \|g_{u_i}\|_\varphi\big) \leqslant \sum_{i=1}^\infty C_{\varphi,\mu}(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. Hence $C_{\varphi,\mu}$ is a capacity.

It remains to prove that $C_{\varphi,\mu}$ is outer. Indeed,

$$C_{\varphi,\mu}(E) \leqslant \inf\big\{C_{\varphi,\mu}(O) : E \subset O,\ O \text{ open}\big\}.$$

For the reverse inequality, let $\varepsilon > 0$ and $u \in A_\varphi(E)$ such that $\|u\|_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon$. Since $u \in A_\varphi(E)$, there is an open set O, $E \subset O$, such that $u \geqslant 1$ on O. Then $C_{\varphi,\mu}(O) \leqslant \|u\|_{M^1_{\varphi,\mu}} \leqslant C_{\varphi,\mu}(E) + \varepsilon$. The claim follows by letting $\varepsilon \to 0$. □

4.5. Theorem. *If (K_n) is a decreasing sequence of compact sets and $K = \bigcap_n K_n$, then*

$$\lim_{n\to\infty} C_{\varphi,\mu}(K_n) = C_{\varphi,\mu}(K).$$

Proof. First, we observe that $\lim_{n\to\infty} C_{\varphi,\mu}(K_n) \geqslant C_{\varphi,\mu}(K)$. Let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore

$$\lim_{n\to\infty} C_{\varphi,\mu}(K_n) \leqslant C_{\varphi,\mu}(O)$$

and since $C_{\varphi,\mu}$ is an outer capacity, we obtain the claim by taking infimum over all open sets O containing K. □

4.6. Theorem. *Let $E \subset X$. If there exists $f \in M^1_{\varphi,\mu}(X)$ such that $f = \infty$ on an open set containing E, then $C_{\varphi,\mu}(E) = 0$.*

Proof. If there exists $f \in M^1_{\varphi,\mu}(X)$ such that $f = \infty$ on an open set O containing E, then $f \geqslant \alpha$ on O for all $\alpha > 0$. Therefore, $C_{\varphi,\mu}(E) \leqslant \alpha^{-1}\|f\|_{M^1_{\varphi,\mu}}$ for every $\alpha > 0$. Letting $\alpha \to \infty$, we obtain $C_{\varphi,\mu}(E) = 0$. □

4.7. Theorem. *Consider the following propositions:*

(i) $f_n \to f$ *in* $M^1_{\varphi,\mu}(X)$ *strongly.*

(ii) $f_n \to f$ *in* $C_{\varphi,\mu}$*-capacity.*

(iii) *There is a subsequence* (f_{n_j}) *such that* $f_{n_j} \to f$ $C_{\varphi,\mu}$*-q.u.*

(iv) $f_{n_j} \to f$ $C_{\varphi,\mu}$*-q.e.*

We have (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Proof. Let us show that (i) ⇒ (ii). By Theorem 4.6, we have f and f_n are finite $C_{\varphi,\mu}$-q.e. for every n. Let $\varepsilon > 0$. Then

$$C_{\varphi,\mu}\big(\{x : |f_n - f|(x) > \varepsilon\}\big) \leqslant \frac{1}{\varepsilon}\|f - f_n\|_{M^1_{\varphi,\mu}}.$$

For (ii) ⇒ (iii), let $\varepsilon > 0$. Then there exists a subsequence (f_{n_j}) of (f_n) such that

$$C_{\varphi,\mu}\big(\{x : |f_{n_j} - f|(x) > 2^{-j}\}\big) < \varepsilon 2^{-j}.$$

We put

$$E_j = \{x : |f_{n_j} - f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j.$$

We have $C_{\varphi,\mu}(G_m) \leqslant \sum_{j \geqslant m} \varepsilon 2^{-j} < \varepsilon$.

On the other hand for any $x \in G_m^c$ and $j \geqslant m$

$$|f_{n_j} - f|(x) \leqslant 2^{-j}.$$

Thus $f_{n_j} \to f$ $C_{\varphi,\mu}$-q.u.

Finally, let us show that (iii) ⇒ (iv). We have that for all $j \in \mathbb{N}$ there exists X_j such that $C_{\varphi,\mu}(X_j) \leqslant \frac{1}{j}$ and $f_{n_j} \to f$ on X_j^c. We put $X = \bigcap_j X_j$, then $C_{\varphi,\mu}(X) = 0$ and $f_{n_j} \to f$ on X^c. □

4.8. Theorem. *Let φ be a Musielak–Orlicz function such that φ and φ^* satisfy the Δ_2-condition. If $f_n, f \in L_\varphi(X)$ are such that $f_n \to f$ weakly in $L_\varphi(X)$ then*

$$\liminf f_n \leqslant f \leqslant \limsup f_n \quad C_{\varphi,\mu}\text{-q.e.}$$

Proof. The space $(L_\varphi(X), \|\cdot\|)$ is reflexive. By the Banach–Saks theorem, there is a subsequence denoted again (f_n) such that the sequence $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converges to f strongly in $L_\varphi(X)$. By Theorem 4.7, there is a subsequence of (g_n) denoted again (g_n) such that

$$\lim_{n\to\infty} g_n = f \quad C_{\varphi,\mu}\text{-q.e.}$$

On the other hand,

$$\liminf f_n \leqslant \lim_{n\to\infty} g_n.$$

Therefore,

$$\liminf f_n \leqslant f \quad C_{\varphi,\mu}\text{-q.e.}$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality. □

4.9. Theorem. *Let φ be a Musielak–Orlicz function such that φ and φ^* satisfy the Δ_2-condition. Let (O_n) be an increasing sequence of open subsets of X and $O = \bigcup_n O_n$, then*

$$\lim_{n\to\infty} C_{\varphi,\mu}(O_n) = C_{\varphi,\mu}(O).$$

Proof. We have $\lim_{n\to\infty} C_{\varphi,\mu}(O_n) \leqslant C_{\varphi,\mu}(O)$. For the reverse inequality, if

$$\lim_{n\to\infty} C_{\varphi,\mu}(O_n) = \infty$$

there is nothing to show.

Assuming that $\lim_{n\to\infty} C_{\varphi,\mu}(O_n) < \infty$, we have that for any $n \in \mathbb{N}$ there exists $f_n \in M^1_{\varphi,\mu}(X)$ such that

$$f_n \geqslant 1 \text{ on } O_n \quad \text{and} \quad \|f_n\|_{M^1_{\varphi,\mu}(X)} \leqslant C_{\varphi,\mu}(O_n) + \frac{1}{n}.$$

Then

$$\|f_n\|_\varphi + \|g_n\|_\varphi \leqslant C_{\varphi,\mu}(O_n) + \frac{1}{n}.$$

Where $g_n \in D(f_n) \cap L_\varphi(X)$. Now (f_n) and (g_n) are bounded sequences in $L_\varphi(X)$, hence there exist subsequences, which we denote again by (f_n) and (g_n), such that $f_n \to f$ and $g_n \to g$ weakly in $L_\varphi(X)$. Thus

$$\|f\|_\varphi \leqslant \liminf_n \|f_n\|_\varphi \quad \text{and} \quad \|g\|_\varphi \leqslant \liminf_n \|g_n\|_\varphi.$$

Hence

$$\|f\|_{M^1_{\varphi,\mu}(X)} \leqslant \lim_{n\to\infty} C_{\varphi,\mu}(O_n).$$

On the other hand, by Theorem 4.8, we have for any $n \in \mathbb{N}$

$$f \geqslant 1 \quad \text{on } O_n \; C_{\varphi,\mu}\text{-q.e.}$$

Therefore, $f \geqslant 1$ on O $C_{\varphi,\mu}$-q.e. Let Y be a subset of O where $f \geqslant 1$, then $C_{\varphi,\mu}(O) = C_{\varphi,\mu}(Y)$. Thus

$$C_{\varphi,\mu}(O) \leqslant \lim_{n\to\infty} C_{\varphi,\mu}(O_n).$$

□

4.10. Theorem. *Let φ be a Musielak–Orlicz function such that the function $x \mapsto 1$ belongs to $L_{\varphi^*}(X)$. We have that there exists $\alpha > 0$ such that for any $E \subset X$*

$$\mu(E) \leqslant \alpha C_{\varphi,\mu}(E).$$

Proof. Let $E \subset X$ and $f \in A_\varphi(E)$. Then $f \geqslant 1$ on an open set O containing E and $f \geqslant 0$. We have

$$\mu(E) \leqslant \mu(O) \leqslant \int_O f(x)d\mu(x) \leqslant \int_X f(x)d\mu(x).$$

By Hölder's inequality,

$$\int_X f(x)d\mu(x) \leqslant 2\|f\|_\varphi \|1\|_{\varphi^*}.$$

Therefore

$$\mu(E) \leqslant 2\|1\|_{\varphi^*} \|f\|_{M^1_{\varphi,\mu}(X)}.$$

We obtain the claim by taking infimum over $f \in A(E)$. □

4.11. Corollary. *Let φ be a Musielak–Orlicz function such the function $x \mapsto 1$ belongs to $L_{\varphi^*}(X)$. If (f_n) is a sequence which converges to f in $M^1_{\varphi,\mu}(X)$, then there exists a subsequence of (f_n) which converges to f μ-a.e.*

Proof. It is an immediate consequence of Theorem 4.7 and Theorem 4.10. □

4.12. Theorem. *Let φ be a Musielak–Orlicz function satisfying the Δ_2-condition and such the function $x \mapsto 1$ belongs to $L_{\varphi^*}(X)$. For each $f \in M^1_{\varphi,\mu}(X)$, there is a $C_{\varphi,\mu}$-quasi continuous function $g \in M^1_{\varphi,\mu}(X)$ such that $f = g$ μ-a.e.*

Proof. Let $f \in M^1_{\varphi,\mu}(X)$. By Theorem 3.9, there exists a sequence (f_n) of Lipschitz functions such that $f_n \to f$ in $M^1_{\varphi,\mu}(X)$. By Theorem 4.7, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \to f$ C_φ-q.u. The claim now follows by Theorem 4.10. □

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