DIDACTICA MATHEMATICAE
Vol. 38, 2016, p. 149–174

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On formal and informal notation of calculation during the early learning of arithmetic by young students

Abstract: This article refers to one of the important objectives in teaching mathematics – shaping its formal language. It raises the question of teaching methods in crossing the decimal threshold in addition and subtraction, used in Poland for over 50 years, that are too formal for students of the first educational stage, and that boil down to successive transformations on a symbolic level. On one hand, there will be presented a theoretical analysis of literature concerning the formation of the language of mathematics and inefficient methods associated with a too early transition to the level of symbolic writing, that can lead to the danger of formalism in teaching mathematics. On the other hand, based on fragments of the research, there will be presented extreme difficulties of a second grade student in a primary school in the process of crossing of the threshold of ten in symbolic writing and examples of the use of less formal writing for children, adapted to their natural development. Considerations in the article suggests leaving a rigid school methodology, while encouraging the use of less formal writing in computational mathematics in early childhood education, as a transitional stage in the development of a proper formal language for students.

On the language of Mathematics

The significance of symbols for mathematics is invaluable – symbols are universal and the effectiveness of symbols simplifies the mathematical thinking

Key words: early learning of arithmetic, symbolic language of mathematics, constructivist concepts of teaching, informal notation of calculation.

and is used to process information. Krygowska wrote: "The symbolic language of mathematics is characterized by this peculiarity, that it is the specific tool of mathematics, because – as Poincaré said – there are moments when language itself works for a mathematician" (Krygowska, 1977, p. 32), as well as "automatic symbols allow a mathematician to take a rest from thought, when they can thus creatively attack difficult problems appropriately" (Krygowska, 1955, p. 32). Konior (1996) notes that: "a variable is the basic element of the written language of mathematics, used to express generality; it is in mathematics that we find such an economical and yet extremely simple way to represent the general and express change" (Konior, 1996, p. 74). Turnau (1990) indicates that the language of mathematics consists of written language, spoken language and the language of mental mathematical activity, which is characterized by words, symbols and drawings.

Shaping the formal language plays a specific role in teaching mathematics. Among the general objectives of teaching, as skills needed by contemporary people, we can distinguish:

> - The ability to learn using various sources of information, in particular, textbooks that use the language of mathematics,
> - The ability of "mathematization", ie. schematization and idealization of reality using terms and language of mathematics.
>
> (Turnau, 1990, p. 30)

Many educators of mathematics (Krygowska, 1955, 1956; Turnau, 1990; Konior, 1994, 1996; Ćwik, 1984; Demby, 2000, 2003) agree that teaching symbolism, particularly algebraic symbolism, is not easy and requires a long, thought-out and consistently applied methodical coverage. At the same time, the authors point out the difficulties inherent in an understanding and assimilation by the student of this symbolism:

> The transition from the concrete language to the language of symbols, from words to characters, is the essential difficulty for a student, of which failure is associated with a qualitative change in the thinking of the student.
>
> (Krygowska 1955, p. 29)

The author also points out the technical and psychological difficulties related to the formal character of the symbolic calculation. Among them she mentions: the ambiguity of symbols, the conventionality of writing algebraic expressions, a letter as a sign of a variable and constant, the difficulty of reconciling the automatic and conscious use of symbols.

Important in the use of mathematical language, among other things, is knowledge of reading and transforming of mathematical information: "School manipulates mathematical language, and different modes of representation of concepts, theorems and mathematical procedures, created for the purpose of teaching, it which is associated with the assimilation by the students of certain conventions" (Konior, 1994, p. 327).

The literature, next to the difficulty in controlling the student's formal language of mathematics, draws attention to a certain type of danger. Lack of understanding of the student's letter symbol may indicate a deep-rooted formalism and verbalism in the thinking of the student, "one way to uncover the sources of formalism and verbalism in the teaching of mathematics can be an analysis of typical errors. It is not just for the noting of these errors; the purpose of such analysis should be primarily the explanation of why a student makes a mistake and what is a student's reasoning that leads to an error" (Krygowska, 1955, p. 27). Deep analysis of errors in the understanding of algebraic symbolism can be found in these works (Krygowska, 1955, 1956; Ćwik, 1984; Konior, 1994; Demby, 2003), wherein the authors indicate the source of errors occurring and formulate recommendations for school teaching of algebra.

Beginning of the early learning of school arithmetic

Analysis of the beginnings of the formation of mathematical language, from the point of child development opportunities at an early-stage, still requires a thorough and deep research. Educators of mathematics (Gruszczyk-Kolczyńska, 1992, 2013, 2014; Hejny, 2012, 2013; Semadeni, 2015) accordingly believe that the initial symbolic records in shaping arithmetic among younger students should be introduced in accordance with the constructivist approach, which was based on the pedagogical and psychological research of Piaget and followers of Piaget.

According to constructivism, knowledge is created by a human being, in an integration with the environment, who builds the structure of knowledge based on the information available. So, according to constructivist concepts of teaching, the learner actively constructs knowledge on their own and for themselves.

Gruszczyk-Kolczyńska notes that at the beginning of school education maturity for learning mathematics is required, and in the meantime "it is tacitly assumed that all children beginning their education in school understand operationally and are able to freely use the symbolic and iconic representations" (Gruszczyk-Kolczyńska, 1992, p. 16).

An innovative method of teaching mathematics developed by the Czech professor Hejný'ego, based on constructivism (2012, 2013) is known under the name derived from the name of the author – method of Hejný. The basic idea of this method is to build patterns of mathematical concepts, phenomena, processes and events in the mind of every student. These schemes are made by students in appropriately selected didactic situations, so called "environments" that are close to the child's experience.

Wide considerations, both theoretical and practical, for the early teaching of mathematics at school that takes into account the constructivist approach have been described by Semadeni (2015). The teaching of mathematics that takes into account the views of constructivism should be based on organizing teaching situations "that allow students to construct an important concepts and logical – mathematical schemes in their mind, so that they become the foundation of their knowledge" (Semadeni, 2015, p. 14).

Often educators of mathematics draw attention to the difficult and almost impossible way of the transition to the level of symbolic thinking in a situation where the child is not ready for it, "One of the most common, widespread errors in didactics is a too early transition to the level of symbolic writing, without a sufficiently long period of calculation using concrete things" (Semadeni, 2015, p. 45). Passive transfer of knowledge, e.g. through verbal explanation is not sufficient. The author believes that, at the beginning of the learning of arithmetic, concepts should be formed intuitively and symbolic writing should derive from specific activities performed by a child and in the context of comprehensible situations.

Gruszczyk-Kolczyńska (1992) in deeply analyzing the process of developing of the concept of number in children's minds, clearly shows that students who shaped the concept of numbers in their minds, for many years experienced situations of converting, manipulating objects, estimating and comparing who has more. Without these long-term tests, the development of abstract thinking would not be possible.

Dąbrowski (2007) notes that "the understanding of symbols and manipulating them is by far the hardest part of mathematical education – it is symbols, or rather lack of their understanding, that generates the highest school failures also in terms of arithmetic. Hence, caution and diligence are important in the course of the gradual development of a learner's symbolic language: first the sense – then the symbol, first the meaning - then enrichment of the dictionary with a new concept" (Dąbrowski, 2007, p. 20).

Kalinowska (2010) believes that among many students – future teachers of mathematics and some other teachers – there is still a myth that early abandonment of concrete examples e.g. banning finger calculation, and a qu-

ick transition to the symbolic writing would speed up the memory mastery of arithmetic. The author writes: "it seems that the focus of school mathematics teaching is merely student's symbolic thinking. It is a target phase of development and, as such, is in some sense identified with the 'right' thinking. This is indeed the highest level of development of thinking, which also depends on its previous steps" (Kalinowska, 2010, p. 21). According to the author, the child's cognitive base is formed by action, observing and giving meaning, and only through the use of this base, is it possible to develop the abstract thinking apparatus – "abstract understanding needs to 'grow' from earlier experience" (Kalinowska, 2010, p. 21).

Symbolic writing $5 + 3$, easy from the point of view of an adult, has two meanings: "Process (in terms of time) of adding the numbers 5 and 3 and the object which is the result of the process, i.e. the number of $5 + 3$" (Semadeni, 2015, p. 66). It is worth noting that much more complicated calculations created by a student while exceeding the threshold of ten are the result of the conceptualised addition and subtraction that emerged much earlier. Authors (Gray & Tall, 1994; Semadeni, 2015) distinguish three levels of conceptual addition. First one, the level of addition by counting all of the elements is "entirely based on procedure level: calculating procedure from 1 to 5, already known to a child, combined with another: calculating from 1 to 3, resulting in – while joining two calculated sets into one larger – to a long procedure of calculating from 1 to 8" (Semadeni, 2015, p. 67). At the second level, the level of counting, a child considers the first number as a whole, 5 became the object of thought for a child, but the continuation is at a level of counting procedure, therefore the child counts 6, 7, 8. The third level is a high one, at this level the symbol $8 + 5$ becomes the object of thought and the child just knows that 5 and 3 equals 8. Although not every adult observes these levels in the child's mind, the author believes that every child has to go through all of them and notes: "Trying to teach your child the results of adding by verbally repeating them, does not cause the transition to a higher level and in the worst case can block the growth of a child's thinking. This requires time, and the artificial acceleration of this process is inefficient and may be harmful to a child" (Semadeni, 2015, p. 67). Formation of the concept of subtraction is also associated with the child's transition: first level of procedures, then gradually the level where at least one number becomes the object, but still accompanied by the procedure up to the level of understanding of writing e.g. 9–4 as object number 5. It is worth mentioning that the conceptual mastery of subtraction is more difficult than adding because of the need for a dynamic presentation of subtraction in the context of a decrease.

The crossing of the threshold of ten in the teaching of mathematics

Approximately 50 years ago in Poland the school methodology associated with the introduction of crossing the threshold of ten was based on a strict schedule: complement (or subtract) to ten, and then add (or subtract) the missing part of the number. In the books (Figures 1–3), this method was communicated to a student by way of illustration and various provisions of symbolic:

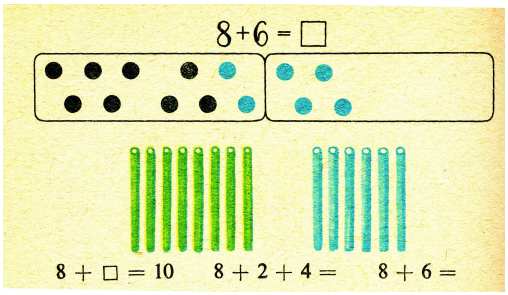

Figure 1. Schoolbook for class I (Rusiecki & Schayer, 1968, p. 102).

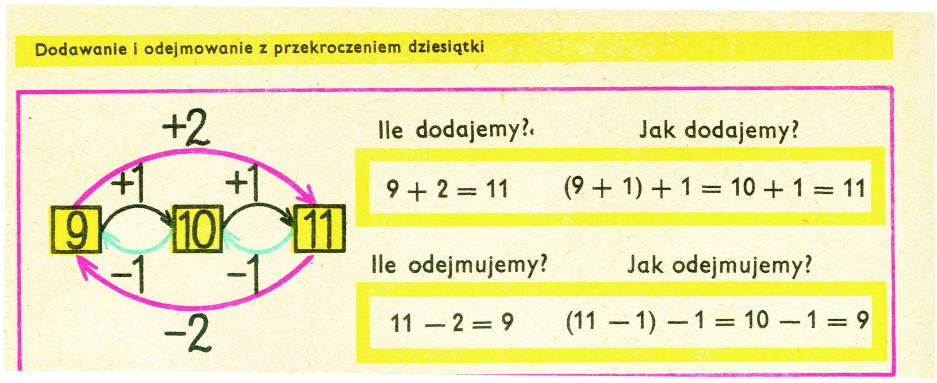

Figure 2. Schoolbook for class I (Cydzik 1973, p. 86).

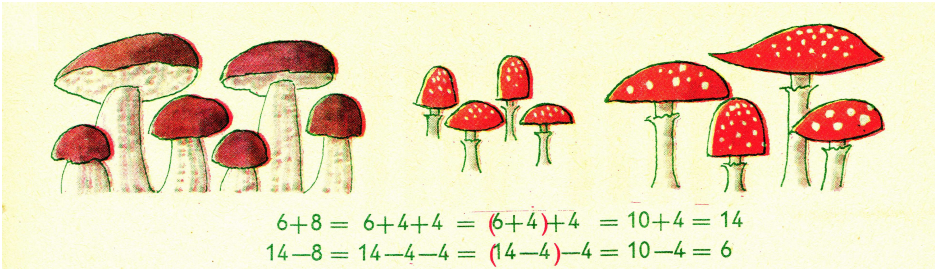

Figure 3. Schoolbook for class I (Puchalska & Ryger 1975, p. 89).

Analyzing the figures presented in the textbooks, it is clear that the drawing was not the carrier (the paradigmatic example) of the method. The drawings do show a full ten immediately, you have to count the dots, sticks or mushrooms to determine that they represent ten. Analyzing symbolic methods of calculations appearing in books, we may see different ways of finding the complement of the first component 10 (with a window or decomposition of the second component for two respective components). In the cited school book (Figure 2), with the addition of a banal $9+2$, there also appears a complicated graph that shows adding and subtracting as mutually opposite actions.

Semadeni (2015), analyzing the method of crossing the threshold of ten textbooks by Cydzik, writes: "strict adherence to this scheme of calculation leads to a peculiar situation. For instance, in the workbook made by Cydzik, a student could find a whole page of such calculations to be performed along with the task: *Calculate according to the formula*. It ended with a forced calculation $10 = 9 + 9 + 1 + 9 = 10 + 9 = 19$, whose absurdity was perceived by many parents. After all, it should be obvious to the student that 9 and 10 equals19. That is the way of explaining to the students the number 19, when they learn to write dozens of other" (Semadeni, 2015, p. 28).

Research conducted by Pisarski (1996) draws attention to the "apparent methodical ease". This term the author defines as the treatment methodology "used by the teacher wanting to help children in the absorption of specific issues" (Pisarski, 1996, p. 144). Such methods are often presented by the authors in textbooks of mathematics. As a result of research the author has identified many apparent methodology facilitations, among them – as an imposed method of reasoning, strange to a child's, logic – he lists *Adding in excess of ten*. He writes: "In the task of adding the numbers 6 and 7 it helps to prepare the following notation: $7 = 6+6+(4+3) = (6+4)+3 = 10+3 = 13$. The teacher explains this notation like this for example: First I decompose the second component of the sum in the right way to the sum of two numbers. Then I add a component to the first distribution of the given numbers, completing it in this way for ten. After this operation, I add the second component of distribution to ten. The intention of the teacher introducing this methodical facilitation is to provide a reliable method of addition. But both symbolic writing, requiring many formal arithmetic operations using parentheses and cited herein verbal explanations, are too difficult for an average student" (Pisarski, 1996, p. 148).

Using two parentheses in the symbolic writing quoted by the author was the result of the associative law of addition – obvious for mathematicians and totally incomprehensible for students at this age, and even for most parents. The required detailed scheme of operations often becomes an unnecessary and arduous effort for a student, "and the development of understanding of what

it is to add, ceases to interest. For it turns out, that the most important need is to remember the given pattern of conduct" (Kalinowska, 2010, p. 11).

It turns out that that methodology of crossing the threshold of ten applies not only in Poland. The following example from a German school is revealing:

> The teacher explained to students the crossing of the threshold of ten with the example $7 + 5$:
>
> Teacher: First, fill up to 10 or $7 + 3 = 10$, then $10 + 2 = 12$.
> He wants to practise this strategy and asks Timo.
>
> Teacher: What is $9 + 4$?
>
> Timo: If it were 10, it would be 14, because $5 + 5$ is 10, and 4 to this is 14, but it's $5 + 4$ so...
> [will be 13] – Timo did not say the result because he was most likely interrupted,
>
> Teacher: Who can explain it again to Timo?
>
> Sina: You have to calculate $9 + 1 =$, and aditionally 3, you have 13.
>
> Teacher: Timo, do you understand?
>
> (Spiegel & Selter, 2003, p. 27)

Analysis of this example clearly shows that Timo meant a different way of calculating this example, not complementing to ten. He saw 5 and 4 in the number of 9, and he wanted to use that fact. Most likely, if he was not stopped he would give the correct result – 13. His way of thinking has not only not been understood by the teacher, but the boy received a clear signal that his strategy is not accepted.

Semadeni (2015) clearly states that we should move away from a rigid methodology from half a century ago, which requires students to pass through a few steps at a symbolic level. In his view, following a detailed path of several steps is too difficult for many children and didactically not very effective. Detailed analysis of the mental operations that a child must conduct comes down to a few steps, and additionally requires the retention of intermediate results in short-term memory. Strategies of calculation of 6 +7 through the completion to ten may here be different. One of them is described above by Pisarski, consisting in the fact that in the first step we properly spread the second component, the number 7. When a child wants to meet this challenge they may spread the number 7 into 2 and 5, but will not receive the full ten. A student must also think about the first number so the distribution was correct. Another strategy may extend to four consecutive steps: complete 6 to 10, if you have already added 4, how much must be added (that is, to complete the number 4 to 7), and further after another equal sign write 10 as a result of $6 + 4$ and to this ten, add the missing 3, and as a result you will get the number 13. Often, however, the number that a student obtained as the final

result of such complicated mental procedures, written in one part with a few equal signs, do not associate with the result of the initial calculation. For them, it is rather a sequence of operations (often incomprehensible), which must be learned as shown on the scheme.

In the literature (Gruszczyk-Kolczyńska, 1999; Semadeni, 2015), you can find a description of the use of the support for the creation of writings discussed here. It is a twenty piece abacus (Figure 4), in which there are 10 beads in one color and 10 beads in a different color strung on its line. The next picture (Figure 5), shows the operation of $8+5 = 13$ through an appropriate deduction and moving beads.



Figure 4. The abacus: 10 and 10.



Figure 5. The abacus: $8 + 5 = 13$.

Analogically, abacuses can be found in some of the current school textbooks, however, it is important to remember that a static analysis of the figure presented in the manual is not sufficient. It is important to move or place beads in two colors on the abacus.

Research methodology

Below there will be discussed parts of longitudinal studies, i.e. longitudinal studies conducted since 2013. The survey covered one student in the period of early childhood education within the grades I-III, and now attending grade IV. The studies showed the difficulty he had in mastering mathematics. At the beginning of second grade the student received psycho-pedagogical assessment, that suggested a risk of dyslexia. It is therefore worth stating that the description of his behaviour outlined above will not be that of the average student of today's school. The aim of the research is to penetrate deep into mathematical cognitive processes of one student. The observed processes can shed light on similar thought processes, which may not so obviously appear in the daily school practice, but are important for understanding certain aspects

of mathematical thinking. The main research method is case study where the researcher makes the current diagnosis, while strongly stimulating the child's efforts. Associated methods are the participant observation (during which notes have been collected) and analysis of the child's work (homework, additional tasks, school tests, and the student's reaction to the way of teaching.

Description and analysis of the results of research

Excerpts of research results presented here are related to a mathematical problem, namely crossing the decimal threshold for addition and subtraction in the field of 100. First the methodology of work with the student will be briefly outlined and then follows a deep analysis of the four tests which are a summary of the skills acquired by the student and also one test of grade III.

Characteristics of the student's experience

Since the first grade of primary school the student used textbooks and workbooks from the series"*A do Z*" (Białobrzeska, 2014). The methodology for crossing the decimal threshold in the range 20 is proposed at the end of grade I and is based on a story made in the form of drawing abacuses, where each step of the algorithm can be used: complementation to 10 is shown in Figure 6.

In Figure 6 are linear operations with a few equality signs and brackets that take into account all the required steps. The teacher introducing the algorithm gives up the proposed static drawings and bases the methodology on the aforementioned teaching tool – a base twenty abacus, 10 beads in two colors (Figure 4). Each student had his abacus at his disposal. After all necessary operations on the abacus are done by the student, he then follows the linear calculations proposed by the author of the textbook in the type: $8 + 6 = (8 + 2) + 4 = 10 + 4 = 14$. The student described in the study also used an abacus while doing the homework. It is worth mentioning that the calculations in the range of 20 with the abacus did not cause any difficulty for him. What's more, with this help he was able to complete the required exercise book linear calculations and even felt some success when he managed to write the calculation correctly on the blackboard in the math class. At the end of the first grade when the teacher was checking the memory calculations in the range of 10 (also 20) the student had not mastered memory accounts, his skill in counting remained at the second stage of complementation, supported by calculations on his fingers.

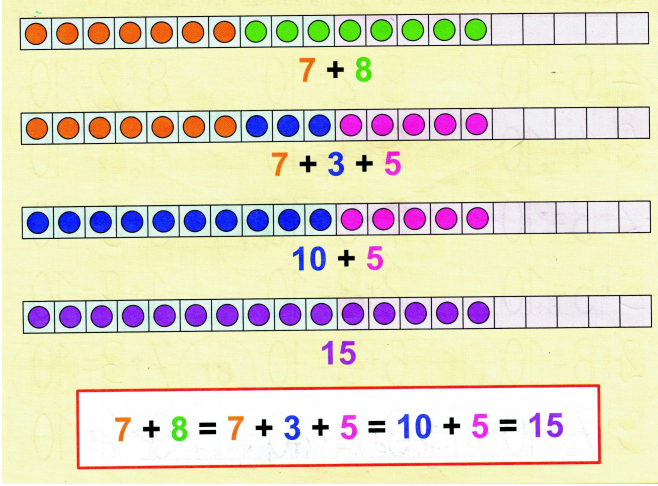

Figure 6. Schoolbook for class I (Białobrzeska, 2013, p. 34).

At the beginning of the second class, while reminding the students about calculations in the range of 20, the teacher no longer supported the learning process with the base twenty abacus. After a brief reminder of calculation in the range of 20 the teacher tests writing skills for linear symbols when crossing the decimal threshold. A month later, the teacher also checks skills on "writing out" when crossing the decimal threshold regarding calculations to 100.

During individual work with the student as part of the research, the student still had access to the abacus. With its help, he was able to write in the way that is so-called. "writing out" in the field of 20. Before the test, there was organized some exercises of conducting calculations without asking for help. Verbal language while writing out the calculations e.g. $8 + 7$ was as follows: Let's write a parenthesis, then let's rewrite the first number (shows it on fingers and looking for completion to 10) add 2, close parenthesis, (now look at the second number) and says 2 and adds so that the second number is obtained, notes the exact number of fingers, that is 5, now equals $10 + 5$ equals 15. This procedure is not always performed correctly, it is reproduced mechanically – sometimes the student forgets what should be in the next step, he is not sure when to close parenthesis, or if he should write an equality sign at the time.

Works presented below (Figures 7–9) are tests the student performed in lessons of mathematics in the second half of September in the second grade. At the exams there was no access to help, counting on his fingers, we might say was not overt, but rather discreet. Another presented work (Figure 10) is the test at the end of the third grade on the calculations in the range of

100. At that time there was no need to write the calculations in one note, in "writing out". Yet still the student helps himself by doing calculations on his fingers. All deletion marks appearing in the student's work are made by the teacher.

Analysis of test 1

In the first exercise (Figure 7) which was on the addition of single-digit numbers with the "writing out" when crossing the decimal threshold, the student obtained only one correct result and it was probably completely by accident:

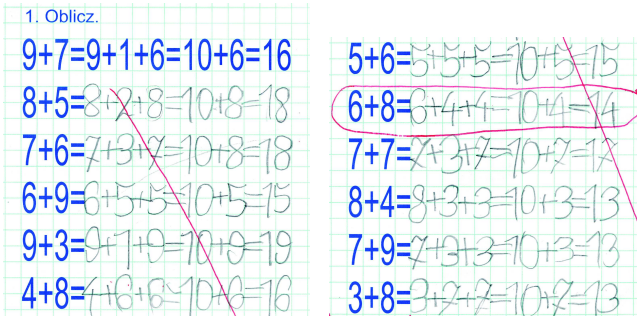

Figure 7. Second grade (9 years old), September 2015.

The analysis of this work shows that the student put a lot of effort to transcribe as many as 11 examples, and numbers appearing in calculations, although not correct, are not accidental. Almost all examples (except two) the student correctly reproduced the first step of the algorithm, "you have to make up to ten". The next correct step of the algorithm could be, for example this: "I had to add 5, I already added 2 to 5 so I miss 3, so I will add 3". Meanwhile, in half of these cases, after the second sign "+" the student entered simply the first component of the primary sum (in example 2, 3, 5, 7, 9). This surprising fact, of rewriting the symbol of the first component of the sum, was commented on by the student after the test in an oral interview as follows: "So I was supposed to look at the second number?". This indicates clearly the mechanical step of "looking at a second number" due to dispersion of the second component. In the other half of the cases after the second "+" sign the student for some reason enters the same number, which occurs in the vicinity. He remembered that after the second equal sign in each example he should complete ten and thus enters a number 10 (even where 6 and $8+5+3$ does not result in 10). Then to 10 he adds the number from the last operation which, however, is not a "complement" to the second component and, consequently, receives erroneous final results.

Analysis of test 2

A week later, the student writes another test about subtraction in the range of 20 over the decimal threshold (Figure 8):

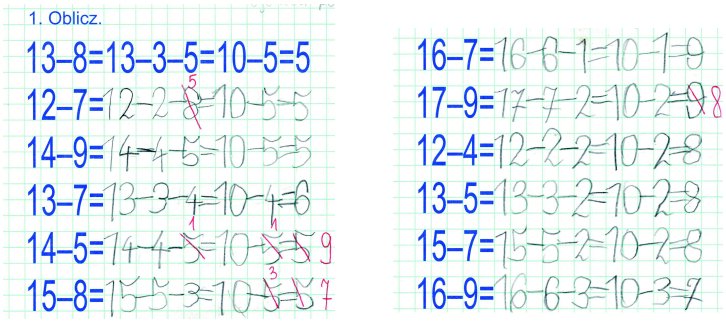

Figure 8. 2nd grade (9 years old), September 2015.

This time, the number of correct results is relatively higher compared to the first test. You can see the clear progress resulting in remembering (perhaps partial understanding) the second step of the algorithm, "if you have to subtract 9 from 17 and 7 is already subtracted you have to subtract another 2". Only a few mistakes appeared here.

Analysis of test 3

A month after learning the notation of calculation when crossing the decimal threshold in the range of 20, a student writes a test of adding a single digit number to a two – digit number (Figure 9):

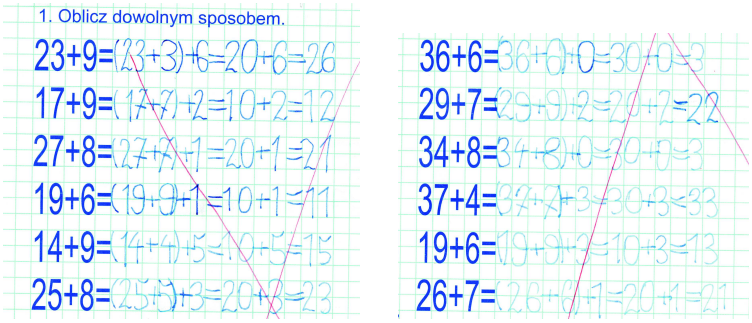

Figure 9. 2nd grade (9 years old), November 2015.

The student writes the second digit of the first number almost always after "+" sign! As if in the student's memory there was still: "look at the second number". His understanding of the second written number in the example of

$27 + 8$ is 7, or it is a reaction to visual stimulus. Next, after the second "+" sign the student tries to enter the complement of the second primary component of addition. And so on: 3 plus 6 equals 9, 2 plus 7 equals 9, 7 plus 1 equals 8 and further 4 and 5 equals 9 and 3 plus 5 equals 8. And finally another example of logic that the student developed: 6 and 0 to 6. A modification of this logic can be seen in the following example: after 9 plus 2 not equals 7! But the student, through the erroneous execution of the first step, which is the number 9, tried to find meaning to, we could say, symbol juggling. Thus, he completed the number 7 with two, and he obtained nine. Similarly, he did two penultimate examples, completing: the number 4 to seven with 3, number 6 to nine with 3. The values entered after the second equal sign may surprise us: $23 + 3 = 20$, $7 + 17 = 10$, $27 + 7 = 20$ and the same way in all examples. One possible hypothesis is that the student has done subtraction here. But looking at the overall work of this student, it is very likely that symbolic notation that was too early for him allowed him to remember only certain steps of the algorithm, perhaps he remembered a fragment of the strategy: "numbers with zero at the end must appear everywhere" – we would say "it must appear in full tens" therefore he simply writes down 20, 10 and 20. As before, the final step in the calculation is almost correct, but as a result of erroneous previous steps no final result is correct. It is worth noticing the difficulty of symbolic writing when adding the number $0 : 30 + 0 = 3$ appears in two examples.

Analysis of test 4

In the fourth test (Figure 10) of subtraction of a single digit number from a two-digit one, we might think that the student has mastered the next steps of the algorithm:

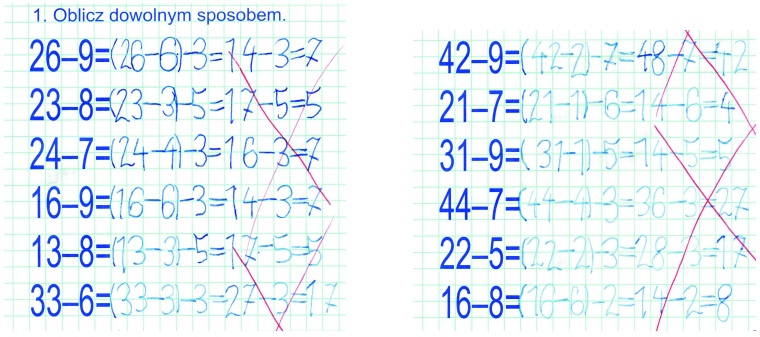

Figure 10. 2nd grade (9 years old), November 2015.

He faultlessly subtracts in 12 examples to get a full 10, and then subtracts the missing "part" of the subtrahend. Surprising, however, are the results that appear after the second equal sign. They show an "odd" function, that the student performs: probably he subtracts numbers in brackets from full tens: 20, 10, 30 or 40. Moreover, giving the final result he repeats the procedure: again he subtracts the subtrahend from full tens, and not from the written numbers. Saved calculations seem absurd: $26-6$ equals 14 as if he is counting $20-6=14$, and $14-3$ to 7, as if: $10-3$ to 7. The observations of the student's behaviour shows that it is the effect of a "not complete" procedure of counting on fingers subtracting the appropriate number of fingers from a full ten, e.g. by performing subtraction: $26-6$ the student opens his palm twice with 10 fingers and closes 6, he sees the other four fingers and he remembers one ten is gone so he writes the result of 14.

Analysis of test 5

Two other examples of works of the same student in the second semester third grade are listed here (Figures 11, 12):

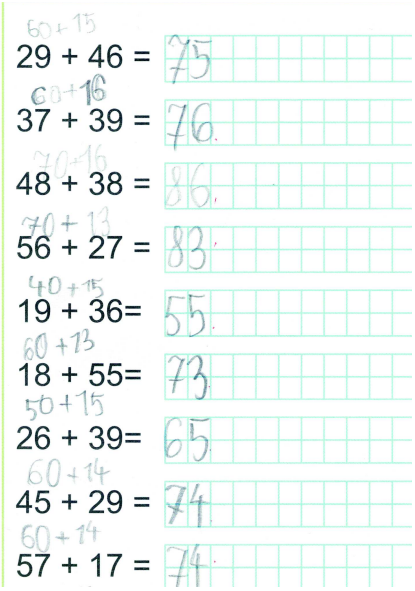

Figure 11. 3nd grade (10 years old), April 2016.

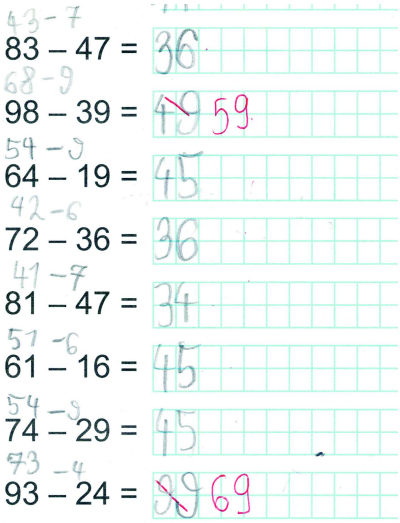

Figure 12. 3nd grade (10 years old), April 2016.

All calculations presented in these works are still supported by counting on fingers individually decimal and unity. Without this support the student would be completely helpless in obtaining the result. It turned out that writing partial

results of the activities down was extremely helpful in obtaining correct results and in the student's success. But it was not a formal notation with triple equal sign, that would look like this: $29 + 46 = (20 + 40) + (9 + 6) = 60 + 15 = 75$; $83-47 = (8340)-7 = 43-7 = 36$ (such notation was proposed in a textbook). In addition, it is worth noting that the calculation method used by the student in these works is completely different than one that is in the range of 20, where the student had to "reach" a full ten.

Research Report no. 2

The second study was also a longitudinal one. The methodology of this study was the same as in the first research. The Participant of the research was also one student who showed interest in mathematics and even clear mathematical ability. The study began in 2013, when the boy was 5 years old. Currently, he is a student of second grade. Individual research was conducted 3 or 4 times a month, with a duration of 1 hour (excluding holidays). During the study we observed the student's behaviour and we stimulated his cognitive processes in the field of mathematical thinking. The collected material was full of student behaviour records in the form of his written work as well as video recordings. A detailed description of these studies wont be presented in the article, merely, for the sake of the subject considered in the article, an indication of the importance of the transitional phase, such as informal, spontaneous notes of the arithmetic calculations in shaping the origins of arithmetic.

Characteristics of the student experience

The method of individual work with students was based on the constructivist approach, and the main goal was to create an understanding and an exploration of the concept of number, basic arithmetic and geometric concepts. The student worked on a particular material in the early stages of work, gradually moving into the phase of iconic representation. In terms of development of addition and subtraction in the range of 100 researchers referred to adding tokens representing given components, in the range of 1000 the researchers worked on paper bills. Multiplication was formed in specific situations, such as the formation of several groups with the same number of components (in simulated real situations, e.g. putting 3 cakes on each of 4 plates), laying several dice with the same number of faces. Division was shown by simulating real situations both in terms of the fitting and dividing. Gradually, these activities were accompanied by symbolic writing of calculation with the final result that was found by using concrete things. After a series of performed activities the

student was encouraged to move to the level of iconic representation, then the symbolic ones while focusing on understanding the meaning of the action, for example: $7 \cdot 4$ to seven fours, or presented with the drawing of cubes, or writing four symbols of number 7, or $24 : 6$ is counting how many sixes can fit in 24.

An indispensable aid when performing the calculations was the student's fingers. A common procedure of a researcher was to encourage the student to independently note and try to solve mathematical operations that were new to him.

Below is illustrated some informal notes made by the student. Some of them are made spontaneously, others are suggested by the researcher through relating them to the previously performed operations – the reader will find further details during the analysis of works of the student. Works are from different periods and presented in order to show the way a mathematical experience becomes more and more frequent and refined through a student's achievements moving in the direction of fluency in symbolic language.

Notes on adding

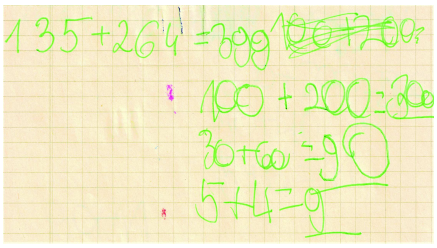

Figure 13. Work 1: 5 years 6 months.

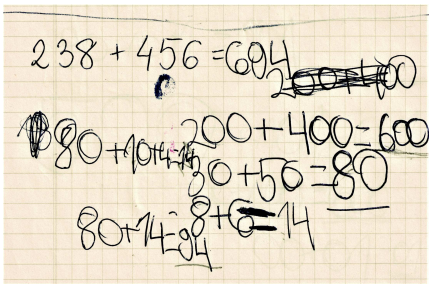

Figure 14. Work 2: 5 years 6 months.

To find the result of adding two three-digit numbers, the student makes some side notes of additional calculations (Figure 13). The notation method used for calculation is spontaneous and is not dictated by the researcher. He adds 100 to 200 and performs deletion due to lack of space on the sheet. At the age of five years this is a natural lack of planning skills of records on paper, so that there is enough space for the whole note. Below, more on the left side he repeats addition and writes the result 300. Next, he adds 30 to 60 and 5 to 4. He underlines all the partial results and writes 399 as the final result. Clearly the boy understands the sense of a three-digit number, for example the number 135 as 100 plus 30 plus 5. This knowledge is illustrated in the following auxiliary calculation, where he separately added hundreds, tens and ones, and finally he enters the correct result. In the second example (Figure 14), which is more

difficult because it exceeds the decimal threshold, the researcher intervenes slightly by stressing partial results, which must be taken into account in the final result. The suggestion is understood by the student and it carries him to the correct result.

Records of multiplication

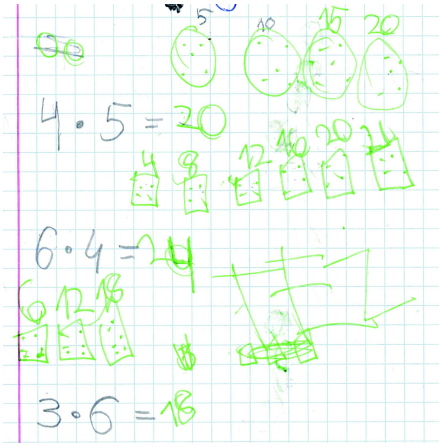

Figure 15. Work 3: 6 years 5 months.

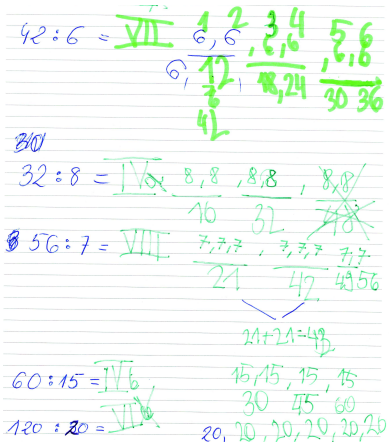

Figure 16. Work 4: 8 years and 1 month.

Notes (Figures 15, 16) show the notation of the calculation of the results of multiplication, the meaning of which the student already understands. In Figure 15 finding the products is supported by additional drawings of dots in loops or dice and additional numbers, which are successive multiples of one of the factors. Counting the dots on dice was one of the methods of individual work with the student and he spontaneously started to perform such drawings instead of composing them. In another work, instead of drawings the student writes a relevant number of identical components and enters indirect multiple data numbers still using his fingers. This is the way suggested by the researcher: to count four sevens, we can write down these sevens.

Notation of dividing

The next two works of the student (Figures 17, 18) refers to finding the results of dividing. The way to find the result of dividing, if the student has not mastered the multiplication tables by heart, is proposed by the researcher: we have to check if 6 fits into 42, so all we have to do is just write 6's until we have 42 (the first two symbols of the number 6 was written by the researcher). The

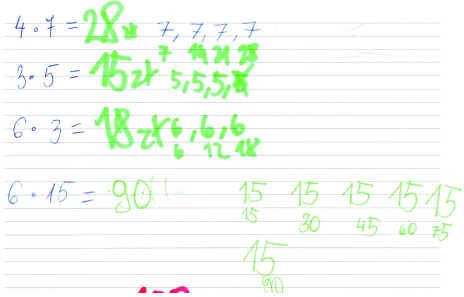

Figure 17. Work 5: 8 years and 1 month.

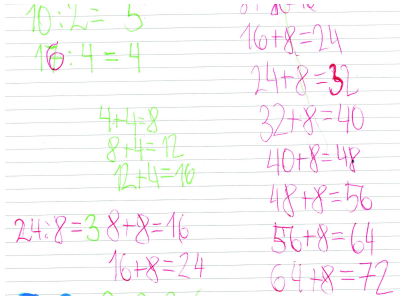

Figure 18. Work 6: 8 years, 1 month.

student not only wrote six correctly, he then wrote multiples, spontaneously at the top of the counted six, entering the numbers $1, 2, \ldots 7$. Calculating $32 : 8$ clearly shows that he understands the situation and he makes notes freely. You can see that after writing too many eights and finding the number 48 he comes back to the result of 32, he deletes two eights and enters number 4 as the quotient (as an additional attraction note the final results using roman characters). The note accompanying dividing in the next task is completely different. It arose as a result of joint work with the researcher. The initial difficulty that occurred was to count how many eights were there in the calculation, as in the first action two eights should be taken into account. In further examples he easily counts repeating numbers and writes the result at the end, next to the initial operation.

The conclusions of the study

The results of this research not only confirm considerations presented in the article and described in the quoted literature, but drastically demonstrate the problem taking a student to the level of use of mathematical symbols too early.

The student, when writing calculations in a formal way, dutifully approached the task. However, he found himself in a situation where a symbolic writing is needed, this type of calculation was beyond him. It is worth mentioning that if the student was not required to write out a formal account he could, in most cases, give correct results of operations. Not that it would be given from his memory, but thanks to understanding the sense of operations, he could create his own strategies for calculations on his fingers, even in situations of calculating in the range of 100, exceeding ten. However, he was required to use the complicated writing in the prescribed form. He tried to mechanically reproduce some steps and algorithms and in lack of understanding these steps, he created his own erroneous but consistent strategies. The difficulty in under-

standing the symbolic writing observed here can be described in Krygowaska's words, who examined errors in the context of the use of the student's letter symbol in the science of algebra: "A balance between conscious and automatic use of symbols is often absent, of which the ultimate result is verbalism and formalism in mathematical thinking" (Krygowska, 1955, p. 34).

Analysis of results in later periods of education shows that a clear impact of the initial success of the student encouraged him to use an auxiliary, less formal way of calculation in the range of addition and subtraction to 100. The primary difference in these notes was the appearance of only one equal sign, instead of three, after which there was the final result. Obviously, the time interval between the difficulties with the formal writing with a use of brackets presented here and correct calculations without a formal writing in the range of 100 was more than a year. So we can hypothesize that the additional impact on the positive effects of operating on numerical material could be an effect of the constructivist approach used in the individual work of a child.

Analysis of the results of the second research clearly shows that informal notes can be an important intermediate step in the development of formal, proper symbolic language. In the initial stage, they are far more understandable for the student than a too soon and rigidly imposed formal notation of activities. This is a very important phase of the development of the correct mathematical language. Not infrequently, these notes are original, surprising for an adult and well illustrating the process of children's thinking.

It is worth noting that spontaneous auxiliary calculations of the second student in addition process contained only one equal sign. Notation with multiple equal signs is not consistent with a child's logic and requires a long period of shaping the understanding of the meaning of such notation. That could be indicated by a broad discussion on difficulties in assigning an equal sign in a mathematical notation and the process of formation of the concept of the equal sign, says Konior (1994). The author argues that students in the early years of education treat the "=" sign as a signal to perform operations occurring on its left: "Very often the only given data is an expression "A", in which "there is something to do'', and the equal sign, and results appear as a complete record of "$A = B$" appears later. This one-sided understanding of the symbol "=" has its origin in the dominance of one direction: from a given operation (A) to the result (B). One direction is of course also a culturally conditioned way of writing and commonly used habit of reading from left to right" (Konior, 1994, p. 330). An important clue could be a methodical treatment of notation with a single equal sign, which is consistent with the natural development of a child, as a transitional stage in the application of more complicated formal writing.

Summary

Mathematics cannot be a set of schematic tricks for mechanical remembering for a student. Promoting a single, well-defined method for crossing the threshold of ten, not only cannot become a universal method that a student can apply in every case, but also can be unavailable for a student because of its formalism when making calculation notes. The requirement of "writing out" when crossing the threshold of ten becomes useless not only for the weak students, it is also senseless for gifted students, because "the specifics of mental activity of students is based on the trend of using 'shortcuts in thinking'. Students solve the task without the proposed methodology by 'facilitating' intermediate steps. It's not a facilitation for them. For those children who are mentally ahead of average expectations of school, it is difficult to understand the meaning of writing out, for example, exceeding ten through completion, if after reading the description of the task, they immediately know what the answer would be" (Klus-Stańska, 2004, p. 111).

Using one acceptable strategy of calculation does not apply only to the problem of crossing the threshold of ten in mathematics. For example, the results of research conducted among third graders (Dąbrowski, 2008; Kalinowska, 2010) indicate that in the calculation of 199 + 87, 1007 – 999 the majority of students (approx. 80%) use one, learned strategy: "It turned out that the high level of difficulty of performing calculations written in these examples, did not discourage most of the students and in the first operation over 80% (average 85.15%) of them tried to apply a written addition" (Kalinowska, 2010 p. 94).

Meanwhile, many studies indicate that the appropriate approach to teaching mathematics allows children to explore their own often surprising strategy for calculations (Dąbrowski, 2006, 2009; Kalinowska, 2010; Rasch, 2001). Informal notation often reveals the student's mind and is understandable to the child. Unfortunately, sometimes from the point of view of the convention of mathematical writing, it contains mistakes. It takes a lot of teacher's care to gradually show what is the mistake and not to do any harm to the child's logic.

> First of all, you need to let the students do it and encourage them, because the use of one "best" method is often a source of cognitive limiting. Creating many different ways of calculating makes the students better understand the role of numbers in operations and their interdependencies. ... Calculations made by students using personal strategies on the one hand gives the teacher a deeper "look" into the ways of understanding the students' actions on numbers, and on the other hand it is beneficial to the development of their thinking. This allows constructing new ways of counting, that generates mathematical knowledge. This

> makes it more flexible and useful because the means of calculation can be modified, depending on the "look" of a specific situation.
>
> (Kalinowska, 2010, p. 93)

The use of informal notation by a student may, however, be overlooked in school teaching for various reasons. One of them is the use of workbooks, commonly used in the first stage of education – which in its editorial structure does not provide space for free, non-schemed notation, and often they leave only lines, grids and spaces for notes provided by the author. There is no place there for additional notes or drawings.

Forming a precise, formal mathematical language in the students' minds is one of the important goals of mathematics education. The stage of informal notes can be an important transitional phase in teaching formal language associated with the symbolism of arithmetic, and also, in a longer term of algebra.

While it is worthwhile to appreciate the natural thinking of children in this transition phase, we must be also aware of certain dangers that can create or preserve some ways of formalsation in the mind of a student, and furthermore it can lead to a mistaken understanding of arithmetic expressions. The learning process of informal writing should be organized with great vigilance, so that it will not turn into a formally "facilitated" teaching, described by Krygowska (1956), indicating significant dangerous moments for learning, such as calculation schemes without conceptual content, imperfectly developed responses to visual stimuli. In terms of arithmetic, the author calls for a balance between the written calculation and memory: "Memory calculation and considerations about quality counterbalance conditional responses to visual stimuli, if they are wrongly shaped by the absence of our vigilance" (Krygowska, 1956, p. 53) The fundamental idea of the author in the search for teaching methods is to take care in organizing a learning process so that in future there will be no need to remove from a student's mind what was already shaped there: "formalism and verbalism distorts student's thinking and it will be difficult to undo this destructive process" (Krygowska, 1955 , p. 39).

Seeking for methodological guidelines in order to assist in the development of mathematical language, taking into account the phase transition of informal notes, it is worth mentioning two important principles of teaching formulated by Turnau (1990), (which – as the author himself states – are rarely applied consistently):

1. Introduction of a formal algorithm or rule may be made only after the assimilation of operations, after which the child knew how to perform this operation in simple cases with the use of specific materials or activities performed in the mind;

> 2. After introducing the algorithm or rule, in the course of using it, you should return to the intuitive, that is, referring to specifics, understanding of the operation so often that such a return is always possible.
>
> (Turnau, 1990, p. 124)

The author formulates these principles in the context of creating and mastering the rules and formal algorithms of students' operations, and also refers to them while shaping the language of algebra, where the "updating of the right kind of intuitive understanding of an algebraic object might give the idea of the proper operation and prevent error" (Turnau, 1990, p. 163). It seems that the use of the aforementioned principles of teaching, may be relevant also in the enviroment of the development of mathematical language in the initial stage of teaching school arithmetic.

In the first phase, before formal notes are used, it's worth it to encourage the idea of making informal, spontaneous notes by students and to discuss their mathematical correctness. In the second phase it's worth it to specify a student's notes in the mature forms of notation, as well as in the situation of an error, to give the opportunity to return to specific activities, from which the note originates.

It also seems reasonable that in the process of teaching of symbolic language of arithmetic, we should take into account the guidelines formulated by Demby (2003), although the author speaks in the context of teaching skills of transformation of algebraic expressions:

> Didactic operations of teachers are likely to be effective only if their aim is to convert students' rules to the rules that lead to correct algebraic calculation and towards building a system of rules – among other things, by analogy or contrasting with other rules. ... It is important not to block the creation of their own rules by students, but to support the process of their improvement, as well as to strengthen the confidence of students about their own thinking.
>
> (Demby, 2003, p. 106)

Informal, own notes used by students should be modified in the direction of becoming more transparent, formal and to become an important stage in the development of the symbolic language of mathematics.

References

B i a ło b r z e s k a, J.: 2013, *Od A do Z, Ćwiczenia do matematyki, Klasa I, Część 4*, Didasco, Warszawa.

B i a ło b r z e s k a, J.: 2014, *Od A do Z, Ćwiczenia do matematyki, Klasa II, Część 5*, Didasco, Warszawa.

C y d z i k, Z.: 1973, *Matematyka dla I klasy szkoły podstawowej*, Państwowe Zakłady Wydawnictw Szkolnych, Warszawa.

Ć w i k, M.: 1984, Zdegenerowany formalizm w myśleniu niektórych uczniów szkoły średniej, *Dydaktyka Matematyki* **3**, 45–84.

D ą b r o w s k i, M.: 2006, *Pozwólmy dzieciom myśleć*, Centralna Komisja Egzaminacyjna, Warszawa.

D ą b r o w s k i, M.: 2009, *Badanie umiejętności podstawowych uczniów trzecich klas szkoły podstawowej. Trzecioklasista i jego nauczyciel. Raport z badań ilościowych 2008*, Centralna Komisja Egzaminacyjna, Warszawa.

D e m b y, A.: 2000, Typy procedur algebraicznych, stosowanych przez uczniów w wieku 13–15 lat, *Dydaktyka Matematyki. Roczniki PTM seria 5* **22**, 45–74.

D e m b y, A.: 2003, Kształtowanie się umiejętności przekształcania wyrażeń arytmetycznych u uczniów w wieku 13–15 lat, *Dydaktyka Matematyki. Roczniki PTM seria 5* **25**, 69–109.

G r a y, E. M., T a l l, D. O.: 1994, Duality, ambiguity and flexibility: A proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, **25(2)**, 115–141.

G r u s z c z y k – K o l c z y ń s k a, E.: 1992, *Dzieci ze specyficznymi trudnościami w uczeniu się matematyki*, WSiP, Warszawa.

G r u s z c z y k – K o l c z y ń s k a, E.: 2013, *Nauczycielska diagnoza edukacji matematycznej dzieci*, Nowa Era, Warszawa.

G r u s z c z y k – K o l c z y ń s k a, E. (ed.): 2014, *Edukacja matematyczna w klasie I*, CEBP 24.12 Sp. z o.o., Kraków.

G r u s z c z y k – K o l c z y ń s k a E., Z i e l i ń s k a, E.: 1999, *Dziecięca matematyka. Program dla przedszkoli, klas zerowych i placówek integracyjnych*, WSiP, Warszawa.

H e j n ý, M.: 2012, *Exploring the Cognitive Dimension of Teaching Mathematics throughScheme-oriented Approach to Education*, ORBI SCHOLAE **6(2)**, 41–55.

H e j n ý M., Z e m a n o v á, R.: 2013, Vyučování orientované na budování schemat v praxi, in. B. Tomková, M. Mokriš (eds.) *Matematika v primárnej škole. Rôzne cesty, rovnaké ciele*, Prešov, 82–86.

K a l i n o w s k a, A.: 2010, *Pozwólmy dzieciom działać – mity i fakty o rozwijaniu myślenia matematycznego*, Centralna Komisja Egzaminacyjna, Warszawa.

K l u s - S t a ń s k a, D.: 2004, Dziecko uzdolnione matematycznie. Adrian – odsłona pierwsza, in: D. Klus-Stańska (ed.) *Świat dziecięcych znaczeń*, Wydawnictwo Akademickie Żak, Warszawa.

K o n i o r, J.: 2002, Posługiwanie się przez uczniów znakiem równości w wyrażeniach matematycznych, in: J. Żabowski (ed.) *Materiały do studiowania dydaktyki matematyki, Tom IV, Prace prof. dr hab. Jana Koniora*, WN Novum, Płock (przedruk z Matematyka, Nr 4, 1994).
K o n i o r, J.: 1996, O pojęciu zmiennej w nauczaniu szkolnym matematyki, *Dydaktyka Matematyki. Roczniki PTM seria 5* **18**, 71–102.
K r y g o w s k a, Z.: 2003, O poprawne rozumienie przez uczniów symbolu literowego w nauce algebry, in: J. Żabowski (eds.) *Materiały do studiowania dydaktyki matematyki, Tom I, Prace prof. dr Anny Zofii Krygowskiej*, WN Novum, Płock, (przedruk z Matematyka, Nr 4, 1955).
K r y g o w s k a, Z.: 2003, O niebezpieczeństwie formalizmu w nauczaniu algebry w szkole, in: J. Żabowski (eds.) *Materiały do studiowania dydaktyki matematyki, Tom I, Prace prof. dr Anny Zofii Krygowskiej*, WN Novum, Płock (przedruk z Matematyka, Nr 4, 1956).
K r y g o w s k a, Z.: 1997, *Zarys dydaktyki matematyki*, cz. 2, WSiP, Warszawa.
K r y g o w s k a, Z.: 1999, *Zarys dydaktyki matematyki*, cz. 1, WSiP, Warszawa.
P i s a r s k i, M.: 1996, Ułatwienia metodyczne w nauczaniu początkowym matematyki, in: B. Nowecki (ed.) *Problemy Studiów Nauczycielskich*, nr 6, Kraków, 144–158.
P u c h a l s k a, E., R y g e r, M.: 1975, *Matematyka dla klasy 1 szkoły podstawowej*, WSiP.
R a s c h, R.: 2001, *Zur Arbeit mit problemhaltigen Textaufgaben im Mathematikunterricht der Grundschule*, Franzbecker, Berlin.
R u s i e c k i, A. M., S c h a y e r, W.: 1968, *Arytmetyka I*, Państwowe Zakłady Wydawnictw Szkolnych, Warszawa.
S e m a d e n i, Z.: 2015, Matematyka w edukacji początkowej – podejście konstruktywistyczne, in: Z. Semadeni (ed.) *Matematyczna edukacja wczesnoszkolna. Teoria i praktyka*, Wydawnictwo Pedagogiczne ZNP, Kielce.
S i w e k, H.: 1998, *Czynnościowe nauczanie matematyki*, WSiP, Warszawa.
S i w e k, H.: 2005, *Dydaktyka matematyki. Teoria i zastosowania w matematyce szkolnej*, WSiP, Warszawa.
S p i e g e l, H., S e l t e r, C h.: 2003, *Kinder & Mathematik. Was Erwachsene wissen sollten*, Kallmeyer, Hannover.
T u r n a u, S.: 1990, *Wykłady o nauczaniu matematyki*, PWN, Warszawa.

O formalnych i nieformalnych zapisach obliczeniowych podczas kształtowania się początków arytmetyki u młodszych uczniów

S t r e s z c z e n i e

Artykuł nawiązuje do jednego z ważnych celów nauczania matematyki – kształtowania formalnego języka matematyki. Została tu omówiona, nadal stosowana i propagowana w Polsce od ponad 50 lat, metodyka nauczania w klasach początkowych przekraczania progu dziesiątkowego podczas dodawania i odejmowania. Metoda ta sprowadza się do kolejnych przekształceń na poziomie symbolicznym i jest zbyt formalna dla uczniów I etapu edukacyjnego.

W artykule prowadzone są, na podstawie literatury, rozważania teoretyczne dotyczące kształtowania się języka matematyki oraz nieefektywności metod związanych ze zbyt wczesnym przejściem na poziom symbolicznego zapisu, które mogą rodzić niebezpieczeństwo formalizmu w nauczaniu matematyki. Mocno akcentuje się zalety podejścia konstruktywistycznego w kształtowaniu początków arytmetyki.

Na podstawie fragmentów badań longitudinalnych zostaną pokazane trudności jednego ucznia kl. II w zakresie przekraczania progu dziesiątkowego przy symbolicznych zapisach oraz pozytywne skutki stosowania przez niego w późniejszej edukacji mniej formalnych zapisów. Ponadto przedstawiono raport z drugich badań, w którym zwrócono uwagę na rolę stosowania mniej formalnych zapisów podczas wykonywania 4 podstawowych działań matematycznych. Stosowane przez badanego ucznia obliczenia były czasami spontaniczne, zgodne z jego naturalnym rozwojem oraz pokazujące pojęciowe rozumienie działań.

Postulaty końcowe sformułowane w artykule sugerują odejście od jednej, sztywnej metodyki szkolnej stosowanej podczas zapisów wykonywanych działań arytmetycznych oraz zachęcają do stosowania mniej formalnych zapisów obliczeniowych w matematycznej edukacji wczesnoszkolnej – jako etap przejściowy w kształtowaniu się poprawnego języka formalnego uczniów. Zaleca się przy tym, by w nauczaniu poczatków arytmetyki zmierzać w kierunku przekształcania własnych uczniowskich zapisów w reguły prowadzące do poprawnych, coraz bardziej formalnych zapisów obliczeniowych.

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