On the dependence of parameters in the equivalence theorem for the real method

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Summary. We determine the exact dependence on $\theta, q, p$ of the constants in the equivalence theorem for the real interpolation method $(A_0, A_1)_{\theta,q}$ with pairs of $p$-normed spaces.

Keywords
real interpolation; $K$-functional; $J$-functional; $p$-normed spaces

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 Dedicated to Professor Henryk Hudzik on the occasion of his 70th birthday.

1. Introduction

The real interpolation method $(A_0, A_1)_{\theta,q}$ is very useful in applications of interpolation theory to function spaces, PDEs, operator theory and approximation theory (see, for example, the books by Butzer and Berens [4], Bergh and Löfström [2], Triebel [15,16], König [12], Bennett and Sharpley [1] or Brudnyi and Krugljak [3]). It can be realized either by means

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of Peetre’s $K$-functional $(A_0, A_1)_{θ,q,K}$ or $J$-functional $(A_0, A_1)_{θ,q,J}$, the dual functional to $K$.

In order to be able to combine interpolation techniques with those of extrapolation theory, it is important to work with normalized scales of real interpolation spaces (see, for example, the papers by Jawerth and Milman [9], Milman [13], Karadzhov and Milman [10], Fiorenza and Karadzhov [8] or Cobos and Kühn [7]). In the case of the $K$-scale $(A_0, A_1)_{θ,q,K}$ generated by a Banach couple, this is achieved by multiplying the norm by the factor $(θ(1−θ)q)^{1/q}$. This modification ensures that the embeddings

$$A_0 ∩ A_1 \rightarrow (A_0, A_1)_{θ,q,K} \rightarrow A_0 + A_1$$

and

$$(A_0, A_1)_{θ,q,K} \rightarrow (A_0, A_1)_{θ,r,K} \quad (q < r)$$

have uniformly bounded quasi-norms with respect to $θ$.

It is also of interest to determine the exact dependence on the parameters $θ, q$ for the quasi-norms of the embeddings between $K$-spaces and $J$-spaces, and vice versa. Accordingly, we study such problems here for couples of $p$-normed spaces with $0 < p ≤ 1$.

2. Preliminaries

Let $A$ be a vector space, and let $\| \cdot \|_s$ be a quasi-norm on $A$ with a constant $c ≥ 1$ in the quasi-triangle inequality. Let $0 < p ≤ 1$ be such that $c = 2^{1/p−1}$. It is well known that there is a $p$-norm $\| \cdot \|$ on $A$ equivalent to $\| \cdot \|_s$ (see [11, §15.10] or [12, Proposition 1c.5]). On the other hand, it is clear that any $p$-norm is a quasi-norm with constant $2^{1/p}$ in the quasi-triangle inequality.

Subsequently we work with pairs of $p$-normed quasi-Banach spaces $(A_0, A_1)$. By this we mean two $p$-normed quasi-Banach spaces $A_j$ which are continuously embedded in the same Hausdorff topological vector space. For $t > 0$, Peetre’s $K$- and $J$-functionals are defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf \left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\right\}$$

where $a \in A_0 + A_1$ and

$$J(t, a) = J(t, a; A_0, A_1) = \max \left\{\|a\|_{A_0}, t\|a\|_{A_1} : a \in A_0 \cap A_1\right\}.$$ 

Note that $K(1, \cdot)$ is the quasi-norm of $A_0 + A_1$ and $J(1, \cdot)$ the quasi-norm of $A_0 \cap A_1$. Moreover, $J(t, \cdot)$ is a $p$-norm on $A_0 \cap A_1$. It will be also useful to work with the functional

$$K_p(t, a) = \inf \left\{(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p)^{1/p} : a = a_0 + a_1, a_j \in A_j\right\}$$
which is a $p$-norm on $A_0 + A_1$ and satisfies that

$$K(t, a) \leq K_p(t, a) \leq 2^{1/p} K(t, a), \quad a \in A_0 + A_1. \quad (1)$$

For $0 < \theta < 1$ and $0 < q \leq \infty$, the real interpolation space realized by means of the $K$-functional $(A_0, A_1)_{\theta, q; K}$ consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{(A_0, A_1)_{\theta, q; K}} = \left( \sum_{m=-\infty}^{\infty} \left( 2^{-\theta m} K(2^m, a) \right)^q \right)^{1/q}$$

(as usual, the sum should be replaced by the supremum if $q = \infty$). The corresponding space defined in terms of the $J$-functional $(A_0, A_1)_{\theta, q; J}$ is the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $(u_m) \subseteq A_0 \cap A_1$ and

$$\left( \sum_{m=-\infty}^{\infty} \left( 2^{-\theta m} J(2^m, u_m) \right)^q \right)^{1/q} < \infty.$$

The quasi-norm on $(A_0, A_1)_{\theta, q; J}$ is given by

$$\|a\|_{(A_0, A_1)_{\theta, q; J}} = \inf \left\{ \left( \sum_{m=-\infty}^{\infty} \left( 2^{-\theta m} J(2^m, u_m) \right)^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

The equivalence theorem shows that $(A_0, A_1)_{\theta, q; K} = (A_0, A_1)_{\theta, q; J}$ with equivalent quasi-norms (see [2,14,15] or [3]). Next we describe the dependence of the constants involved in the equivalence of quasi-norms on the parameters $\theta, q, p$.

3. The results

We start with the embeddings from $K$-spaces into $J$-spaces. We write $[x]$ for the largest integer which is less than or equal to $x$. In the proof we use similar decompositions as in [5,6].

3.1. Theorem. Let $(A_0, A_1)$ be a pair of $p$-normed quasi-Banach spaces, let $0 < \theta < 1$ and $0 < q \leq \infty$. Then

$$\|a\|_{(A_0, A_1)_{\theta, q; J}} \leq 2^{1/p} \theta (1 - \theta) \|a\|_{(A_0, A_1)_{\theta, q; K}}.$$

Proof. Take any $a \in (A_0, A_1)_{\theta, q; K}$. Since

$$\min(1, t^{-1}) K(t, a) = \min(t^\theta, t^{\theta-1}) \frac{K(t, a)}{t^\theta} \leq c \min(t^\theta, t^{\theta-1}) \|a\|_{(A_0, A_1)_{\theta, q; K}},$$

we have that $\min(1, t^{-1}) K(t, a) \to 0$ as $t \to 0$ or $t \to \infty$. Given $v \in \mathbb{Z}$, let $\mu_v = 2^{v[1/\theta]}$. Take any $\varepsilon > 0$. We can decompose $a = a_{0,v} + a_{1,v}$ in such a way that $a_{j,v} \in A_j$ and

$$\|a_{0,v}\|_{A_0} + \|a_{1,v}\|_{A_1} \leq (1 + \varepsilon) K_p(\mu_{v+1}, a)^p.$$
Put \( u_v = a_{0,v} - a_{0,v-1} = a_{1,v-1} - a_{1,v} \). Then

\[
\|a - \sum_{v=-N}^{M} u_v\|_{A_0+A_1} = \|a - a_{0,M} + a_{0,-N-1}\|_{A_0+A_1} \leq \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} \to 0
\]

as \( M \to \infty \) and \( N \to \infty \). Hence \( a = \sum_{v=-\infty}^{\infty} u_v \) in \( A_0 + A_1 \). Moreover, using that \( (\mu_v) \) is an increasing sequence and that \( f(t, \cdot) \) is a \( p \)-norm, we obtain

\[
\begin{align*}
J(\mu_v, u_v)^p & \leq \|a_{0,v}\|^p_{A_0} + \|a_{0,v-1}\|^p_{A_0} + \mu_v^p \|a_{1,v-1}\|^p_{A_1} + \mu_v^p \|a_{1,v}\|^p_{A_1} \\
& \leq (1 + \varepsilon) K_p(\mu_{v+1}, a)^p + (1 + \varepsilon) K_p(\mu_v, a)^p \\
& \leq 2(1 + \varepsilon) K_p(\mu_{v+1}, a)^p.
\end{align*}
\]

Let \( I_v = [\mu_{v-1}, \mu_v) \). The number of integer numbers \( m \) such that \( 2^m \in I_v \) is

\[
\| \{ m \in \mathbb{Z} : 2^{(v-1)[1/\theta]} \leq 2^m < 2^{v[1/\theta]} \} \| = [1/\theta].
\]

Put \( w_m = [1/\theta]^{-1} u_v \) if \( 2^m \in I_v, v \in \mathbb{Z} \). This sequence is also contained in \( A_0 \cap A_1 \) and, by (1),

\[
K_p(1, a - \sum_{m=-N}^{M} w_m)^p \leq K_p(1, a - \sum_{m=-P}^{Q} u_m)^p + d_1 K_p(1, u_Q) + d_2 K_p(1, u_P) \\
\leq 2 \left\| a - \sum_{m=-P}^{Q} u_m \right\|^p_{A_0+A_1} + d_1 \left\| u_Q \right\|^p_{A_1} + d_2 \left\| u_P \right\|^p_{A_0}
\]

for some \( 0 \leq d_1, d_2 < 1 \). This yields that \( a = \sum_{m=-\infty}^{\infty} w_m \) is in \( A_0 + A_1 \).

We obtain that

\[
\begin{align*}
\|a\|_{(A_0+A_1)_p,q} & \leq \left( \sum_{m=-\infty}^{\infty} \left( 2^{-\theta m} J(2^m, w_m) \right)^q \right)^{1/q} \\
& = \left( \sum_{v=-\infty}^{\infty} \sum_{2^v \leq m \leq 2^{v+1}} \left( 2^{-\theta m} J(2^m, w_m) \right)^q \right)^{1/q} \\
& \leq \left( \sum_{v=-\infty}^{\infty} \sum_{2^v \leq m \leq 2^{v+1}} 2^{-\theta m p [1/\theta]^{-p}} (\mu_v, u_v)^p \right)^{1/q} \\
& \leq \left( \sum_{v=-\infty}^{\infty} \sum_{2^v \leq m \leq 2^{v+1}} 2^{-\theta m q [1/\theta]^{-q}} (2(1 + \varepsilon))^{1/q} K_p(\mu_{v+1}, a) \right)^{1/q}.
\end{align*}
\]

Note that \( 2^m \in I_v \) if and only if \( 2^{m+2[1/\theta]} \in I_{v+2} \). So

\[
\sum_{2^v \leq m \leq 2^{v+1}} 2^{-\theta m q} = 2^{2\theta [1/\theta] q} \sum_{2^{v+2} \leq m \leq 2^{v+3}} 2^{-\theta n q} \leq 2^q \sum_{2^v \leq m \leq 2^{v+1}} 2^{-\theta n q}.
\]
Therefore
\[
\|a\|_{(A_0, A_1)_{\theta,q,K}} \leq \left( \sum_{v=-\infty}^{\infty} \frac{1}{\theta} \right)^{\theta/p} K_p(\mu_{v+1}, a)^{q/\theta} 2^{2q} \sum_{v'=-\infty}^{\infty} 2^{-\theta q v'} \right)^{1/q} \leq 2 \frac{2^{1+1/p} (1 + \epsilon)^{1/p} [1/\theta]^{-1}}{1 - 2^{-\theta q}} \right)^{1/q}. 
\]
Since \([1/\theta]^{-1} \leq 2\theta\), using (1) and passing to the limit as \(\epsilon \to 0\), we get
\[
\|a\|_{(A_0, A_1)_{\theta,q,K}} \leq 2^{1+1/p} \theta \|a\|_{(A_0, A_1)_{\theta,q,K}}. 
\]  
(2)

On the other hand,
\[
2^{-\theta m} K(2^m, a; A_0, A_1) = 2^{(1-\theta) m} K(2^m, a; A_1, A_0), \quad a \in A_0 + A_1. 
\]
This yields that \((A_0, A_1)_{\theta,q,K} = (A_1, A_0)_{-(1-\theta),q,K}\) with equality of quasi-norms. Similarly,
\[
2^{-\theta m} f(2^m, u_m; A_0, A_1) = 2^{-(1-\theta)(-m)} f(2^m, u_m; A_1, A_0), \quad a \in A_0 \cap A_1. 
\]
So, if \(a = \sum_{m=\infty}^{\infty} u_m\) is any \(J\)-representation of \(a \in (A_0, A_1)_{\theta,q,J}\), working with \(v_m = u_{-m}\), which also satisfies \(a = \sum_{m=-\infty}^{\infty} v_m\), we obtain that \((A_0, A_1)_{\theta,q,J} (A_1, A_0)_{-(1-\theta),q,J}\) with equality of quasi-norms. Consequently, by (2), for any \(a \in (A_0, A_1)_{\theta,q,K}\) we derive
\[
\|a\|_{(A_0, A_1)_{\theta,q,J}} = \|a\|_{(A_1, A_0)_{(1-\theta),q,J}} \leq 2^{1+1/p} (1 - \theta) \|a\|_{(A_0, A_1)_{\theta,q,K}} = 2^{1+1/p} \theta \|a\|_{(A_0, A_1)_{\theta,q,K}}. 
\]
This and (2) yield that
\[
\|a\|_{(A_0, A_1)_{\theta,q,J}} \leq 2^{1+1/p} \min\{\theta, 1 - \theta\} \|a\|_{(A_0, A_1)_{\theta,q,K}}. 
\]
Finally, since \(\min\{\theta, 1 - \theta\} \leq 2\theta(1 - \theta)\), the desired estimate follows.

Before passing to the embeddings of \(J\)-spaces into \(K\)-spaces we establish an auxiliary result.

3.2. Lemma. Let \(0 < \theta < 1, 0 < q < \infty\) and put \(J = \left( \sum_{m=-\infty}^{\infty} 2^{-\theta m} q \min(1, 2^m)^q \right)^{1/q} \). Then
\[
2^{-1/2}(\theta(1 - \theta) q \log 2)^{-1/4} \leq J \leq 2^{1/4}(\theta(1 - \theta) q \log 2)^{-1/4}. 
\]

Proof. We have
\[
J^q = 1 + \sum_{m=1}^{\infty} 2^{-\theta m} q + \sum_{m=1}^{\infty} 2^{-(1-\theta) m} q
= 1 + \frac{2^{-\theta q}}{1 - 2^{-\theta q}} + \frac{2^{-(1-\theta) q}}{1 - 2^{-(1-\theta) q}}
= 1 + \frac{1}{2^{\theta q} - 1} + \frac{1}{2^{(1-\theta) q} - 1}. 
\]
For all $x \geq 0$ it holds that $e^x - 1 \geq x$, whence $(2^x - 1)^{-1} \leq (x \log 2)^{-1}$. This implies
\[
J^q \leq 1 + \frac{1}{\theta q \log 2} + \frac{1}{(1-\theta)q \log 2}
= 1 + \frac{1}{\theta(1-\theta) q \log 2}.
\]
Now using $1 + \frac{1}{x} = \frac{x+1}{x} \leq \frac{e^x}{x}$ for $x > 0$ and $\theta(1-\theta) \leq 1/4$ for $0 < \theta < 1$, we obtain the upper estimate
\[
J^q \leq \frac{e^{\theta(1-\theta)q \log 2}}{\theta(1-\theta) q \log 2} \leq \frac{2^{q/4}}{\theta(1-\theta) q \log 2}.
\]

For the proof of the lower estimate we use the inequality $e^x - 1 \leq x e^x$, which gives $(2^x - 1)^{-1} \geq (2^x \log 2)^{-1}$ for all $x > 0$. Therefore, we obtain
\[
J^q = 1 + \frac{1}{2^{\theta q} - 1} + \frac{1}{2^{(1-\theta)q} - 1}
\geq \frac{1}{2^{\theta q} \theta q \log 2} + \frac{1}{2^{(1-\theta)q} (1-\theta) q \log 2}
= \frac{(1-\theta) 2^{(1-\theta)q} + \theta 2^{\theta q}}{2^{q} (1-\theta) \theta q \log 2}.
\]

From the arithmetic-geometric mean inequality
\[
(1-\theta) a + \theta b \geq a^{1-\theta} b^\theta \quad \text{for} \quad 0 < \theta < 1 \quad \text{and} \quad a, b > 0,
\]
we get
\[
(1-\theta) 2^{(1-\theta)q} + \theta 2^{\theta q} \geq 2^{(1-\theta)^{q} + \theta^q q} \geq 2^{q/2},
\]
where we have also used that $(1-\theta)^2 + \theta^2 = 2 \left( \theta - \frac{1}{2} \right)^2 + 1 \geq \frac{1}{2}$. This implies the lower estimate
\[
J^q \geq 2^{-q/2} \left( \theta(1-\theta) q \log 2 \right)^{-1}
\]
and completes the proof. \hfill \square

Note that if $q = \infty$, then
\[
\sup_{m \in \mathbb{Z}} \{2^{-dq} \min(1, 2^m) \} = 1.
\]

3.3. Theorem. Let $(A_0, A_1)$ be a pair of $p$-normed quasi-Banach spaces, let $0 < \theta < 1$, $0 < q \leq \infty$ and put $r = \min(p, q)$. Then
\[
\| a \|_{(A_0, A_1)_{\theta, p, q}} \leq 2^{1/4} \left( \theta(1-\theta) r \log 2 \right)^{-1/r} \| a \|_{(A_0, A_1)_{\theta, r}}.
\]
Dependence of the parameters in the equivalence theorem

Proof. First note that if \( u \in A_0 \cap A_1 \) and \( n, m \in \mathbb{Z} \), then

\[
K_p(2^n, u) \leq \min(1, 2^{n-m})/\!(2^m, u).
\]

Besides, since \( r \leq p \), the functional \( K_p(\cdot, \cdot) \) is also an \( r \)-norm.

Given any \( a \in (A_0, A_1)_{\theta, q, l} \) and any \( l \)-representation \( a = \sum_{m=-\infty}^{\infty} u_m \) of \( a \), we have

\[
K_p(2^n, a) \leq \sum_{m=-\infty}^{\infty} K_p(2^n, u_m) \leq \sum_{m=-\infty}^{\infty} \min(1, 2^{n-m})/\!(2^m, u_m) \leq \sum_{m=-\infty}^{\infty} \min(1, 2^{n-m})/\!(2^m + n, u_{m+n}) \leq \sum_{m=-\infty}^{\infty} \min(1, 2^{-m})/\!(2^m + n, u_{m+n}) \leq \sum_{m=-\infty}^{\infty} \min(1, 2^{-m})/\!(2^m + n, u_{m+n}) \leq 2^{1/q} /\!(\theta(1 - \theta) q log 2)^{-1/q} /\!(\sum_{v=-\infty}^{\infty} 2^{-\theta q} /\!(2^v, u_v)^{q})^{1/q}.
\]

If \( r = p < q \), so that \( 1 < q/p \), we derive with the help of the triangle inequality in \( \ell_{q/p} \) that

\[
\|a\|_{(A_0, A_1)_{\theta, q, l}} \leq \left( \sum_{n=-\infty}^{\infty} 2^{-\theta n p} /\!K_p(2^n, a)^{q/p} \right)^{1/q} \leq \left( \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} 2^{mp \theta} /\!\min(1, 2^{-m})^p 2^{-\theta(m+n) p} /\!(2^m + n, u_{m+n})^{p/q} \right) \right)^{1/q} \leq 2^{-1/q} /\!(\theta(1 - \theta) p log 2)^{-1/p} /\!(\sum_{v=-\infty}^{\infty} 2^{-\theta q} /\!(2^v, u_v)^{q})^{1/q}.
\]

Consequently,

\[
\|a\|_{(A_0, A_1)_{\theta, q, l}} \leq 2^{-1/q} /\!(\theta(1 - \theta) r log 2)^{-1/r} /\!\|a\|_{(A_0, A_1)_{\theta, q, l}}.
\]
In the case of a pair of Banach spaces with \( p = 1 \), \( K(t, \cdot) \) is a norm and we can establish Theorems 3.1 and 3.3 working directly with \( K(t, \cdot) \).

We obtain the following result.

**3.4. Corollary.** Let \((A_0, A_1)\) be a pair of Banach spaces, let \( 0 < \theta < 1 \) and \( 1 < q \leq \infty \). Then

\[
\frac{1}{24} \|a\|_{(A_0, A_1)_{\theta, q, K}} \leq \theta(1 - \theta) \|a\|_{(A_0, A_1)_{\theta, q, K}} \leq \frac{2^{1/4}}{\log 2} \|a\|_{(A_0, A_1)_{\theta, q, K}}.
\]

Comparing the estimates given in Theorems 3.1 and 3.3 and focusing on the exponent of the term in \( \theta \), we observe that if \( q < 1 \), then \( (\theta(1 - \theta))^{1/q} \) decreases faster than \( \theta(1 - \theta) \) as \( \theta \to 0 \) or \( \theta \to 1 \). We finish the paper by showing that the exponent of \( \theta(1 - \theta) \) in Theorem 3.1 can be improved if the \( p \)-normed couple \((A_0, A_1)\) is a mutually closed pair, that is

\[
A_j = \{ a \in A_0 + A_1 : \sup_{m \in \mathbb{Z}} \{ 2^{-mj} K(2^m, a) \} < \infty \} \quad \text{for} \ j = 0, 1.
\]

The proof is based on the strong fundamental lemma. As usual, we write \((A_0 + A_1)^o\) for the closure of \( A_0 \cap A_1 \) in \( A_0 + A_1 \).

**3.5. Theorem.** Let \((A_0, A_1)\) be a mutually closed pair of \( p \)-normed quasi-Banach spaces. Let \( 0 < \theta < 1 \) and \( 0 < q \leq p \). Then there is a constant \( c_{p,q} \) depending only on \( p \) and \( q \) such that

\[
\|a\|_{(A_0, A_1)_{\theta, q, K}} \leq c_{p,q}(\theta(1 - \theta))^{1/q} \|a\|_{(A_0, A_1)_{\theta, q, K}}.
\]

**Proof.** According to [14, Theorem 3.2], there is a constant \( d_{p,q} \) depending only on \( p \) and \( q \), such that for any \( a \in (A_0 + A_1)^o \) there is \((u_n) \subseteq A_0 \cap A_1 \) such that \( a = \sum_{n=\infty}^{\infty} u_n \) and

\[
\left( \sum_{n=\infty}^{\infty} \min(1, t/2^n) J(2^n, u_n)^q \right)^{1/q} \leq d_{p,q} K(t, a), \quad t > 0.
\]

Take any \( a \in (A_0, A_1)_{\theta, q, K} \). By the above representation and Lemma 3.2, we derive that

\[
\|a\|_{(A_0, A_1)_{\theta, q, K}} \leq \left( \sum_{n=\infty}^{\infty} \left( 2^{-\theta q} J(2^n, u_n) \right)^q \right)^{1/q} \leq \sqrt{2}(\theta(1 - \theta)q \log 2)^{1/4} \left( \sum_{n=\infty}^{\infty} \sum_{m=\infty}^{\infty} 2^{-\theta(q-m)v} \min(1, 2^{m-v}) q 2^{-\theta v} J(2^n, u_n)^q \right)^{1/q} = \sqrt{2}(\theta(1 - \theta)q \log 2)^{1/4} d_{p,q} \left( \sum_{n=\infty}^{\infty} 2^{-\theta q} \sum_{n=\infty}^{\infty} \min(1, 2^{m-v}) q J(2^n, u_n)^q \right)^{1/4} \leq \sqrt{2}(\theta(1 - \theta)q \log 2)^{1/4} d_{p,q} \left( \sum_{m=\infty}^{\infty} 2^{-\theta q m} K(2^m, a)^q \right)^{1/4} = c_{p,q}(\theta(1 - \theta))^{1/q} \|a\|_{(A_0, A_1)_{\theta, q, K}}.
\]
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References


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